Proceedings of the 2002 Winter Simulation Conference
E. Yücesan, C.-H. Chen, J. L. Snowdon, and J. M. Charnes, eds.

AN EMPIRICAL EVALUATION OF SAMPLING METHODS IN RISK ANALYSIS SIMULATION: QUASI-MONTE CARLO, DESCRIPTIVE SAMPLING, AND LATIN HYPERCUBE SAMPLING

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ABSTRACT

This paper compares the performance, in terms of convergence rates and precision of the estimates, for six Monte Carlo Simulation sampling methods: Quasi-Monte Carlo using Halton, Sobol, and Faure numeric sequences; Descriptive Sampling, based on the use of deterministic sets and Latin Hypercube Sampling, based on stratified numerical sets. Those methods are compared to the classical Monte Carlo. The comparison was made for two basic risky applications: the first one evaluates the risk in a decision making process when launching a new product; the second evaluates the risk of accomplishing an expected rate of return in a correlated stock portfolio. Descriptive sampling and Latin Hypercube sampling have shown the best aggregate results.

1 INTRODUCTION

Recently, there is a widespread use of Monte Carlo Simulation in Finance, like for risk management (Value at Risk, for example) or for derivatives price valuation. In order to cope with the poor precision of the standard Monte Carlo approach, more efficient sampling methods are being suggested in such applications. Two of such approaches are here considered:

1. Pseudo-probabilistic simulation, like Latin Hypercube Sampling (McKay, Beckman, and Conover 1979) and Descriptive Sampling (Saliby 1990 and 1997);
2. Deterministic methods, also known as quasi-Monte Carlo methods, based on low discrepancy sequences and generated without any randomness feature (Paskov and Traub 1995, Traub and Papageorgiou 1996).

After a brief presentation of such methods, our study compares their efficiency in two finance applications: a project Risk Analysis and a correlated stock portfolio evaluation. Monte Carlo sampling was used as a standard benchmarking for comparison.

The comparisons were based on performance indexes defined by two features: the rate of convergence and the error magnitude of the results after the application of a common stopping rule.

2 SAMPLING METHODS

2.1 Latin Hypercube Sampling: Stratified Sets

Latin Hypercube Sampling (McKay, Beckman, and Conover 1979) was suggested as a Variance Reduction Technique, in which the selection of sample values is highly controlled, although still letting them to vary. The basis of LHS is a full stratification of the sampled distribution with a random selection inside each stratum. Sample values are randomly shuffled among different variables or dimensions. Input samples of size n are generated based on the inverse transform method, given by

$$xh_i = F^{-1}[(i-1+R_i)/n)] \ , \ i=1,.., n, \qquad (1)$$

Where R_i stands for an independent random uniform in [0,1] , i=1,.., n, and $F^{-1}(R)$, $R \in (0,1)$ is the inverse transform for the modeled input distribution.

LHS has been widely used in risk analysis, and already implemented in softwares like @RISK and Crystal Ball, among others. In general, it produces substantial Variance Reductions over Standard Monte Carlo in Risk Analysis applications. However, its use in non-terminating simulations, like in queuing systems, is still uncommon.

2.2 Descriptive Sampling: Deterministic Sets

Descriptive sampling (DS) was proposed in order to avoid set variability in simulation studies (Saliby 1990). When using the standard Simple Random Sampling (SRS) or

Monte Carlo approach, two kinds of variation are present in a randomly generated sample - one related to the set of values and the other to their sequence. But, of these two kinds of variability, only the sequence variability is really pertinent, while the set variability, according to the author, is unnecessary and, thus, spurious. In fact, the author's claim is that the introduction of a random set selection in any Monte Carlo application implies a methodological mistake: the introduction of unnecessary noise in the solution process.

Symbolically, it can be written:

Descriptive Sampling = deterministic set x random sequence.

Although an independent development of LHS, Descriptive Sampling can be seen as a limiting LHS case when input sample sizes and the number of input variables (Problem dimensionality) increases towards infinite. In fact, Saliby (1997) has shown that both methods are practically equivalent when n, the input sample size, is over say 100, a typically low value for simulation sample sizes. Moreover, DS is totally equivalent to a variation of LHS known as Central or Midpoint LHS (Owen 1998). Like with LHS, DS sample values are also generated based on the inverse transform method, so that

$$xd_i = F^{-1}[(i-0.5)/n)]\ ,\ \ i=1,.., n, \qquad (2)$$

Where $F^{-1}(R)$, $R \in (0,1)$ is the inverse transform for the particular input distribution.

Again here, sample values are randomly shuffled among different variables or dimensions.

2.3 Discrepancy

Discrepancy measures the departure from the uniformity property (Traub and Papageorgiou 1996; Morokoff and Caflisch 1995).

Discrepancy is defined as:

$$D_N = \sup_{Q \in I^s} \left| \frac{\#\ of\ points\ in\ Q}{N} - v(Q) \right| \qquad (3)$$

Where:

- *N*...is the number of points in the sample or the sample size;
- I^s...is the *s*-dimensional unit-cube in the Universe of the experiment;
- *v(Q)*...is the volume of a sub-region "*Q*" in the unit-cube I^s.

One simple way to understand the discrepancy concept – and the uniform property – is geometrically. As such, two sets of fifty two-dimensional points are plotted in a unit-square. Figure 1 presents a set of fifty points randomly generated, while Figure 2 presents fifty points generated using a low discrepancy sequence for each dimension.

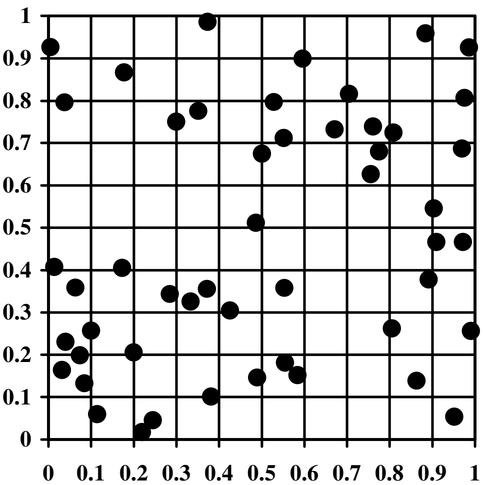

Figure 1: 50 Points in the Unity Square Based on a Simple Random Sampling Generation

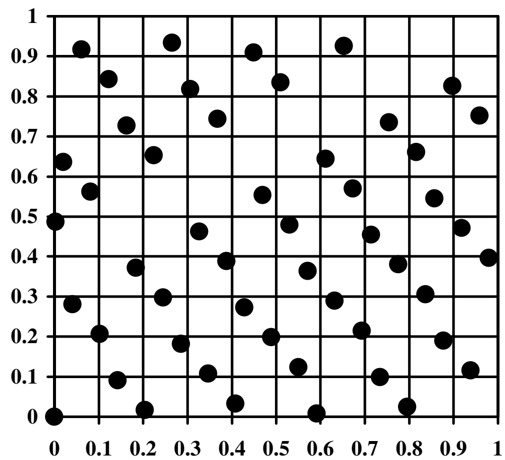

Figure 2: 50 Points in the Unity Square Based on a *Low-Discrepancy* Sequence Generation

Given the importance of the uniformity property in any Monte Carlo application, and that *low-discrepancy* sequences are usually more uniform than random ones, it is reasonable to expect that *low-discrepancy* series will generate better results in simulation experiments than random sampling. The use of *low-discrepancy* sequences is the basis of the Quasi-Monte Carlo sampling method.

2.4 Quasi-Monte Carlo

Morokoff and Caflisch (1995) compared the use of Halton, Sobol, and Faure sequences with simple random sampling. As a rule, Quasi-Monte Carlo performed better than the classic Monte Carlo, but this advantage reduces when the number of variables – problem dimensionality - increases.

The Halton sequence in one dimension is based on a choice of a prime number p and the expansion of a sequence of integers into base p notation. Once defined the prime seed *p*, the *k*-th element of the sequence is derived based on the following two-step procedure.

1. Decompose the number (*K-1*) into *p*-base;

$$k\text{-}1 = [\, a_n \;\; a_{n-1} \;\; \ldots \;\; a_2 \;\; a_1 \;\; a_0 \,]_p. \tag{4}$$

2. Add the terms from the decomposition, divided by increasing powers of prime *p.*

$$h_k = \frac{a_0}{p} + \frac{a_1}{p^2} + \ldots + \frac{a_{n-1}}{p^n} + \frac{a_n}{p^{n+1}} \tag{5}$$

For Halton sequences, a different prime seed is used for each dimension.

The Sobol sequence follows the same principle and algorithm as in the Halton sequence, but the prime number in every sequence is the same and equal to *2* (*p = 2*). The only difference between two Sobol sequences of the same size is the order in which the elements appear. In order to impose a random property in the Sobol series and differentiate one from another, permutation techniques are applied. In simulation experiments with more than one variable, each dimension uses a different permutation of the same set of values when applying the Sobol sampling method.

The Faure sequence, like the Sobol, basically follows the same algorithm as Halton. Now, the prime number, although the same for each dimension, is not fixed and depends on the final sampling size. The prime number in Faure sequences is defined as the lowest prime greater or equal to the sampling size. As in Sobol applications, samples used for two different input variables are based on the same set of values but different permutations.

Preliminary studies by Niederreiter (1988 and 1992) have shown that Halton, Sobol and Faure sequences presented much more uniformity than simple random sampling number series.

Traub and Papageorgiou (1996) have tested the low-discrepancy sequences on a Collateralized Mortgage Obligation application and concluded that Faure performed better than Sobol. The authors also confirmed significant results related to the evaluation of derivative assets with more than 1,500 dimensions using the Faure sequence.

3 EMPIRICAL COMPARISONS

Although comparisons between Quasi-Monte Carlo Methods are usually reported (see for instance Morokoff and Caflisch 1995), their comparison with other more controlled sampling approaches is not readily available. To help filling this gap, this work extends this comparison to Latin Hypercube Sampling and Descriptive Sampling, mainly to evaluate if there is any extra benefit from the use of deterministic sampling methods over the benefits already achieved with the use of controlled sample sets as in LHS or DS.

Both the rate of convergence and the precision of the results were used to compare sampling methods performances.

Ten different sample sizes, ranging from 50 to 500 in steps of 50, were used. The number of simulation runs for each sample size was not fixed, being determined by the stopping rule. Although this approach can lead to less reliable estimates of the relative efficiency of each method, it allows a measurement of their convergence speed. In order to compute an overall efficiency index, the number of "wins" was computed for each sampling method, over all tested sample sizes, for both the rate of convergence and the precision. According to this idea, the method with the best result for each sample size regarding the criterion under analysis is the "winner" on that evaluation. In case of ties, all winner methods are marked. For each criterion, the total number of "wins" indicates the winner sampling method on this criterion.

3.1 Application A1: Risk in Launching a New Product

This application refers to a risk analysis in the launching of a new product in the market, evaluating the Net Present Value (NPV) risk profile, in particular, the mean NPV.

For the basic case, initial investment is fixed, product market life ranged from 2 to 3 years, product cost and selling price are fixed, and annual sales are independent and identically distributed according to a given empirical cumulative distribution function. The opportunity capital cost for the Company is also known.

Experiment variations increased the number of input variables: the problem dimensionality. Variations increased the range of market life (yearly sales) and also transformed some fixed values in random ones: initial investment and product cost. Whilst the basic case had 5 random variables (dimensions), the other three variations tested had 8, 12 and 16.

The number of wins in terms of the convergence rate was about the same for all sampling methods. However, the number of wins regarding the precision of the results was quite different, as shown in Table 1. As seen, Quasi-Monte Carlo sampling methods performed badly. On the other hand, Descriptive Sampling and Latin Hypercube Sampling performed better, with a slight advantage to LHS, but not strong enough to distinguish both methods.

Table 1: Number of "Wins" in Terms of Precision of the Results for the Sampling Methods in the First Application (A1)

# Vars	Sampling Method					
	Halton	Sobol	Faure	Descript	HyperC	Rand
5				6	4	
8				3	6	1
12				5	3	2
16				1	7	2
Total				15	20	5

3.2 Application A2: Stock Portfolio Evaluation

In this application, the return of a balanced stock portfolio, composed by four equally weighted and correlated stocks, was simulated. The stocks returns are identically distributed, according to a multivariate normal distribution, with known parameters: mean, standard deviation and correlation matrix. The simulation estimate was the probability that the portfolio return exceeds a given investor cost of capital.

Derived from the basic case, two experiment variations were also considered. In the first variation, the stock portfolio was composed by eight equally weighted and correlated stocks, whilst in the second variation the number of such stocks was ten. Again here, the idea was to vary problem dimensionality.

Correlated samples under Quasi-Monte Carlo, LHS and DS were simulated using a procedure described in Iman and Conover (1982) and based on the Cholesky transformation.

Again here, the number of wins, for each sampling method, in terms of convergence rate was about the same for all of them. As in the first application, the number of wins regarding the precision of the results was quite different, as shown in Table 2. Here again, Quasi-Monte Carlo performed badly. Descriptive Sampling , Latin Hypercube Sampling and, unexpectedly, Simple Random Sampling performed better.

Table 2: Number of "Wins" in Terms of Precision of the Results for the Sampling Methods in the Second Application (A2)

# Vars	Sampling Method					
	Halton	Sobol	Faure	Descript	HyperC	Rand
4	1	1	1	1	2	4
8			1	3	3	3
10	1			4	3	2
Total	2	1	2	8	8	9

4 SUMMARY OF RESULTS

The results for both applications and all variations were consolidated in Table 3 for the convergence rate criterion and in Table 4 for the precision of the results criterion.

Table 3: Consolidated Number of "Wins" in Terms of Convergence Rate for Each Sampling Method (Basic-Cases and Variations)

App	Sampling Method					
	Halt	Sob	Faure	Descr	HyperC	Rand
A1	7	8	8	8	8	1
A2	7	4	6	7	5	3
Tot	14	12	14	15	13	4

Table 4: Consolidated Number of "Wins" in Terms of the Precision of the Results for Each Sampling Method (Basic-Cases and Variations)

App	Sampling Method					
	Halt	Sob	Faure	Descr	HyperC	Rand
A1				15	20	5
A2	2	1	2	8	8	9
Tot	2	1	2	23	28	14

Although a more careful study is still necessary, the results as a whole indicates that both Descriptive Sampling and Latin Hypercube Sampling performed far better than the others, in particular Quasi-Monte Carlo.

5 CONCLUSION

The Latin Hypercube Sampling method have shown the best aggregate performance index closely followed by Descriptive Sampling. In fact, as already mentioned, both methods are practically equivalent, being the observed performance differences certainly due to chance variantions.

On the other hand, the poor relative performance of Quasi-Monte Carlo Methods deserves a further confirmation in more extensive experiments. However, it seems that all advantages from their use are in fact only related to the input set control and, as such, Latin Hypercube Sampling and Descriptive Sampling, in particular, are more efficient and easier to apply.

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