On Global Transverse Feedback Linearization

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Abstract

Necessary and sufficient conditions for global transverse feedback linearization are presented, together with easy-to-check sufficient conditions.

1 Problem Formulation

Consider the smooth dynamical system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1}$$

defined on $\mathbb{R}^n$, with f and g smooth vector fields, $h : \mathbb{R}^n \to \mathbb{R}^p$ $(p \geq 2)$[1] smooth, and $u \in \mathbb{R}$.

Given a smooth parameterized curve $\sigma : \mathbb{D} \to \mathbb{R}^p$, where $\mathbb{D}$ is either $\mathbb{R}$ or S^1, the maneuver regulation problem entails finding a smooth control $u(x)$ making the output of the system approach the set $\sigma(\mathbb{D})$ and making sure that the curve is traversed in one direction. The feedback $u(x)$ should also be such that $\sigma(\mathbb{D})$ is invariant under the output dynamics, meaning that

$$\begin{pmatrix} h(x(0)) \in \sigma(\mathbb{D}) \\ L_f h(x(0)) + L_g h(x(0)) u(x(0)) \in T_{h(x(0))}\sigma(\mathbb{D}) \end{pmatrix} \Longrightarrow (\forall t \geq 0)\ h(x(t)) \in \sigma(\mathbb{D}).$$

When $\mathbb{D} = S^1$, $\sigma(\mathbb{D})$ is a periodic curve. Banaszuk and Hauser in [1] provide a solution to this problem in the special case when $\mathbb{D} = S^1$ and $h(x) = x$. Notice that one particular instance of maneuver regulation is the case when a controller is designed to make $y(t)$ asymptotically track a specific time *parameterization* of the curve $\sigma(t)$. Thus asymptotic tracking and maneuver regulation are closely related problems. In some cases, however, it may be undesirable to pose a maneuver regulation problem as one of tracking because tracking controllers don't make $\sigma(\mathbb{D})$ invariant under the output dynamics. Moreover, even if a maneuver regulation problem admits a solution, its time parameterized version may not (consider, for instance, the problem of maneuvering a wheeled vehicle with bounded translational speed by means of steering).

[1] We do not allow single output systems ($p = 1$) because in such case the problem investigated in this paper degenerates to asking that y follows the entire real line.

We impose geometric restrictions on the class of curves $\sigma(\cdot)$.

Assumption 1: The curve $\sigma : \mathbb{D} \to \mathbb{R}^p$ enjoys the following properties

(i) σ is C^r, $(r \geq 1)$

(ii) σ is regular, i.e., $\|\dot{\sigma}\| \neq 0$

(iii) $\sigma : \mathbb{D} \to \sigma(\mathbb{D})$ is injective (when $\mathbb{D} = S^1$ we instead require σ to be a Jordan curve)

(iv) σ is proper, i.e. for any compact $K \subset \mathbb{R}^p$, $\sigma^{-1}(K)$ is compact (automatically satisfied when $\mathbb{D} = S^1$)

Assumption 1 guarantees that $\sigma(\mathbb{D})$ is a submanifold of $\mathbb{R}^p$ of dimension 1.

Assumption 2: There exists a C^1 map $\gamma : \mathbb{R}^p \to \mathbb{R}^{p-1}$ such that 0 is a regular value of γ and $\sigma(\mathbb{D}) = \gamma^{-1}(0)$. Moreover, the *lift* of $\gamma^{-1}(0)$ to $\mathbb{R}^n$, $\Gamma := (\gamma \circ h)^{-1}(0)$, is a submanifold of $\mathbb{R}^n$.

A sufficient condition for

$$\Gamma = \{x : \gamma_1(h(x)) = \ldots = \gamma_{p-1}(h(x)) = 0\} \tag{2}$$

to be a submanifold of $\mathbb{R}^n$ is that h be *transversal* to $\gamma^{-1}(0)$, i.e., [2, 3]

$$(\forall x \in \Gamma) \ \operatorname{Im}(dh)_x + T_{h(x)}\gamma^{-1}(0) = \mathbb{R}^p. \tag{3}$$

The codimension of Γ is equal to the codimension of $\gamma^{-1}(0)$ which implies $\dim \Gamma = n - p + 1$ [4]. A weaker sufficient condition is

$$(\text{on a neighborhood of } \Gamma) \ \dim(\operatorname{Im}(dh)_x + T_{h(x)}\gamma^{-1}(0)) = \text{constant}. \tag{4}$$

The problem of maneuvering y to $\gamma^{-1}(0)$ is thus equivalent to maneuvering x to Γ and can be cast as an output stabilization problem for the system

$$\dot{x} = f(x) + g(x)u$$
$$y' = (\gamma \circ h)(x). \tag{5}$$

In general one may be able to maneuver x to the subset of Γ which can be made invariant by a suitable choice of the control input. Accordingly, let Γ^* be the largest controlled invariant submanifold of Γ under (1) and let $n^* = \dim \Gamma^*$ ($n^* \leq \dim \Gamma = n - p + 1$). Further, let u^* be a *friend of* Γ^*, i.e., a smooth feedback rendering Γ^* invariant, and define $f^* := (f + gu^*)|_{\Gamma^*}$.

Assumption 3: Γ^* is a closed connected submanifold (with $n^* \geq 1$) and the following conditions hold

(i) $(\exists\ \epsilon > 0)(\forall\ x \in \Gamma^*)\quad \|L_{f^*}h(x)\| > \epsilon.$

(ii) $f^* : \Gamma^* \to T\Gamma^*$ is complete

In [1], $\Gamma^* = \Gamma = \sigma(S^1)$, and it is assumed that $f(x) \neq 0$ on Γ^*. Thus in that work Assumption 3 is automatically satisfied (the completeness of f^* follows from the periodicity of $\sigma(S^1)$).

We first focus on the well-definiteness part of the assumption. In order to derive conditions guaranteeing that Γ^* is a closed submanifold, associate with each constraint in (2) the single input, single output system $\{f, g, \gamma_i \circ h\}$ where $i \in \{1, \ldots, p-1\}$ and a corresponding zero dynamics manifold Γ_i^*.

Lemma 1.1 *If $\bigcap_k \Gamma_k^*$ is a non-empty, closed, controlled invariant submanifold, then Γ^* exists and it is given by* $\Gamma^* = \bigcap_k \Gamma_k^*$.

Proof : ($\subset$) Choose any point $x \in \Gamma^*$. Since $\Gamma^* \subset \Gamma$,

$$(\forall k \in \{1, \ldots, p-1\})\quad \gamma_k(h(x)) = 0.$$

This, together with the fact that, by definition, Γ^* is locally invariant around x, implies that

$$(\forall k \in \{1, \ldots, p-1\}) \;\; x \in \Gamma_k^*$$

or $x \in \bigcap_k \Gamma_k^*$.

($\supset$) Since $\bigcap_k \Gamma_k^*$ is controlled invariant and output zeroing, and since $\Gamma^* \subset \bigcap_k \Gamma_k^*$, one has that, by the maximality of Γ^*, $\Gamma^* = \bigcap_k \Gamma_k^*$.

Let r_i be the relative degree of system $\{f, g, \gamma_i \circ h\}$ and define $\mathcal{H}_i : x \mapsto \mathrm{col}(\gamma_i \circ h(x), L_f(\gamma_i \circ h(x)), \ldots, L_f^{r_i-1}(\gamma_i \circ h)(x))$. If each r_i is well-defined and uniform over Γ, one has that each Γ_i^* is a closed submanifold and $\Gamma_i^* = \mathcal{H}_i^{-1}(0)$. This is not enough to guarantee that $\bigcap_k \Gamma_k^*$ is a submanifold, as the intersection of two submanifolds need not be a submanifold. A sufficient condition for the intersection $\Gamma_i^* \cap \Gamma_j^*$, $i \neq j$, to be a submanifold is that [3]

$$(\forall x \in \Gamma_i^* \cap \Gamma_j^*) \;\; T_x\Gamma_i^* + T_x\Gamma_j^* = \mathbb{R}^n$$

or, equivalently, $\ker\,(d\mathcal{H}_i)_x + \ker\,(d\mathcal{H}_j)_x = \mathbb{R}^n$. Using the fact that $T_x(\Gamma_i^* \cap \Gamma_j^*) = T_x\Gamma_i^* \cap T_x\Gamma_j^*$ one easily arrives at the following result.

Corollary 1.2 Γ^* *is a closed submanifold if each system* $\{f, g, \gamma_i \circ h\}$*,* $i \in \{1 \ldots p-1\}$ *has a uniform relative degree* r_i *over* Γ *and, if* $p > 2$*, the following conditions are satisfied.*

(i) For $k = 1, \ldots, p-2$*,*

$$\left(\forall x \in \bigcap_{j=1}^{k+1} \Gamma_j^*\right) \;\; H_x^k + \ker(d\mathcal{H}_{k+1})_x = \mathbb{R}^n,$$

where H_x^k is defined inductively as

$$H_x^1 := \ker(d\mathcal{H}_1)_x, \qquad k = 1$$

$$H_x^k := H_x^{k-1} \cap \ker(d\mathcal{H}_k)_x, \quad k > 1.$$

(ii) Letting $u_k^* := -\frac{L_f^{r_k}(\gamma_k \circ h)}{L_g L_f^{r_k - 1}(\gamma_k \circ h)}$, $1 \le k \le p-1$,

$$(u_1^*)\big|_{\bigcap_i \Gamma_i^*} = \cdots = (u_{p-1}^*)\big|_{\bigcap_i \Gamma_i^*}.$$

In this case, $n^* = n - \sum_{i=1}^{p-1} r_i$.

Remark 1.1 *Rather than using transversality to derive the sufficient conditions of Corollary 1.2, one can employ a slight modification of the zero dynamics algorithm of [5] (see also [6]) or the constrained dynamics algorithm presented in [7]. In both cases a feasible initial condition for the algorithm should be defined to be any point* $x_0 \in \Gamma^*$ *such that* $f(x_0) + g(x_0)u_0 \in T_{x_0}\Gamma^*$ *for some real number* u_0. *If the sufficient conditions of Corollary 1.2 are not satisfied, the zero dynamics algorithm may still find a* locally *maximal controlled invariant submanifold of* Γ.

Remark 1.2 *Condition (i) in the Corollary above can be weakened by assuming, instead that, for* $k = 1, \ldots, p-2$,

$$\left(\text{on a neighborhood of } \bigcap_{j=1}^{k+1} \Gamma_j^*\right) \ \dim(H_x^k + \ker(d\mathcal{H}_{k+1})_x) = \text{constant}.$$

The condition, in Assumption 3, that $\|L_{f^*}h(x)\| > \epsilon$ on Γ^* implies that there are no equilibria on Γ^* and that, whenever $x \in \Gamma^*$, $\|\dot{y}\| = \|L_{f^*}h(x)\| > \epsilon$. This condition ensures that the output of (1) traverses the curve $\sigma(\mathbb{D})$. The next example illustrates that this condition is not strictly necessary for the feasibility of the maneuver regulation problem.

Example 1.1 *Consider the dynamical system and path*

$$\dot{x} = \mathrm{col}(x_2, u, x_3)$$

$$y = \mathrm{col}(x_1, x_2), \quad \sigma : \lambda \in \mathbb{R} \mapsto \mathrm{col}(\lambda, \lambda)$$

Here $\mathbb{D} = \mathbb{R}$ *and* $\sigma(\mathbb{D}) = \{y : y_1 - y_2 = 0\}$. *The lift* Γ *is given by* $\Gamma = \{x : x_1 - x_2 = 0\}$ *and it is readily seen that* $\Gamma^* = \Gamma$ *and a friend of* Γ^* *is* $u^* = x_1$. *Assumption 3 is not satisfied since there exists a single point on* Γ^* *where* $L_{f^*}h(x) = \mathrm{col}(x_2, x_1) = 0$. *Yet, almost all initial conditions on* Γ^* *result in path traversal. Specifically, the only case where the path is* not *traversed is when* $x_1(0) = x_2(0) = 0$. △

Example 1.1 shows that even if Assumption 3 is violated, it may still be possible to traverse the path. However, if $\|L_{f^*}h(x)\|$ fails to be bounded away from zero, then the situation becomes problematic.

Example 1.2 *Consider the dynamical system and path*

$$\dot{x} = \mathrm{col}\left(x_1 x_3, \frac{-2x_1^2 x_3}{\left(x_1^2+1\right)^2}, x_3\right) + \mathrm{col}(0, u, 0)$$

$$y = \mathrm{col}(x_1, x_2), \quad \sigma : \lambda \in \mathbb{R} \mapsto \mathrm{col}\left(\lambda, \frac{1}{\lambda^2+1}\right)$$

Here $\mathbb{D} = \mathbb{R}$, $\Gamma = \Gamma^* = \{x : x_2 - \frac{1}{x_1^2+1} = 0\}$, *and* $u^* = 0$. *Assumption 3 is not satisfied since*

$$L_f h(x) = \mathrm{col}\left(x_1 x_3, \frac{-2x_1^2 x_3}{\left(x_1^2+1\right)^2}\right)$$

is zero on the set $\{x : x_1 x_3 = 0\}$. *Let* $u = -x_2 + \frac{1}{x_1^2+1}$. *The result is a closed-loop system where any initial condition*

$$x^0 = \mathrm{col}\left(\delta, *, \epsilon\right)$$

where $\epsilon\delta = 0$ *will not result in path traversal. However, initial conditions with* $\epsilon\delta \neq 0$ will *result in path traversal. This example illustrates the fact that Assumption 3(i) avoids pathological situations whereby some phase curves originating outside of* Γ^* *may approach points of* Γ^* *where* $L_{f^*}h = 0$, *thus not traversing the path* $\sigma(\mathbb{D})$. △

We are now ready to formulate the main problems investigated in this paper. The following are a direct generalization of analogous statements found in [1].

Problem 1: *Find, if possible, a single coordinate transformation* $T : x \mapsto (z, \xi) \in \Gamma^* \times \mathbb{R}^{n-n^*}$ *valid in a neighborhood* $\mathcal{N}$ *of* Γ^* *such that in* (z, ξ) *coordinates*

(i) $\Gamma^* = \{(z, \xi) \in \Gamma^* \times \mathbb{R}^{n-n^*} : \xi = 0\}$

(ii) The dynamics of system (1) *take the form*

$$\begin{aligned}
\dot{z} &= f_0(z, \xi) \\
\dot{\xi}_1 &= \xi_2 \\
&\vdots \\
\dot{\xi}_{n-n^*-1} &= \xi_{n-n^*} \\
\dot{\xi}_{n-n^*} &= b(z, \xi) + a(z, \xi)u
\end{aligned} \tag{6}$$

where $a(z, \xi) \neq 0$ *in* $\mathcal{N}$.

It is clear that if one can solve Problem 1, then the smooth feedback

$$u = -\frac{1}{a(z, \xi)}(b(z, \xi) + K\xi). \tag{7}$$

achieves local output stabilization of (5) and hence local stabilization of (1) to Γ^* (resp., $\Gamma^* \cap U^0$). In turn, stabilization of (1) to Γ^* implies, by Assumption 3(i), traversal of $\sigma(\mathbb{D})$ in output coordinates.

It is also not difficult to see that (7) makes $\sigma(\mathbb{D})$ invariant under the output dynamics (see Section 1). In light of this, the main focus of Problem 1 is the output stabilization of (5).

Remark 1.3 *The autonomous feedback control* (7) *achieves exponential stabilization to* Γ^**, however,* (7) *does not prevent the closed-loop system from exhibiting finite escape time (i.e., the entire* $\sigma(\mathbb{D})$ *is traversed in finite time), even though the vector field of the closed-loop system is complete on* Γ^**. A similar problem is encountered in feedback linearization when stabilizing a minimum phase system in normal form. There are various ways to modify* (7) *to avoid finite escape time. Discussing them is beyond the scope of this paper.*

2 Solution to Problem 1

Theorem 2.1 *Problem 1 is solvable if and only if there exists a function* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *such that*

1. $\Gamma^* \subset \{x \in \mathbb{R}^n : \alpha(x) = 0\}$
2. α *yields a uniform relative degree* $n - n^*$ *over* Γ^**.*

Proof : ($\Rightarrow$) Consider system (6) and let $\alpha = \xi_1$. Conditions (i) and (ii) follow immediately.

($\Leftarrow$) From a slight modification[2] of the proof of [6, Proposition 9.1.1] one obtains a coordinate transformation $T : \mathbb{R}^n \to \mathcal{Z}^* \times \mathbb{R}^{n-n^*}$, valid in a neighborhood of $\mathcal{Z}^*$, yielding the normal form (6), where $\mathcal{Z}^* := \{(z, \xi) : \xi = 0\}$ is the zero dynamics manifold associated with the output function α. We are left to show that $\mathcal{Z}^* = \Gamma^*$. First notice that $\Gamma^* \subset \mathcal{Z}^*$ for if $\bar{x} \in \Gamma^*$ then $\alpha(\bar{x}) = 0$. Since through $\bar{x}$ there passes a controlled invariant submanifold, Γ^*, and $\bar{x}$ is output zeroing, it follows that $\bar{x} \in \mathcal{Z}^*$ as well. Finally, since Γ^* and $\mathcal{Z}^*$ are two connected, closed submanifolds of the same dimension and $\Gamma^* \subset \mathcal{Z}^*$, one has that $\Gamma^* = \mathcal{Z}^*$.

[2] Here the main difference is that we do not require that the vector fields $\{\tau_i\}_{i\in\{1\ldots n-n^*\}}$ in [6, Proposition 9.1.1] be complete. This implies that the normal form (6) is valid over a neighborhood $\mathcal{N}$ of Γ^*, rather than $\mathbb{R}^n$. If the vector fields τ_i $i \in \{1 \ldots n - n^*\}$ are complete, then the transformation is globally valid on $\mathbb{R}^n$.

The function α is used to generate the feedback (7) by setting

$$a(T(x)) = L_g L_f^{n-n^*-1}\alpha(x)$$
$$b(T(x)) = L_f^{n-n^*}\alpha(x).$$

The conditions in Theorem 2.1, although rather intuitive, are difficult to check in practice. In what follows we present sufficient conditions for the existence of a solution to Problem 1 which are easier to check.

Corollary 2.2 *If one of the* path constraints *in* (2), $\gamma_{\bar{k}} \circ h$, *yields a uniform relative degree* $n - n^*$ *over* Γ^*, *then Problem 1 is solved by setting* $\alpha = \gamma_{\bar{k}} \circ h$.

Thus, it may be possible to solve Problem 1 by performing input-output linearization choosing as output one of the path constraints. However, Problem 1 may be solvable even when *none* of the path constraints yields a well-defined relative degree. We postpone this discussion until Example 2.1.

Lemma 2.3 *If there exists a function* $\alpha : \mathbb{R}^n \to \mathbb{R}$ *which satisfies the conditions of Theorem 2.1, then for all* $x \in \Gamma^*$

$$T_x\Gamma^* + \operatorname{span}\{g, \ldots, ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n. \tag{8}$$

Proof : By its definition, f^* is such that, for all $x \in \Gamma^*$, $f^*(x) \in T_x\Gamma^*$. Also, by Assumption 3(i), $\operatorname{span}\{f^*\}$ is a one dimensional, hence involutive, distribution on Γ^*. Fixing $x \in \Gamma^*$, the facts above imply that, in a neighborhood of x in Γ^*, there exist $n^* - 1$ one-forms $d\phi_2, \ldots, d\phi_{n^*}$ such that

$$(\operatorname{span}\{f^*\})^{\perp} = (T_x\Gamma^*)^{\perp} \oplus \operatorname{span}\{d\phi_2 \ldots d\phi_{n^*}\}.$$

Define the map $t \mapsto \Phi_t^{f^*}(x)$, which, for some positive δ, is a diffeomorphism of $(-\delta, \delta) \subset \mathbb{R}$ onto its image and denote by ϕ_1 its inverse. By construction, $(L_{f^*}\phi_1)(x) = 1$, implying that $d\phi_1(x) \notin (\text{span}\{f^*(x)\})^\perp$ and thus that

$$(\mathbb{R}^n)^* = (\text{span}\{f^*\}(x))^\perp \oplus \text{span}\,\{d\phi_1\}(x)$$

or, equivalently,

$$\mathbb{R}^n = T_x\Gamma^* \oplus \text{span}\{d\phi_1\ d\phi_2 \ldots d\phi_{n^*}\}^\perp(x). \tag{9}$$

Consider a set of linearly independent vectors $\{v_1, \ldots, v_{n^*}\}$ spanning $T_x\Gamma^*$. Let $V = \begin{bmatrix} v_1 & \cdots & v_{n^*} \end{bmatrix}$ and define a matrix S as follows [1]

$$S = \begin{bmatrix} d\phi_1 \\ d\phi_2 \\ \vdots \\ d\phi_{n^*} \\ dL_f^{n-n^*-1}\alpha \\ \vdots \\ \alpha \end{bmatrix} \begin{bmatrix} V & g & \cdots & ad_f^{n-n^*-1}g \end{bmatrix}$$
$$= \begin{bmatrix} L_V\phi & * \\ 0 & \Delta \end{bmatrix}$$

(all vectors and one-forms are evaluated at x) where $\{L_V\phi\}_{ij} = \langle d\phi_i, v_j\rangle$, $i, j = 1, \ldots, n^*$, and $\Delta \in \mathbb{R}^{n-n^* \times n-n^*}$ is upper triangular with non-zero diagonal (this follows from condition (2) in Theorem 2.1). It is clear that if the matrix $L_V\phi$ is nonsingular then S is nonsingular as well, implying that $\text{Im}([V\ g(x)\ \cdots\ ad_f^{n-n^*-1}g(x)]) = T_x\Gamma^* + \text{span}\{g, \ldots, ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n$ and the proof is complete. To prove that $L_V\phi$ is nonsingular, we use the fact that if A and B are two matrices such that the product AB makes sense then

$$\operatorname{rank}(AB) = \operatorname{rank}(B) - \dim(\ker(A) \cap \operatorname{Im}(B)).$$

Applying this to $L_V\phi$, all we have to show is that

$$\operatorname{Im} V \cap \ker\left(\operatorname{col}(d\phi_1(x), \ldots, d\phi_{n^*}(x))\right) = 0$$

or, equivalently,

$$T_x\Gamma^* \cap \ker\left(\operatorname{col}(d\phi_1(x), \ldots, d\phi_{n^*}(x))\right) = 0,$$

and this follows directly from (9).

Remark 2.1 *Condition* (8) *is a generalization of the notion of transverse linear controllability to the case of controlled invariant submanifolds of any dimension. It is useful in deriving checkable sufficient conditions for the existence of a solution to Problem 1. The notion of transverse linear controllability was originally introduced in [8] and later used in [1] for transverse feedback linearization. In both papers,* $n^* = 1$, $\mathbb{D} = S^1$, *and* $T_x\Gamma^* = \operatorname{span}\{f^*(x)\}$.

Theorem 2.4 *Problem 1 is solvable if*

1. Γ^* *is parallelizable* ($T\Gamma^* \cong \Gamma^* \times \mathbb{R}^{n^*}$)
2. $T_x\Gamma^* + \operatorname{span}\{g \ldots ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n$ *on* Γ^*
3. $(n - n^* \geq 2)^3 \implies$ ($\operatorname{span}\{g \ldots ad_f^{n-n^*-2}g\}$ *is involutive*).

[3] *Notice that since* $p \geq 2$, *one has that* $n - n^* \geq 1$.

Proof : We will show that if the above conditions hold, then a function α can be constructed satisfying the conditions of Theorem 2.1. Let $\{v_1, \ldots, v_{n^*}\}$ be a set of independent vector fields defined on Γ^* such that $T_x\Gamma^* = \text{span } \{v_1, \ldots, v_{n^*}\}(x)$. Global existence of these vector fields is guaranteed by the hypothesis that Γ^* is parallelizable. Condition (1) can be rewritten as

$$\text{span}\{v_1, \ldots, v_{n^*}, g, \ldots, ad_f^{n-n^*-1}g\}(x) = \mathbb{R}^n.$$

We use the flows of these vector fields to generate s-coordinates. Choose any point $x^0 \in \Gamma^*$ and consider the mapping F defined as

$$s \mapsto \Phi^g_{s_n} \circ \cdots \circ \Phi^{ad_f^{n-n^*-1}g}_{s_{n^*+1}} \circ \Phi^{v_{n^*}}_{s_{n^*}} \circ \cdots \circ \Phi^{v_1}_{s_1}(x^0).$$

The map $F : F^{-1}(\mathcal{N}) \to \mathcal{N}$, where $\mathcal{N}$ is a neighborhood of Γ^*, is a diffeomorphism. Let $T_1 = \text{col}(s_1, \ldots, s_{n^*+1})$, $T_2 = \text{col}(s_{n^*+2}, \ldots, s_n)$, and define

$$H_1^{T_1}(x^0) := \Phi^{ad_f^{n-n^*-1}g}_{s_{n^*+1}} \circ \Phi^{v_{n^*}}_{s_{n^*}} \circ \cdots \circ \Phi^{v_1}_{s_1}(x^0)$$

$$H_2^{T_2}(x^1) := \begin{cases} \Phi^g_{s_n} \circ \cdots \Phi^{ad_f^{n-n^*-2}g}_{s^{n^*+2}}(x^1) & \text{if } n - n^* \geq 2 \\ x^1 & \text{if } n - n^* = 1. \end{cases}$$

With these definitions, rewrite $F(s)$ as

$$F(s) = H_2^{T_2} \circ H_1^{T_1}(x^0).$$

Choose $\alpha(x) = s_{n^*+1}(x)$. By construction, any point $x \in \Gamma^*$ can be reached by flowing along

$v_1, \ldots v_{n^*}$. Therefore, in s-coordinates, any point $x \in \Gamma^*$ is represented as

$$F^{-1}(x) = \mathrm{col}\left(\underbrace{* \cdots *}_{n^* \text{ elements}} \; 0 \cdots 0 \right).$$

Thus, on Γ^*, $\alpha(x) = 0$, which proves that condition (1) of Theorem 2.1 is satisfied. Further, notice that, by construction

$$L_{ad_f^{n-n^*-1}g}\alpha = 1$$

on $\mathcal{N}$. If $n - n^* = 1$, this shows that α yields a uniform relative degree 1 over Γ^*, as required. If $n - n^* \geq 2$, let $D = \mathrm{span}\,\{ad_f^i g\}_{i \in \{0,\ldots,n-n^*-2\}}$. By assumption, D is a non-singular and involutive distribution. Let S denote the integral manifold of D passing through the point $H_1^{T_1}(x^0)$. In s-coordinates

$$S = \{s \in V \subset \mathbb{R}^n : s_1 = c_1, \ldots, s_{n^*+1} = c_{n^*+1}\}$$

where c_i, $i = 1, \ldots, n^* + 1$ are constants and V is an open set containing the point $H_1^{T_1}(x^0)$. Thus, for any $s \in S$,

$$T_s S = \mathrm{Im} \begin{bmatrix} 0 \\ I_{n-n^*-1} \end{bmatrix}.$$

Since S is an integral manifold of D it follows that, in s-coordinates, $D(s) = T_s S$, implying that in s-coordinates the vector fields $ad_f^i g$, $i \in \{0, \ldots, n - n^* - 2\}$, have the form:

$$ad_f^i g = \mathrm{col}\left(0 \cdots 0 \; \underbrace{* \cdots *}_{n-n^*-1 \text{ elements}} \right).$$

It readily follows that, on $\mathcal{N}$, $L_{ad_f^i g}\alpha(x) = 0$, $i = 0, \ldots, n - n^* - 2$. Thus, α yields a uniform relative degree $n - n^*$ over Γ^*.

Corollary 2.5 *If $n - n^* \in \{1, 2\}$ and Γ^* is parallelizable, then Problem 1 is solvable if and only if*

(1) *is transversely linearly controllable.*

Proof : This follows directly from Lemma 2.3 and Theorem 2.4.

Corollary 2.2 shows that a sufficient condition to solve Problem 1 is that one of the path constraints yields a uniform relative degree $n - n^*$ over Γ^*. In the next example we use Theorem 2.4 to show that this condition is not necessary and, in particular, Problem 1 may be solvable even when *none* of the path constraints yields a well-defined relative degree.

Example 2.1 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= x_3 + x_1 u \\ \dot{x}_2 &= 1 \qquad\qquad y = \mathrm{col}(x_1, x_2), \\ \dot{x}_3 &= u \end{aligned}$$

and the path $\sigma : \mathbb{R} \to \mathbb{R}^2$ *defined by* $\lambda \mapsto \mathrm{col}(0, \lambda)$*. Then* $\sigma(\mathbb{R}) = \{y \in \mathbb{R}^2 : y_1 = 0\}$*. Let* $\gamma(y) := y_1$*. Then, the lift of the path to the state space is*

$$\Gamma = (\gamma \circ h)^{-1}(0) = \{x \in \mathbb{R}^3 : x_1 = 0\}.$$

The SISO system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 u \\ \dot{x}_2 &= 1 \qquad\qquad y' = \gamma \circ h(x) := x_1, \\ \dot{x}_3 &= u \end{aligned} \tag{10}$$

does not have a well-defined relative degree anywhere on the set $\{x_1 = 0\}$ *because* $L_g(\gamma \circ h) = x_1$ *changes sign in any neighborhood of* $\{x_1 = 0\}$*.*

Application of the zero dynamics algorithm gives that the largest controlled invariant submanifold *contained in* Γ *is*

$$\Gamma^* = \{x : x_1 = x_3 = 0\}.$$

Thus $n^* = 1$ *and the friend of* Γ^* *is* $u^* = 0$*, yielding* $f^* = \frac{\partial}{\partial x_2}$*. We now check the sufficient conditions of Theorem 2.4. We have*

$$g = x_1 \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \quad ad_f g = (x_3 - 1)\frac{\partial}{\partial x_1}.$$

Thus, for all $x \in \Gamma^*$*,*

$$T_x\Gamma^* + \mathrm{span}\{g, ad_f g\}(x) = \mathrm{span}\left\{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, -\frac{\partial}{\partial x_1}\right\} = \mathbb{R}^3$$

showing that the system is transversely linearly controllable and condition (1) in Theorem 2.4 is satisfied. Condition (2) is automatically satisfied by Corollary 2.5. We conclude that, despite the fact that (10) does not have a well-defined relative degree, by Theorem 2.4 there exists a solution to Problem 1. By following the semi-constructive procedure outlined in the proof of Theorem 2.4 we can actually compute an output function α *solving the problem. To this end, we choose* $x^0 = 0$ *and construct the map*

$$s \mapsto x := \Phi^g_{s_3} \circ \Phi^{ad_f g}_{s_2} \circ \Phi^{f^*}_{s_1}(0).$$

We have

$$s_1 \mapsto \Phi^{f^*}_{s_1}(0) = \mathrm{col}(0, s_1, 0)$$

$$(s_2, p) \mapsto \Phi^{ad_f g}_{s_2}(p) = \mathrm{col}((p_3 - 1)s_2 + p_1, p_2, p_3)$$

$$(s_3, q) \mapsto \Phi^g_{s_3}(q) = \mathrm{col}(e^{s_3} q_1, q_2, s_3 + q_3).$$

The composition of the maps above gives

$$\Phi^{g}_{s_3} \circ \Phi^{ad_f g}_{s_2} \circ \Phi^{f^*}_{s_1}(0) = \mathrm{col}(-s_2 e^{s_3}, s_1, s_3).$$

We next find the inverse $x \mapsto s$,

$$s_1(x) = x_2, \quad s_2(x) = -x_1 e^{-x_3}, \quad s_3(x) = x_3.$$

Finally, we pick $\alpha(x) = s_2(x) = -x_1 e^{-x_3}$. *We indeed verify that this output meets the two necessary and sufficient conditions to solve Problem 1: (i)* $\Gamma^* \subset \{x : \alpha(x) = 0\}$ *(when* $x_1 = x_3 = 0$ *one has* $\alpha = 0$*) and, (ii), the system with output* α *has a well-defined relative degree* $n - n^* = 2$ *in a neighborhood of* Γ^*. *This example satisfies Assumption 3 since* f^* *is complete and* $L_{f^*}h = 1$. *This indicates that stabilizing the auxiliary output* $y = \gamma \circ h(x)$*, makes the original system output* $y = h(x)$ *approach and follow the desired path.* △

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