Efficiency Loss in a Cournot Mechanism for Network Resource Allocation

Ramesh Johari and John N. Tsitsiklis

***Abstract*— We consider a resource allocation problem where individual users wish to send data across a network to maximize their utility, and a cost is incurred at each link dependent on the total rate sent on that link. We analyze a model where users choose the rates at which they want to send along available paths, and each link sets a price equal to the marginal cost of the rate through it. As long as users do not anticipate the effect of their actions on prices, such a scheme can maximize the sum of users' utilities minus the sum of the links' costs (called aggregate surplus). Continuing previous efforts to quantify the effects of selfish behavior in network pricing mechanisms, our research considers the possibility that users may anticipate the effect of their actions on the link price. While the efficiency loss is generally arbitrarily high, we establish bounds for several special cases. In particular, if each link has an affine marginal cost function, we establish that the aggregate surplus is at least $2/3$ of the optimal aggregate surplus, even for general networks; thus, the efficiency loss when users are selfish is no more than approximately 33%.**

I. Introduction

In this paper we consider a *congestion pricing* approach to resource allocation in communication networks; see, e.g., [1] for an overview. Congestion pricing views the network as a collection of scarce resources, and allocates these resources through a market mechanism. We apply this framework to a problem where many users require service from links in a network. We will consider a simple market mechanism for link data rate allocation, where users choose the rates they desire from the network. Such mechanisms, where market participants choose their desired quantities, are known as *Cournot* models [2], [3].

Our basic model consists of a collection of links as well as a collection of users. Each link of the network has a cost function describing the cost of allocating resources, as a function of the total rate allocated at the link; we assume this cost function is convex, with convex marginal cost. This cost may be related to the delay or loss incurred by traffic at the link, though we do not specify any fixed interpretation of the cost. Each user has a set of paths available through the network; each path uses a subset of the links. The users choose the rate they wish to send along each available path; it is this feature which makes this a Cournot model. Each link then sets a price equal to the *marginal cost* of the total rate passing through it.

We assume that each user is endowed with a concave *utility function*, which describes the value to the user of a given rate allocation. We start by assuming that the users do not anticipate the effects of their actions on link prices; that is, the users act as *price takers*. In this case it is straightforward to show, using the *first fundamental theorem of welfare economics* [2], that at the resulting competitive equilibrium the users achieve an efficient allocation, i.e., an allocation that maximizes the sum of their utilities minus the sum of the links' costs (known as the *aggregate surplus*).

While the price taking assumption is a reasonable starting point for analysis, in general users acting in their own self interest may actually be able to anticipate the dependence of path prices on their choice of rates. In this case, the model becomes a game, and it is not necessarily true aggregate surplus is maximized at a Nash equilibrium of this game.

The main contribution of this paper is an analysis of the loss of efficiency when users are price anticipating. Specifically, we evaluate the worst case ratio of aggregate surplus achieved at the Nash equilibrium to the maximum possible aggregate surplus. In Section II, we show by example that in general, the loss of efficiency may be arbitrarily high. Nevertheless, we establish in several special cases that the efficiency loss can be bounded. In particular, we show for a single link that if all users share the same utility function and the Nash equilibrium is unique, then the loss of efficiency is no more than $1/(2N+1)$, where N is the number of users. This result is then applied to two cases: first, a model where only a single user competes for a single link; and second, a model where several users with identical utility functions compete for a link with differentiable marginal cost. Using a related proof technique, we also show the efficiency loss is no worse than 33% in the special case where the marginal cost of the link is affine (although users are allowed to have general concave utility functions).

We then consider general networks in Section III. It is again the case that users maximize aggregate surplus if they act as price takers. We then establish that as long as all links have affine marginal cost functions, the worst case aggregate surplus at a Nash equilibrium is no worse than 33% of the maximal aggregate surplus; this is an extension of the corresponding result for a single link. We consider this to be the most applicable result of the paper, because it does not make any assumptions about users' preferences, and also continues to hold in a network setting.

Our investigation forms part of a broader body of work on quantifying efficiency loss in environments where participants may be selfish. Results have been obtained for routing, [4], traffic networks [5], [6], and network design problems [7], [8]. These results all compare the value of a global performance metric at the Nash equilibrium to the optimal value of that metric.

Our work is most closely related to the network pricing model proposed by Kelly et al. [9], [10]. In that model, users choose the total amount they are willing to pay, rather than their desired rate. Kelly established that this model maximizes aggregate surplus if link capacities are inelastic [9], and Kelly et al. subsequently extended this result to links characterized by cost functions [10]. Subsequent papers have analyzed the efficiency loss of these models [11], [12], [13].

We briefly compare the results of this paper with these previous works. Johari et al. established that for general networks where links have convex marginal cost functions, the efficiency loss of the mechanism of [10] is no more than approximately 34% when users are price anticipating, as long as their strategies are individual payments *to each link* [12]. However, Hajek and Yang have shown that in networks where links have inelastic capacity, and users submit only a single bid along *each path* available through the network, the efficiency loss may be arbitrarily high [13]. Ideally, we would like a network mechanism that can guarantee low efficiency loss, but where the strategy space of the users is relatively simple—that is, rather than submitting individual choices to each link, a more scalable approach would require the users to make only a single choice per path. In this regard, we have considered a simple Cournot mechanism, in which users choose rates across paths; this scales better than mechanisms which associate decision variables with each user-link pair. Our contribution is that the efficiency loss of this mechanism is no more than 33%, as long as links have affine marginal cost.

II. A Single Link

In this section, we will consider a game where multiple users compete for a single link, and where the strategies of the users are their desired rates. We will find that in general such games can yield arbitrarily high efficiency loss, though we will also establish bounds on efficiency loss for several special cases of interest.

Formally, we consider the following model. We assume that N users compete for a single link. We assume that each user n has a utility function U_n, and that allocation of data rate through the link incurs a cost characterized by a cost function C. We make the the following assumptions.

Assumption 1 *For each n, over the domain $x_n \geq 0$ the utility function $U_n(x_n)$ is concave, nondecreasing, and continuously differentiable (where we interpret $U_n'(0)$ as the right directional derivative of U_n at 0).*

Assumption 2 *There exists a continuous, convex, nondecreasing function $p(q)$ over $q \geq 0$, with $p(0) \geq 0$ and $p(q) \to \infty$ as $q \to \infty$, such that for $q \geq 0$:*

$$C(q) = \int_0^q p(z)dz.$$

In particular, $C(q)$ is convex and nondecreasing.

We assume that both utility and cost are measured in monetary units, so that an efficient allocation is characterized as an optimal solution of the following optimization problem:

$$\text{maximize} \quad \sum_n U_n(x_n) - C\left(\sum_n x_n\right) \qquad (1)$$

$$\text{subject to} \quad x_n \geq 0, \quad n = 1, \ldots, N. \qquad (2)$$

We refer to the objective function (1) as the *aggregate surplus* [2]. Since $p(q) \to \infty$ as $q \to \infty$, while U_n only grows at most linearly, it follows that an optimal solution exists. We now consider the following pricing scheme for resource allocation. Each user n chooses a desired rate x_n. Given the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, the link sets a single price $\mu(\boldsymbol{x}) = p(\sum_n x_n)$. User n then pays $x_n \mu(\boldsymbol{x})$. We first consider the case where, given a price $\mu > 0$, user n chooses x_n to maximize:

$$P_n(x_n; \mu) = U_n(x_n) - \mu x_n. \qquad (3)$$

Notice that in the previous expression, each user is acting as a *price taker*; that is, he does not anticipate the effect of a change in his strategy on the market-clearing price. Since we are using *marginal cost pricing*, i.e., since $\mu(\boldsymbol{x}) = p(\sum_n x_n)$, we expect that price taking users will maximize aggregate surplus at a competitive equilibrium. This is formalized in the following proposition, a special case of the first fundamental theorem of welfare economics [2].

Proposition 1 *Suppose Assumptions 1 and 2 hold. There exists a* competitive equilibrium*, that is, a vector* $\boldsymbol{x}$ *and a scalar* μ *such that* $\mu = p(\sum_n x_n)$*, and:*

$$P_n(x_n; \mu) = \max_{\overline{x}_n \geq 0} P_n(\overline{x}_n; \mu), \quad n = 1, \ldots, N. \tag{4}$$

Any such vector $\boldsymbol{x}$ *solves* (1)-(2). *If the functions* U_n *are strictly concave, such a vector* $\boldsymbol{x}$ *is unique as well.*

Proposition 1 shows that with marginal cost pricing, and if the users of the link behave as price takers, there exists a vector of rates $\boldsymbol{x}$ where all users have optimally chosen their x_n, with respect to the given price $\mu = p(\sum_n x_n)$; and at this "equilibrium," the aggregate surplus is maximized. However, when the price taking assumption is violated, the model changes into a game and the guarantee of Proposition 1 is no longer valid.

Consider, then, an alternative model where the users of a single link are price anticipating, rather than price taking, and play a Cournot game to acquire a share of the link. We use the notation $\boldsymbol{x}_{-n}$ to denote the vector of all rates chosen by users other than n; i.e., $\boldsymbol{x}_{-n} = (x_1, x_2, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N)$. Then given $\boldsymbol{x}_{-n}$, each user n chooses $x_n \geq 0$ to maximize:

$$Q_n(x_n; \boldsymbol{x}_{-n}) = U_n(x_n) - x_n p\left(\sum_m x_m\right). \tag{5}$$

The payoff function Q_n is similar to the payoff function P_n, except that the user now *anticipates* that the price will be set according to $p(\sum_m x_m)$. A *Nash equilibrium* of the game defined by $(Q_1, \ldots, Q_N)$ is a vector $\boldsymbol{x} \geq 0$ such that for all n:

$$Q_n(x_n; \boldsymbol{x}_{-n}) \geq Q_n(\overline{x}_n; \boldsymbol{x}_{-n}), \quad \text{for all } \overline{x}_n \geq 0. \tag{6}$$

We first show that a Nash equilibrium exists for this game. The proof is a standard application of Rosen's existence theorem [14], and is omitted; details may be found in [15].

Proposition 2 *Suppose that Assumptions 1 and 2 hold. Then there exists a Nash equilibrium* $\boldsymbol{x}$ *for the game defined by* $(Q_1, \ldots, Q_N)$*.*

Because the payoff Q_n is concave in x_n for fixed $\boldsymbol{x}_{-n}$, a vector $\boldsymbol{x}$ is a Nash equilibrium if and only if the following first order conditions are satisfied for each n, where $q = \sum_m x_m$:

$$U_n'(x_n) \leq p(q) + x_n \frac{\partial^+ p(q)}{\partial q}; \tag{7}$$

$$U_n'(x_n) \geq p(q) + x_n \frac{\partial^- p(q)}{\partial q}, \quad \text{if } x_n > 0, \tag{8}$$

where $\partial^+ p(q)/\partial q$ and $\partial^- p(q)/\partial q$ denote the right and left directional derivatives of p, respectively. We will use these conditions to investigate the efficiency loss when users are price anticipating. We first show in the following example that, in general, the efficiency loss may be arbitrarily high.

Example 1 Consider a price function p defined as follows:

$$p(q) = \begin{cases} a, & 0 \leq q \leq 1; \\ a + b(q-1), & q \geq 1. \end{cases}$$

Note that this yields:

$$C(q) = \begin{cases} aq, & 0 \leq q \leq 1; \\ aq + \frac{1}{2}b(q-1)^2, & q \geq 1. \end{cases}$$

We assume that $0 < a < 1$, and $b > 1$. We consider a game with $N = 2$ users where $U_1(x_1) = x_1$, and:

$$U_2(x_2) = ax_2.$$

In this case, note that aggregate surplus is maximized when $p(q) = 1$, i.e., when $q = 1 + (1-a)/b$; and furthermore, the rate q should be allocated entirely to user 1, since $a < 1$. Thus the maximal aggregate surplus is $U_1(q) - C(q)$, or:

$$1 + \frac{1-a}{b} - a - \frac{a(1-a)}{b} - \frac{(1-a)^2}{2b} = 1 - a + \frac{(1-a)^2}{2b}. \tag{9}$$

On the other hand, we claim that the vector $\boldsymbol{x}$ defined by:

$$x_1 = \frac{1-a}{b};$$

$$x_2 = 1 - \frac{1-a}{b},$$

is a Nash equilibrium. Observe that $q = x_1 + x_2 = 1$, so $p(q) = a$. Furthermore, $\partial^+ p(q)/\partial q = b$, $\partial^- p(q)/\partial q = 0$. It then follows that (7)-(8) hold for both users 1 and 2. Since $x_1, x_2 > 0$, these conditions are sufficient to ensure that $\boldsymbol{x}$ is a Nash equilibrium. Note that the aggregate surplus at this Nash equilibrium is $U_1(x_1) + U_2(x_2) - C(q) = (1-a)/b + a(1-(1-a)/b) - a = (1-a)^2/b$. Comparing this expression with (9), it is clear that in the limit where $b \to \infty$, the Nash equilibrium aggregate surplus approaches zero, and the maximal aggregate surplus approaches $1-a$; thus the ratio of Nash equilibrium aggregate surplus to the maximal aggregate surplus approaches zero. $\square$

Despite this negative result, we now prove a sequence of results characterizing efficiency loss in more limited environments. We start with the following theorem, which shows that as long as the Nash equilibrium is unique and all users share the same utility function, the efficiency loss is no more than $1/(2N+1)$. We will then use this result to establish bounds on efficiency loss in two special cases.

Theorem 3 *Suppose that $N \geq 1$ users share the same utility function $U_n = U$, such that Assumption 1 holds and $U(0) \geq 0$. In addition, suppose that Assumption 2 holds. Suppose also that the game defined by $(Q_1, \dots, Q_N)$ possesses a unique Nash equilibrium $\boldsymbol{x}$. If $\boldsymbol{x}^S$ is any optimal solution to* (1)-(2)*, then:*

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \geq \left(\frac{2N}{2N+1}\right)\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \quad (10)$$

Proof. We start with a sequence of three lemmas, which will also be useful in the subsequent development. The following lemma lets us assume without loss of generality that $\sum_n U_n(x_n^S) - C(\sum_n x_n^S) > 0$.

Lemma 4 *Suppose that Assumptions 1 and 2 hold. Suppose also that $U_n(0) \geq 0$ for all n. Fix any Nash equilibrium $\boldsymbol{x} = (x_1, \dots, x_N)$ of the game defined by $(Q_1, \dots, Q_N)$, and let $\boldsymbol{x}^S$ be any optimal solution to* (1)-(2)*. If $\sum_n U_n(x_n^S) - C(\sum_n x_n^S) = 0$, then $\sum_n U_n(x_n) - C(\sum_n x_n) = 0$, i.e., $\boldsymbol{x}$ is also an optimal solution to* (1)-(2).

Proof of Lemma. Let $q = \sum_n x_n$, and let $q^S = \sum_n x_n^S$. Note that since $\boldsymbol{x}$ is a Nash equilibrium, for each n we must have $U_n(x_n) - x_n p(q) \geq 0$, since otherwise choosing $x_n = 0$ is a profitable deviation for user n. By convexity of C, we have $\sum_n U_n(x_n) - C(q) \geq \sum_n U_n(x_n) - qp(q) \geq 0$. Thus if $\sum_n U_n(x_n^S) - C(q^S) = 0$, then we must have $\sum_n U_n(x_n) - C(q) = 0$ as well, since $\boldsymbol{x}^S$ is an optimal solution to (1)-(2). $\square$

Thus, if $\sum_n U_n(x_n^S) - C(\sum_n x_n^S) = 0$, the bound (10) trivially holds. We assume without loss of generality, therefore, that $\sum_n U_n(x_n^S) - C(\sum_n x_n^S) > 0$. Now note that we know $U_n'(x_n) \geq p(\sum_m x_m)$ for all n with $x_n > 0$, from (8). In the next lemma, we show that if $U_n'(x_n) = p(\sum_m x_m)$ for all n with $x_n > 0$, then (10) trivially holds.

Lemma 5 *Suppose that Assumptions 1 and 2 hold. Suppose also that $U_n(0) \geq 0$ for all n. Fix any Nash equilibrium $\boldsymbol{x} = (x_1, \dots, x_N)$ of the game defined by $(Q_1, \dots, Q_N)$, and let $\boldsymbol{x}^S$ be any optimal solution to* (1)-(2)*. If $U_n'(x_n) = p(\sum_m x_m)$ for all n with $x_n > 0$, then $\boldsymbol{x}$ is an optimal solution to* (1)-(2).

On the other hand, if there exists at least one n such that $U_n'(x_n) > p(\sum_m x_m)$, then $\sum_n U_n'(x_n)x_n - C(\sum_m x_m) > 0$.

Proof of Lemma. Let $q = \sum_n x_n$, and let $q^S = \sum_n x_n^S$. First suppose that $\boldsymbol{x} = 0$. In this case we have $U_n'(x_n) \leq p(q)$ for all n (from (7)); since $\boldsymbol{x} = 0$, this is a necessary and sufficient optimality condition for (1)-(2). On the other hand, if $x_n > 0$ for at least one n, and $U_n'(x_n) = p(q)$ for all n with $x_n > 0$, then $\boldsymbol{x}$ is again an optimal solution to (1)-(2) (since $U_n'(x_n) \leq p(q)$ for all n with $x_n = 0$, from (7)).

Now suppose that $U_n'(x_n) > p(q)$ for at least one user n. Then $x_n > 0$ (from (7)). For all other $m \neq n$ with $x_m > 0$, we know $U_m'(x_m) \geq p(q)$ (from (8)). Thus we have $\sum_n U_n'(x_n)x_n > qp(q) \geq C(q)$, where the last inequality follows by convexity; so we conclude $\sum_n U_n'(x_n)x_n - C(q) > 0$. $\square$

In the following lemma, we show that linear utility functions give the worst case efficiency loss.

Lemma 6 *Suppose that Assumptions 1 and 2 hold. Suppose also that $U_n(0) \geq 0$ for all n. Fix any rate vector $\boldsymbol{x} \geq 0$, and let $\boldsymbol{x}^S$ be any optimal solution to* (1)-(2)*. Define $\alpha_n = U_n'(x_n)$. If $\sum_n \alpha_n x_n - C\left(\sum_n x_n\right) > 0$ and $\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right) > 0$, then the following inequality holds:*

$$\frac{\sum_n U_n(x_n) - C(\sum_n x_n)}{\sum_n U_n(x_n^S) - C(\sum_n x_n^S)} \geq \frac{\sum_n \alpha_n x_n - C(\sum_n x_n)}{\max_{\overline{q} \geq 0}\left[(\max_n \alpha_n)\overline{q} - C(\overline{q})\right]}. \quad (11)$$

Proof of Lemma. Using concavity, we have $U_n(x_n^S) \leq U_n(x_n) + U_n'(x_n)(x_n^S - x_n)$. Concavity together with the fact that $U_n(0) \geq 0$ also implies $U_n(x_n) - U_n'(x_n)x_n \geq 0$. Furthermore, we have $\sum_n \alpha_n x_n^S - C\left(\sum_n x_n^S\right) \leq \max_{\overline{q} \geq 0}\left[(\max_n \alpha_n)\overline{q} - C(\overline{q})\right]$, as well as $0 < \sum_n \alpha_n x_n - C\left(\sum_n x_n\right) \leq$

$\max_{\overline{q}\geq 0}\left[(\max_n \alpha_n)\overline{q} - C(\overline{q})\right]$. Thus we have:

$$\frac{\sum_n U_n(x_n) - C(\sum_n x_n)}{\sum_n U_n(x_n^S) - C(\sum_n x_n^S)} \geq \frac{\sum_n \big(U_n(x_n) - \alpha_n x_n\big) + \sum_n \alpha_n x_n - C(\sum_n x_n)}{\sum_n \big(U_n(x_n) - \alpha_n x_n\big) + \sum_n \alpha_n x_n^S - C(\sum_n x_n^S)} \geq \frac{\sum_n \alpha_n x_n - C(\sum_n x_n)}{\max_{\overline{q}\geq 0}\left[(\max_n \alpha_n)\overline{q} - C(\overline{q})\right]}.$$

This establishes the claim of the lemma. □

We briefly summarize the assumptions and conclusions to this point. Let $q = \sum_n x_n$, and let $q^S = \sum_n x_n^S$. By symmetry, since $U_n = U$ for all n, the unique Nash equilibrium must be given by $x_n = q/N$ for all n. Also by symmetry, we can assume the optimal solution $\boldsymbol{x}^S$ to (1)-(2) is symmetric, since the objective function (1) is concave. We then have $x_n^S = q^S/N$ for all n.

Lemma 4 shows that we can assume without loss of generality that $\sum_n U(x_n^S) - C(q^S) > 0$; and Lemma 5 shows that we can assume without loss of generality that $U'(x_n) > p(q)$ for all n, and that this implies $\sum_n U'(x_n)x_n - C(q) > 0$. In addition, since $\sum_n U'(x_n)x_n - C(q) > 0$, we must have $q > 0$.

If we now apply Lemma 6 with $\alpha = U'(x_n) = U'(q/N)$, we have:

$$\frac{\sum_n U(x_n) - C(q)}{\sum_n U(x_n^S) - C(q^S)} \geq \frac{\alpha q - C(q)}{\max_{\overline{q}\geq 0}\left[\alpha\overline{q} - C(\overline{q})\right]}.$$

We will compute the worst case value of the right hand side over valid choices of C.

We now argue as follows. Define a new price function $\overline{p}(\overline{q})$ according to:

$$\overline{p}(\overline{q}) = \begin{cases} p(q), & \overline{q} \leq q; \\ p(q) + \dfrac{(\alpha - p(q))N}{q}(\overline{q} - q), & \overline{q} \geq q. \end{cases} \quad (12)$$

(See Figure 1 for an illustration.) Define $\overline{C}(q) = \int_0^q \overline{p}(z)\ dz$. Note that since $\alpha > p(q)$ and $q > 0$, $\overline{p}$ and $\overline{C}$ satisfy Assumption 2. It is also straightforward to check that the maximum $\max_{\overline{q}\geq 0}[\alpha\overline{q} - \overline{C}(\overline{q})]$ is achieved when $\overline{p}(\overline{q}) = \alpha$, i.e., when $\overline{q} = q + q/N$; and furthermore, it is straightforward to check that at this value of $\overline{q}$ we have $\alpha\overline{q} - \overline{C}(\overline{q}) = (\alpha - p(q))(q + q/(2N))$. On the other hand, $\alpha q - \overline{C}(q) = (\alpha - p(q))q$. Thus we have:

$$\frac{\alpha q - \overline{C}(q)}{\max_{\overline{q}\geq 0}\left[\alpha\overline{q} - \overline{C}(\overline{q})\right]} = \frac{2N}{2N+1}.$$

To complete the proof of the theorem, therefore, it suffices to show that:

$$\frac{\alpha q - C(q)}{\max_{\overline{q}\geq 0}\left[\alpha\overline{q} - C(\overline{q})\right]} \geq \frac{\alpha q - \overline{C}(q)}{\max_{\overline{q}\geq 0}\left[\alpha\overline{q} - \overline{C}(\overline{q})\right]}.$$

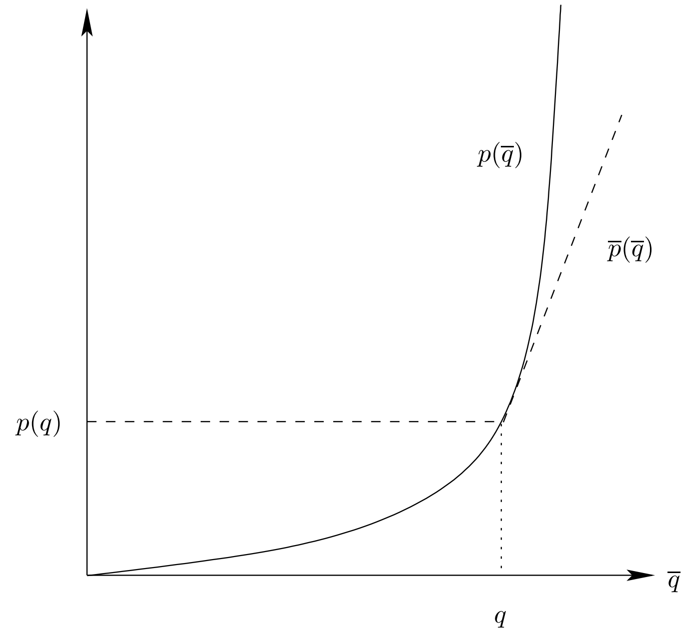

Fig. 1. Proof of Theorem 3: Given a price function p (solid line) and total Nash equilibrium rate q, a new price function $\overline{p}$ (dashed line) is defined according to (12).

Let q^* be an optimal solution to the maximization $\max_{\overline{q}\geq 0}[\alpha\overline{q} - C(\overline{q})]$. Note that we will have $\alpha \leq p(q^*)$. On the other hand, we have assumed that $\alpha = U'(x_n) > p(q)$. Thus we must have $p(q) < p(q^*)$, i.e., $q < q^*$.

To complete the proof, we define an intermediate price function $\hat{p}(\overline{q})$ as follows:

$$\hat{p}(\overline{q}) = \begin{cases} p(q), & \overline{q} \leq q; \\ p(\overline{q}), & \overline{q} \geq q. \end{cases}$$

Define $\hat{C}(q) = \int_0^q \hat{p}(z)\ dz$; then it is straightforward to check that $\hat{p}$ satisfies Assumption 2. Since p is nondecreasing (Assumption 2), we have $p(\overline{q}) \leq p(q)$ for $\overline{q} \leq q$; and thus if we define $\Delta = \hat{C}(q) - C(q)$, then $\Delta \geq 0$. Furthermore, since we have already shown that $q^* > q$, we have $\hat{C}(q^*) = C(q^*) + \Delta$. Because $\alpha > p(q)$, we have $0 < \alpha q - qp(q) = \alpha q - \hat{C}(q) \leq \alpha q^* - \hat{C}(q^*)$ (where the latter inequality follows since q^* is also seen to be an optimal solution to $\max_{\overline{q}\geq 0}[\alpha\overline{q} - \hat{C}(\overline{q})]$). Thus we have:

$$\frac{\alpha q - C(q)}{\alpha q^* - C(q^*)} \geq \frac{\alpha q - C(q) - \Delta}{\alpha q^* - C(q^*) - \Delta} = \frac{\alpha q - \hat{C}(q)}{\alpha q^* - \hat{C}(q^*)}. \quad (13)$$

We now observe that $\hat{C}(q) = qp(q) = \overline{C}(q)$, so the numerator in the last expression is $\alpha q - \hat{C}(q) = \alpha q - \overline{C}(q)$. On the other hand, from (8) it follows that:

$$\alpha = U'\left(\frac{q}{N}\right) \leq p(q) + \frac{q}{N}\cdot\frac{\partial^+ p(q)}{\partial q}. \quad (14)$$

Rearranging, we conclude that:

$$\frac{\partial^+ \overline{p}(q)}{\partial q} = \frac{(\alpha - p(q))N}{q} \leq \frac{\partial^+ p(q)}{\partial q} = \frac{\partial^+ \hat{p}(q)}{\partial q}. \quad (15)$$

Thus since $\hat{p}$ is convex, we have $\hat{p}(\overline{q}) \geq \overline{p}(\overline{q})$ for $\overline{q} \geq q$; on the other hand, we have $\hat{p}(\overline{q}) = \overline{p}(\overline{q})$ for $\overline{q} \leq q$. Since $q^* > q$, we have $\hat{C}(q^*) \geq \overline{C}(q^*)$, so that $\alpha q^* - \hat{C}(q^*) \leq \alpha q^* - \overline{C}(q^*) \leq \max_{\overline{q} \geq 0}(\alpha \overline{q} - \overline{C}(\overline{q}))$. Combining this inequality with (13) yields:

$$\frac{\alpha q - C(q)}{\alpha q^* - C(q^*)} \geq \frac{\alpha q - \overline{C}(q)}{\max_{\overline{q} \geq 0}\left(\alpha \overline{q} - \overline{C}(\overline{q})\right)} = \frac{2N}{2N+1},$$

as required. □

Theorem 3 is useful in settings where uniqueness of the Nash equilibrium can be guaranteed. We now apply Theorem 3 in two special cases: first, a model where only one user demands access to a link; and second, a model where all users share the same utility function, and the price function is differentiable.

Corollary 7 *Suppose that there is a single user (i.e., $N = 1$), with a utility function U such that Assumption 1 holds; in addition, suppose that Assumption 2 holds. Suppose also that $U(0) \geq 0$. If x^S solves* (1)-(2)*, and x maximizes $U(\overline{x}) - \overline{x}p(\overline{x})$ over $\overline{x} \geq 0$, then:*

$$U(x) - C(x) \geq \frac{2}{3}\left(U(x^S) - C(x^S)\right). \quad (16)$$

This bound is tight, i.e., there exists a choice of U and C such that (16) *holds with equality.*

Proof. The proof relies on the following lemma.

Lemma 8 *Suppose that there is a single user (i.e., $N = 1$), with a utility function U such that Assumption 1 holds; in addition, suppose that Assumption 2 holds. Then* at least *one of the following holds: either (a) all optimal solutions to $\max_{\overline{x} \geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$ are also optimal solutions to* (1)-(2)*; or (b) there exists a unique optimal solution to $\max_{\overline{x} \geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$.*

Proof of Lemma. Suppose there exist $x, \hat{x} \in \arg\max_{\overline{x} \geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$ such that $x \neq \hat{x}$. Assume without loss of generality that $x < \hat{x}$; note that this implies $\hat{x} > 0$. By concavity, we have $U'(\hat{x}) \leq U(x)$. By (7), we have $U'(x) \leq p(x) + x\partial^+ p(x)/\partial q$. Since p is nondecreasing and convex, and $x < \hat{x}$, we have $p(x) + x\partial^+ p(x)/\partial q \leq p(\hat{x}) + \hat{x}\partial^- p(\hat{x})/\partial q$. Finally, from (8), we have $p(\hat{x}) + \hat{x}\partial^- p(\hat{x})/\partial q \leq U'(\hat{x})$. Combining these inequalities, we have $U'(\hat{x}) \leq U(x) \leq p(x) + x\partial^+ p(x)/\partial q \leq p(\hat{x}) + \hat{x}\partial^- p(\hat{x})\partial q \leq U'(\hat{x})$. Thus equality must hold throughout; since $x < \hat{x}$, this is only possible if $p(x) = p(\hat{x})$ and $\partial^+ p(x)/\partial x = \partial^- p(\hat{x})/\partial x = 0$. Thus $U'(x) = p(x)$ and $U'(\hat{x}) = p(\hat{x})$, so that both x and $\hat{x}$ are optimal solutions to (1)-(2), as required. □

If all optimal solutions to $\max_{\overline{x} \geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$ are also optimal solutions to (1)-(2), then the bound (16) trivially holds. On the other hand, if there exists a unique optimal solution to $\max_{\overline{x} \geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$, then we can apply Theorem 3 to conclude that (16) holds.

Finally, to see that the bound is tight, let $U(x) = x$, and let $p(q) = (q-1)^+$ (i.e., $p(q) = 0$ for $0 \leq q \leq 1$, and $p(q) = q - 1$ for $q \geq 1$); thus $C(q) = 0$ if $0 \leq q \leq 1$, and $C(q) = (q-1)^2/2$ if $q \geq 1$. Then it is straightforward to verify that $x = 1$ is the unique optimal solution to $\max_{\overline{x} \geq 0}[U(\overline{x}) - \overline{x}p(\overline{x})]$, while $x^S = 2$ is an optimal solution to (1)-(2). Furthermore, we have $U(x) - C(x) = 1$, while $U(x^S) - C(x^S) = 3/2$, matching the bound (16). □

The preceding corollary considered a single user. We now consider a model consisting of multiple users who share the same utility function.

Corollary 9 *Suppose that $N \geq 1$ users share the same utility function $U_n = U$, such that Assumption 1 holds; in addition, suppose that Assumption 2 holds, and that p is differentiable. Suppose also that $U(0) \geq 0$. If $\boldsymbol{x}^S$ is an optimal solution to* (1)-(2)*, and $\boldsymbol{x}$ is a Nash equilibrium of the game defined by $(Q_1, \ldots, Q_N)$, then:*

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \geq \left(\frac{2N}{2N+1}\right)\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \quad (17)$$

Proof. The proof relies on the following lemma.

Lemma 10 *Suppose that Assumption 1 and Assumption 2 hold, and that p is differentiable. Then* at least *one of the following holds: either (a) all Nash equilibria of the game defined by $(Q_1, \ldots, Q_N)$ are also optimal solutions to* (1)-(2)*; or (b) there exists a unique Nash equilibrium of the game defined by $(Q_1, \ldots, Q_N)$.*

Proof of Lemma. The proof is similar to the proof of Lemma 8. Let $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ be two Nash equilibria such that $\boldsymbol{x} \neq \hat{\boldsymbol{x}}$, and let $q = \sum_n x_n$, and $\hat{q} = \sum_n \hat{x}_n$. Assume without loss of generality that $q \leq \hat{q}$. Since $\boldsymbol{x} \neq \hat{\boldsymbol{x}}$, then there must exist a user n such that $x_n < \hat{x}_n$; in particular, $\hat{x}_n > 0$. In this case, we have $U_n'(\hat{x}_n) \leq U_n'(x_n)$ by concavity. By (7) we have $U_n(x_n) \leq p(q) + x_n p'(q)$.

Since p is nondecreasing and convex, $q \le \hat{q}$, and $x_n < \hat{x}_n$, we have $p(q) + x_n p'(q) \le p(\hat{q}) + \hat{x}_n p'(\hat{q})$. Since $\hat{x}_n > 0$, by (7)-(8) we have $p(\hat{q}) + \hat{x}_n p'(\hat{q}) = U_n'(\hat{x}_n)$. Combining these relations yields:

$$U_n'(\hat{x}_n) \le U_n'(x_n) \le p(q) + x_n p'(q)$$
$$p(\hat{q}) + \hat{x}_n p'(\hat{q}) = U_n'(\hat{x}_n). \quad (18)$$

Thus equality must hold throughout; since $x_n < \hat{x}_n$, this is only possible if $p(q) = p(\hat{q})$, and $p'(q) = p'(\hat{q}) = 0$. In this case (7)-(8) imply that for all m, we have $U_m'(x_m) = p(q)$ if $x_m > 0$, and $U_m'(x_m) \le p(q)$ if $x_m = 0$; similarly, $U_m'(\hat{x}_m) = p(\hat{q})$ if $\hat{x}_m > 0$, and $U_m'(\hat{x}_m) \le p(\hat{q})$ if $\hat{x}_m = 0$. These are precisely the optimality conditions for (1)-(2), so we conclude that both $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ are optimal solutions to (1)-(2), as required. □

If all Nash equilibria are also optimal solutions to (1)-(2), then the bound (17) trivially holds. On the other hand, if there exists a unique Nash equilibrium, then we can apply Theorem 3 to conclude that (17) holds. □

We note that Lemma 10 did not require all users to have the same utility function, and thus holds for any game where the price function p is differentiable. We also note that although a tightness result is not claimed in the preceding corollary, such a result may be established by considering a limit of differentiable price functions which approach the worst case price function $\overline{p}$ defined in the proof of Theorem 3. However, defining such price functions requires additional technical complexity, and does not yield additional insight; thus the argument is omitted.

Note that Corollary 9 also yields a competitive limit theorem [2], since as $N \to \infty$ the efficiency loss approaches zero. Indeed, this result is to be expected, since the users are assumed to be identical; thus in the limit of many users no single user should have a significant impact on the market-clearing price.

Corollaries 7 and 9 present bounds on efficiency loss under various restrictions on utility functions and the price function p. We note that these results would continue to hold even if the utility functions were not necessarily differentiable (as we required in Assumption 1). Differentiability of the utility function only eases the presentation of the technical arguments, but is not essential to the results.

By contrast, differentiability of the price function p is essential to the proof of Corollary 9. In particular, in considering the statements of Corollaries 7 and 9, one might expect a more general result to hold: if N users share the same utility function U and Assumption 1 is satisfied, and the price function p satisfies Assumption 2 (but is not necessarily differentiable), then the efficiency loss is no more than $1/(2N+1)$ when users are price anticipating. Such a result would be a generalization of Corollaries 7 and 9.

However, the efficiency loss can be arbitrarily high if the price function is not differentiable, even if all users share the same utility function; we omit the details of such an example, which can be found in Example 3.5 of [15]. The main reason for this negative result is that when the price function is not differentiable, there may exist highly inefficient Nash equilibria which are not symmetric among the players.

To avoid such singular effects, we now search instead for a result that holds regardless of the utility functions of the users. Of course, such a result cannot hold for all price functions. In particular, we prove in the following theorem that if the price function is affine, the resulting efficiency loss is no more than $1/3$ of the maximal aggregate surplus, regardless of the utility functions of the users.

Theorem 11 *Suppose that Assumption 1 holds, and that $p(q) = aq + b$ for some $a > 0, b \ge 0$. Suppose also that $U_n(0) \ge 0$ for all n. If $\boldsymbol{x}^S$ is any solution to (1)-(2), and $\boldsymbol{x}$ is any Nash equilibrium of the game defined by $(Q_1, \dots, Q_n)$, then:*

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \ge \frac{2}{3}\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \quad (19)$$

Furthermore, this bound is tight: for every $a > 0$, $b \ge 0$, and $\delta > 0$, there exists a choice of N and a choice of (linear) utility functions U_n, $n = 1, \dots, N$ such that a Nash equilibrium $\boldsymbol{x}$ exists with:

$$\sum_n U_n(x_n) - C\left(\sum_n x_n\right) \le \left(\frac{2}{3} + \delta\right)\left(\sum_n U_n(x_n^S) - C\left(\sum_n x_n^S\right)\right). \quad (20)$$

Proof. The proof follows in two steps. Using Lemma 6, we first show that the worst case occurs when the utility functions of the users are linear. We then optimize over all games with linear utility functions to determine the worst case efficiency loss.

As in the proof of Theorem 3, using Lemmas 4 and 5 we can assume without loss of generality that

$\sum_n U_n(x_n^S) - C(\sum_n x_n^S) > 0$ and $\sum_n U_n'(x_n)x_n - C(\sum_n x_n) > 0$. If we replace the utility function U_n by a new utility function $\hat{U}_n$ for each n, where $\hat{U}_n(\hat{x}_n) = (U_n'(x_n))\hat{x}_n$, then $\boldsymbol{x}$ continues to be a Nash equilibrium, since the optimality conditions (7)-(8) still hold. Applying Lemma 6, therefore, we see that the ratio of Nash equilibrium aggregate surplus to the maximal aggregate surplus can only be reduced if we replace U_n by $\hat{U}_n$ for all n.

Thus we assume without loss of generality that the utility functions of all users are linear, i.e., $U_n(x_n) = \alpha_n x_n$. Since we have assumed $\sum_n \alpha_n x_n - C(\sum_n x_n) > 0$, we know that $\alpha_n > 0$ for at least one n. Thus, by replacing α_n by $\alpha_n/(\max_m \alpha_m)$, and $C(\cdot)$ by $C(\cdot)/(\max_n \alpha_n)$, we can also assume without loss of generality that $\max_n \alpha_n = 1$. Furthermore, by relabeling if necessary, we can assume that $\alpha_1 = 1$. Note that after rescaling, the new price function p is still affine but may have a different slope.

Since we have restricted attention to settings where $\sum_n \alpha_n x_n - C(\sum_n x_n) > 0$, we must also have $\sum_n x_n > 0$. Thus, from (8) and the fact that $\max_n \alpha_n = 1$ we must have $1 > p(q) = aq + b$; in particular, this implies that $b < 1$.

We start by computing the maximal aggregate surplus under these assumptions. Since the price function is $p(q) = aq + b$, the maximal aggregate surplus is achieved when $p(q^S) = 1$, i.e., when $q^S = (1-b)/a$; this rate is entirely allocated to user 1. The maximal aggregate surplus is thus:

$$\frac{1-b}{a} - \frac{(1-b)^2}{2a} - \frac{b(1-b)}{a} = \frac{(1-b)^2}{2a}.$$

Since the maximal aggregate surplus is fixed as $(1-b)^2/(2a)$, by (7)-(8) the worst case game is identified by solving the following optimization problem (with unknowns $x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n, q$):

$$\begin{aligned}
\text{minimize} \quad & \sum_{n=1}^{N} \alpha_n x_n - C(q) && (21)\\
\text{subject to} \quad & \alpha_n \le p(q) + x_n p'(q), \quad n = 1, \ldots, N; && (22)\\
& \alpha_n \ge p(q) + x_n p'(q), \text{ if } x_n > 0, \\
& \qquad n = 1, \ldots, N; && (23)\\
& \sum_{n=1}^{N} x_n = q; && (24)\\
& \alpha_1 = 1; && (25)\\
& 0 \le \alpha_n \le 1, \quad n = 2, \ldots, N; && (26)\\
& x_n \ge 0, \quad n = 1, \ldots, N. && (27)
\end{aligned}$$

The objective function is the aggregate surplus associated with a Nash equilibrium allocation $\boldsymbol{x}$. The conditions (22)-(23) are equivalent to the Nash equilibrium conditions established in (7)-(8). The constraint (24) ensures that the total allocation made at the Nash equilibrium is equal to q. The constraints (25)-(26) follow since we have restricted without loss of generality to games where $\alpha_1 = \max_n \alpha_n = 1$. The constraint (27) ensures the rate allocated to each user is nonnegative.

We start by assuming that $q > 0$ is fixed, and optimize only over $\boldsymbol{x}$ and $\boldsymbol{\alpha}$. In this case, we start by noting that we may assume without loss of generality that $\alpha_n = p(q) + x_n p'(q)$ for all users $n = 2, \ldots, N$. Indeed, if $(\boldsymbol{\alpha}, \boldsymbol{x})$ is a feasible solution and $x_n > 0$ for some $n = 2, \ldots, N$, then (22)-(23) imply that $\alpha_n = p(q) + x_n p'(q)$. On the other hand, if $x_n = 0$ for some $n = 2, \ldots, N$, we can set $\alpha_n = p(q) = aq + b$; this preserves feasibility, but does not impact the term $\alpha_n x_n$ in the objective function (21). We can therefore restrict attention to feasible solutions for which:

$$\alpha_n = p(q) + x_n p'(q) = aq + b + ax_n, \quad n = 2, \ldots, N. \tag{28}$$

Having done so, observe that the constraint (26), that $\alpha_n \le 1$, may be written as:

$$x_n \le \frac{1 - aq - b}{a}, \quad n = 2, \ldots, N.$$

Finally, the constraint (26) that $\alpha_n \ge 0$ becomes redundant, as it is guaranteed by the fact that $a > 0$, $b \ge 0$, and $q > 0$.

It follows from (26) together with (21)-(22) that a candidate solution satisfying (24) can only exist if $x_1 > 0$, in which case we have $1 = p(q) + x_1 p'(q) = aq + b + ax_1$, so that $x_1 = (1 - aq - b)/a$. In particular, we conclude immediately that for a feasible solution to exist, we must have $0 < (1 - aq - b)/a \le q$. This yields the following reduced optimization problem:

$$\text{minimize} \quad \frac{1 - aq - b}{a} + \sum_{n=2}^{N} (aq + b + ax_n)x_n - C(q) \tag{29}$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{n=2}^{N} x_n = q - \frac{1 - aq - b}{a}; && (30)\\
& x_n \le \frac{1 - aq - b}{a}, \quad n = 2, \ldots, N; && (31)\\
& x_n \ge 0, \quad n = 2, \ldots, N. && (32)
\end{aligned}$$

The objective function (29) is equivalent to (21) upon substitution for α_n (assuming equality in (23)) and x_1 (also by requiring equality in (23)). The constraint (30) is equivalent to the allocation constraint (24); and the constraint (31) ensures $\alpha_n \le 1$, as required in (26).

For fixed $q > 0$, the resulting problem is symmetric in the rates x_n for $n = 2, \ldots, N$. It is clear that a feasible solution exists if and only if:

$$\frac{q}{N} \leq \frac{1 - aq - b}{a} \leq q. \tag{33}$$

In this case the following symmetric solution is feasible:

$$x_n = \frac{q - (1 - aq - b)/a}{N - 1}.$$

Furthermore, since the objective function (29) is strictly convex, this symmetric solution must in fact be optimal. If we substitute in the objective function (29), the resulting optimal value is strictly decreasing as N increases; the worst case occurs as $N \to \infty$, and the optimal objective value (29) becomes:

$$\frac{1 - aq - b}{a} + (aq + b)\left(q - \frac{1 - aq - b}{a}\right) - C(q) =$$
$$\frac{1 - b}{a} - q + (aq + b)\left(2q - \frac{1 - b}{a}\right) - \frac{aq^2}{2} - bq. \tag{34}$$

Furthermore, the feasibility requirements (33) on a, b, and q become $0 < (1 - aq - b)/a \leq q$, or upon rearranging, $(1 - b)/2 \leq aq < 1 - b$.

Until now we have kept the price function and the total rate q fixed, and found the worst case game. We now optimize over all possible choices of price function p (i.e., over $a > 0$ and $b \geq 0$), as well as over possible Nash equilibrium rates (i.e., over $q > 0$). Recall that the maximal aggregate surplus is $(1 - b)^2/(2a)$. Thus, the worst case ratio is identified by the following optimization problem over q, a, and b:

minimize
$$\frac{(1-b)/a - q + (aq + b)\left(2q - (1-b)/a\right) - aq^2/2 - bq}{(1-b)^2/(2a)}$$
subject to
$$(1 - b)/2 \leq aq \leq 1 - b, \quad a > 0, \ b \geq 0, \ q > 0.$$

If we divide numerator and denominator of the objective function by q, and let $\overline{a} = aq$, then this problem becomes equivalent to the following problem:

minimize
$$\frac{(1-b)/\overline{a} - 1 + (\overline{a} + b)\left(2 - (1-b)/\overline{a}\right) - \overline{a}/2 - b}{(1-b)^2/(2\overline{a})}$$
subject to
$$(1 - b)/2 \leq \overline{a} \leq 1 - b, \quad \overline{a} > 0, \ b \geq 0.$$

By substituting $x = \overline{a}/(1 - b)$ and differentiating, it is straightforward to establish that the minimum value of this optimization problem occurs at any pair $\overline{a}$ and b satisfying the constraints, such that $\overline{a}/(1 - b) = 2/3$. One such pair is given by $\overline{a} = 1/3$, and $b = 0$. At any such solution, the minimum objective value is equal to $2/3$. This establishes (19).

We now show (20), for a fixed price function $p(q) = aq + b$ with $a > 0$ and $b \geq 0$. To see this, choose the utility functions so that:

$$U_1(x_1) = \left(\frac{3a}{2} + b\right) x_1;$$
$$U_n(x_n) = \left(a + b + \frac{a}{2(N-1)}\right) x_n, \quad n = 2, \ldots, N.$$

Let $\overline{x} = 1/(2(N-1))$. It is then straightforward to check that for sufficiently large N, if $x_1 = 1/2$ and $x_n = \overline{x}$ for $n = 2, \ldots, N$, the allocation $\boldsymbol{x}$ is a Nash equilibrium. Furthermore, the maximum aggregate surplus is achieved by choosing q^S so that $3a/2 + b = p(q^S) = aq^S + b$, so $q^S = 3/2$, $x_1^S = q^S = 3/2$, and $x_n^S = 0$ for $n = 2, \ldots, N$. Thus the ratio of Nash equilibrium aggregate surplus to maximal aggregate surplus is:

$$\frac{(3a/2 + b)(1/2) + (a + b + a\overline{x})(1/2) - a/2 - b}{(3a/2 + b)(3/2) - (a/2)(3/2)^2 - b(3/2)} = \frac{3/2 + \overline{x}}{9/4}. \tag{35}$$

Now as $N \to \infty$, this ratio approaches $2/3$, as required. $\square$

Note that while the proof makes it appear as if the worst case occurs when the price function satisfies $a/(1 - b) = 2/3$, in fact by an appropriate choice of utility functions the worst case efficiency loss is always *exactly* $1/3$ for *any* affine price function.

We consider the preceding theorem to be the most applicable of the results of this section, since users will generally not have identical utility functions. Furthermore, as we will see in the following section, this result can be extended to a natural network model, where users choose rates on *paths* through the network (rather than at individual links).

III. General Networks

In this section we will consider an extension of the single link model to general networks. We consider a network consisting of J links, or *resources*, numbered $1, \ldots, J$. As before, a set of users numbered $1, \ldots, N$, shares this network of resources. We assume there exists a set of paths through the network, numbered $1, \ldots, P$. By an abuse of notation, we will use J, N, and P to also denote the *sets* of resources, users, and paths, respectively. Each path $q \in P$ uses a subset of the set of resources J; if resource j is used by path q, we will denote this by writing $j \in q$. Each user $n \in N$ has a collection of paths available through the network; if path q serves user n, we will denote this by writing $q \in n$. We will assume without loss of generality that paths are uniquely identified with users, so that for each path q there exists a unique user n such that $q \in n$. (There is no loss of generality because if two users share the same path, that is captured in our model by creating two paths which use exactly the same subset of resources.) For notational convenience, we note that the resources required by individual paths are captured by the *path-resource incidence matrix* $\boldsymbol{A}$, defined by:

$$A_{jq} = \begin{cases} 1, & \text{if } j \in q \\ 0, & \text{if } j \notin q. \end{cases}$$

Furthermore, we can capture the relationship between paths and users by the *path-user incidence matrix* $\boldsymbol{H}$, defined by:

$$H_{nq} = \begin{cases} 1, & \text{if } q \in n \\ 0, & \text{if } q \notin n. \end{cases}$$

Note that by our assumption on paths, for each path q we have $H_{nq} = 1$ for exactly one user n.

Let $y_q \geq 0$ denote the rate allocated to path q, and let $x_n = \sum_{q \in n} y_q \geq 0$ denote the rate allocated to user n; using the matrix $\boldsymbol{H}$, we may write the relation between $\boldsymbol{x} = (x_n, n \in N)$ and $\boldsymbol{y} = (y_q, q \in P)$ as $\boldsymbol{Hy} = \boldsymbol{x}$. Furthermore, if we let f_j denote the total rate on link j, we must have:

$$\sum_{q: j \in q} y_q = f_j, \quad j \in J.$$

Using the matrix $\boldsymbol{A}$, we may write this constraint as $\boldsymbol{Ay} = \boldsymbol{f}$.

We continue to assume that user n receives a utility $U_n(d_n)$ from an allocated rate d_n, and that each link j incurs a cost $C_j(f_j)$ when the total allocated rate at link j is f_j. We will assume that the utility functions satisfy Assumption 1, and each cost function C_j satisfies Assumption 2; we let p_j denote the marginal cost function associated with link j.

The natural generalization of the problem (1)-(2) to a network context is given by the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_n U_n(d_n) - \sum_j C_j(f_j) && (36) \\ \text{subject to} \quad & \boldsymbol{Ay} = \boldsymbol{f}; && (37) \\ & \boldsymbol{Hy} = \boldsymbol{d}; && (38) \\ & y_q \geq 0, \quad q \in P. && (39) \end{aligned}$$

We continue to refer to the objective function (36) as the *aggregate surplus*. Since the objective function is continuous and U_n grows at most linearly while C_j grows superlinearly, an optimal solution $\boldsymbol{y}$ exists. Since the feasible region is convex and the cost functions C_j are each strictly convex, the optimal vector $\boldsymbol{f} = \boldsymbol{Ay}$ is uniquely defined (though $\boldsymbol{y}$ need not be unique). In addition, if the functions U_n are strictly concave, then the optimal vector $\boldsymbol{d} = \boldsymbol{Hy}$ is uniquely defined as well. As in the previous development, we will use the optimal solution to (36)-(39) as a benchmark for the outcome of the network game.

We will consider the following network resource allocation mechanism, a natural generalization of the game considered for a single in the previous section. Each user n chooses a rate y_q for each path $q \in n$; thus the strategy of user n is now a vector $\boldsymbol{y}_n = (y_q, q \in n)$. The total rate demanded at link j is then $\sum_{q: j \in q} y_q$. We continue to assume that each link chooses a price equal to marginal cost, so that given the composite strategy vector $\boldsymbol{y} = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N)$, the price of link j is $\mu_j(\boldsymbol{y}) = p_j(\sum_{q: j \in q} y_q)$. Each user n then pays a total amount $y_q \sum_{j \in q} \mu_j(\boldsymbol{y})$ along each path $q \in n$; thus the total payment by user n is $\sum_{q \in n} y_q \sum_{j \in q} \mu_j(\boldsymbol{y})$.

As in the previous development, if we assume that each user behaves as a price taker, then given a vector of prices $\boldsymbol{\mu} = (\mu_j, j \in J)$, the payoff to user n is:

$$P_n(\boldsymbol{y}_n; \boldsymbol{\mu}) = U_n\left(\sum_{q \in n} y_q\right) - \sum_{q \in n} y_q \sum_{j \in q} \mu_j.$$

Since we are using marginal cost pricing, we again expect price taking users to maximize aggregate surplus at a competitive equilibrium; this is formalized in the following analogue of Proposition 1. The result is again a special case of the first fundamental theorem of welfare economics [2].

Proposition 12 *Suppose that Assumption 1 holds, and that for each price function p_j and cost function C_j Assumption 2 holds. There exists a* competitive equilibrium*, that is, a pair of vectors $\boldsymbol{y}$ and $\boldsymbol{\mu}$ such that $\mu_j = p_j(\sum_{q: j \in q} y_q)$, and:*

$$P_n(\boldsymbol{y}_n; \boldsymbol{\mu}) = \max_{\overline{\boldsymbol{y}}_n \geq 0} P_n(\overline{\boldsymbol{y}}_n; \boldsymbol{\mu}), \quad n = 1, \ldots, N. \quad (40)$$

Furthermore, any such vector $\boldsymbol{y}$ solves (36)-(39).

However, if each user is price anticipating, rather than price taking, the users may not maximize aggregate surplus. Consider, then, an alternative model where the users of a single link are price anticipating, and play a Cournot game to acquire a share of the links of the network. We use the notation $\boldsymbol{y}_{-n}$ to denote the vector of all rates chosen by users other than n; i.e., $\boldsymbol{y}_{-n} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{n-1}, \boldsymbol{y}_{n+1}, \ldots, \boldsymbol{y}_N)$. Then given $\boldsymbol{y}_{-n}$, each user n chooses $\boldsymbol{y}_n \geq 0$ to maximize:

$$Q_n(\boldsymbol{y}_n; \boldsymbol{y}_{-n}) = U_n\left(\sum_{q \in n} y_q\right) - \sum_{q \in n} y_q \sum_{j \in q} p_j\left(\sum_{\overline{q}: j \in \overline{q}} y_{\overline{q}}\right). \tag{41}$$

The payoff function Q_n is similar to the payoff function P_n, except that the user now *anticipates* that the price at link j will be set according to $p_j(\sum_{q:j\in q} y_q)$. A *Nash equilibrium* of the game defined by $(Q_1, \ldots, Q_N)$ is a composite vector $\boldsymbol{y} \geq 0$ such that for all n:

$$Q_n(\boldsymbol{y}_n; \boldsymbol{y}_{-n}) \geq Q_n(\overline{\boldsymbol{y}}_n; \boldsymbol{y}_{-n}), \quad \text{for all } \overline{\boldsymbol{y}}_n \geq 0.$$

As in Proposition 2, we first show that a Nash equilibrium exists for this game. The proof is again a standard application of Rosen's existence theorem [14], and is omitted.

Proposition 13 *Suppose that Assumption 1 holds, and that for each price function p_j and cost function C_j Assumption 2 holds. Then there exists a Nash equilibrium $\boldsymbol{y}$ for the game defined by $(Q_1, \ldots, Q_N)$.*

We would now like to investigate the efficiency loss by comparing the aggregate surplus at a Nash equilibrium to the optimal value of (36)-(39). Of course, since a single link is a special case of a general network, we know from Example 1 that the efficiency loss can be arbitrarily high at a Nash equilibrium. However, it is possible to establish a bound on efficiency loss in the special case where all link price functions are affine. Our basic approach is the same as that adopted in the network model of [11]: we reduce the game to individual games at each link, and then apply Theorem 11. Due to space constraints we only provide a proof outline here; details may be found in Section 3.5.2 of [15].

Theorem 14 *Suppose that Assumption 1 holds, and that for each $j \in J$, $p_j(q_j) = a_j q_j + b_j$ for some $a_j > 0, b_j \geq 0$. Suppose also that $U_n(0) \geq 0$ for all n. If $\boldsymbol{y}^S$ is any solution to (36)-(39), and $\boldsymbol{y}$ is any Nash equilibrium of the game defined by $(Q_1, \ldots, Q_n)$, then:*

$$\sum_n U_n\left(\sum_{q \in n} y_q\right) - \sum_j C_j\left(\sum_{q:j \in q} y_q\right) \geq \frac{2}{3}\left(\sum_n U_n\left(\sum_{q \in n} y_q^S\right) - \sum_j C_j\left(\sum_{q:j\in q} y_q^S\right)\right). \tag{42}$$

Proof Outline. The proof consists of two main steps. First, we show that the game defined by $(Q_1, \ldots, Q_N)$ is equivalent to another game where users choose the rate they expect at *each link*, rather than through each available path. This game can then be analyzed by methods similar to those used in [11] or [12].

First, we define a new game where each user n chooses a rate d_{jn} demanded *at each link*; the strategy of user n is thus $\boldsymbol{d}_n = (d_{jn}, j \in J)$. This strategy determines the rate allocation to user n at each link in the network; given this allocation, user n sends at the maximum rate possible using the paths $q \in n$ which are available. This maximum rate is the optimal value of the following *max-flow* optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{q \in n} y_q && (43)\\ \text{subject to} \quad & \sum_{q \in n: j \in q} y_q \leq d_{jn}, \quad j \in J; && (44)\\ & y_q \geq 0, \quad q \in n. && (45) \end{aligned}$$

We denote the optimal objective value of this optimization problem by $z_n(\boldsymbol{d}_n)$, where $\boldsymbol{d}_n = (d_{jn}, j \in J)$. Finally, the price at each link j is set to $p_j(\sum_n d_{jn})$, and the total payment made by user n is $\sum_j d_{jn} p_j(\sum_n d_{jn})$. Thus, given a composite strategy vector $\boldsymbol{d} = (\boldsymbol{d}_1, \ldots, \boldsymbol{d}_N)$, the payoff to user n is:

$$T_n(\boldsymbol{d}_n; \boldsymbol{d}_{-n}) = U_n(z_n(\boldsymbol{d}_n)) - \sum_j d_{jn} p_j\left(\sum_m d_{jm}\right).$$

The key property of the game defined by $(T_1, \ldots, T_N)$ is that any Nash equilibrium $\boldsymbol{y}$ of the game defined by $(Q_1, \ldots, Q_N)$ is related by a natural transformation to a Nash equilibrium $\boldsymbol{d}$ of the game defined by $(T_1, \ldots, T_N)$; and further, the aggregate surplus remains unchanged under this transformation. This is formalized in the following lemma; the proof is omitted, and may be found in Section 3.5.2 of [15].

Lemma 15 *Suppose that Assumption 1 holds, and that each price function p_j and cost function C_j satisfies Assumption 2. Let $\boldsymbol{y}$ be a Nash equilibrium of the game defined by $(Q_1, \ldots, Q_N)$, and define $d_{jn} = \sum_{q \in n: j \in q} y_q$. Then $\boldsymbol{d}$ is a Nash equilibrium of the game defined by $(T_1, \ldots, T_N)$. Furthermore, there holds:*

$$\sum_n U_n\left(\sum_{q \in n} y_q\right) - \sum_j C_j\left(\sum_{q:j\in q} y_q\right) = \sum_n U_n(z_n(\boldsymbol{d}_n)) - \sum_j C_j\left(\sum_n d_{jn}\right).$$

Given Lemma 15, it suffices to focus on the worst case efficiency loss of the game defined by $(T_1, \ldots, T_N)$. This game can be analyzed using techniques identical to those used in Theorem 7 of [11] and Theorem 14 of [12]. First, we linearize the "composite" utility function $U_n(z_n(\boldsymbol{d}_n))$. Formally, we replace $U_n(z_n(\boldsymbol{d}_n))$ with a linear function $\boldsymbol{\alpha}_n^\top \boldsymbol{d}_n$. The difficulty in this phase of the analysis is that the composite utility function $U_n(z_n(\cdot))$ may not be differentiable, because the max-flow function $z_n(\cdot)$ is not differentiable everywhere; as a result, convex analytic techniques are required.

Finally, we conclude the proof by observing that since the "composite" utility function $\boldsymbol{\alpha}_n^\top \boldsymbol{d}_n$ for user n is linear in the vector of rate allocations $\boldsymbol{d}_n$, the network structure is no longer relevant. In this case the game defined by $(T_1, \ldots, T_R)$ decouples into J Cournot games, one at each link j. We then apply Theorem 11 at each link to arrive at the bound in the theorem. □

The preceding theorem extends Theorem 11 to general networks, where users may have arbitrary utility functions (subject to Assumption 1) and arbitrary paths available through the network. We note here that in general, the result of Theorem 3 does not extend to networks. The reason for this is that even if all users share the same utility function, they may not have the same paths available through the network; and thus their "composite" utilities $U_n(z_n(\cdot))$ from the preceding proof may be not be identical.

IV. Conclusion

This paper has considered a simple model for network resource allocation: users choose the rate at which they want to send data, and links set prices according to the marginal cost of the total rate allocated. While such a scheme is efficient when all users are price taking, there is a loss of efficiency when users are able to anticipate the effects of their choices on the link prices. We established that in the worst case, this efficiency loss is arbitrarily high. However, it may be bounded by 33% in several special cases. When links' marginal costs are affine, this bound extends to general networks as well.

An open question remains concerning the dynamics of such a scheme. This paper has only considered a static equilibrium model of the resource allocation problem. However, in general the change in network state over time will affect the decisions of the market participants, and ensuring convergence to the Nash equilibrium remains a difficult problem.

Acknowledgments

This work was partially supported by the National Science Foundation under a Graduate Research Fellowship and grant ECS-0312921.

References

[1] M. Falkner, M. Devetsikiotis, and I. Lambadaris, "An overview of pricing concepts for broadband IP networks," *IEEE Communications Surveys*, vol. 3, no. 2, 2000.

[2] A. Mas-Colell, M. D. Whinston, and J. R. Green, *Microeconomic Theory*. Oxford, United Kingdom: Oxford University Press, 1995.

[3] C. Shapiro, "Theories of oligopoly behavior," in *Handbook of Industrial Organization*, R. Schmalensee and R. D. Willig, Eds. Amsterdam, The Netherlands: Elsevier Science, 1989, vol. 1, pp. 329–414.

[4] E. Koutsoupias and C. Papadimitriou, "Worst-case equilibria," in *Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science*, 1999, pp. 404–413.

[5] T. Roughgarden and E. Tardos, "How bad is selfish routing?" *Journal of the ACM*, vol. 49, no. 2, pp. 236–259, 2002.

[6] A. S. Schulz and N. Stier Moses, "On the performance of user equilibria in traffic networks," in *Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms*, 2003, pp. 86–87.

[7] E. Anshelevich, A. Dasgupta, E. Tardos, and T. Wexler, "Near-optimal network design with selfish agents," in *Proceedings of the 35th Annual ACM Symposium on the Theory of Computing*, 2003, pp. 511–520.

[8] A. Fabrikant, A. Luthra, E. Maneva, C. Papadimitriou, and S. Shenker, "On a network creation game," in *Proceedings of the 22nd Annual ACM Symposium on Principles of Distributed Computing*, 2003, pp. 347–351.

[9] F. P. Kelly, "Charging and rate control for elastic traffic," *European Transactions on Telecommunications*, vol. 8, pp. 33–37, 1997.

[10] F. P. Kelly, A. K. Maulloo, and D. K. Tan, "Rate control for communication networks: shadow prices, proportional fairness, and stability," *Journal of the Operational Research Society*, vol. 49, pp. 237–252, 1998.

[11] R. Johari and J. N. Tsitsiklis, "Efficiency loss in a network resource allocation game," *Mathematics of Operations Research*, 2004, to appear.

[12] R. Johari, S. Mannor, and J. N. Tsitsiklis, "Efficiency loss in a network resource allocation game: the case of elastic supply," MIT Laboratory for Information and Decision Systems, Publication 2605, 2004.

[13] B. Hajek and S. Yang, "Strategic buyers in a sum-bid game for flat networks," 2004, submitted.

[14] J. Rosen, "Existence and uniqueness of equilibrium points for concave n-person games," *Econometrica*, vol. 33, no. 3, pp. 520–534, 1965.

[15] R. Johari, "Efficiency loss in market mechanisms for resource allocation," Ph.D. dissertation, Massachusetts Institute of Technology, 2004.