

Do We Think and Communicate in Quantum Ways? On the Presence of Quantum Structures in Language¹

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While some of the properties of quantum mechanics are essentially linked to the nature of the micro-world, others are connected to fundamental structures of the world at large and could therefore in principle also appear in other domains than the micro-world. The aim of this article is to show the presence of genuine quantum structures in human language. More in particular, we will point out the violation of Bell's inequalities in specific situations encountered in language. The first sections of this article explain why the violation of Bell's inequalities is proof of the presence of genuine quantum structures, and how over the past decades this insight has increasingly made itself felt in the foundations of quantum mechanics research. This article also contains an overview of earlier work of ours discussing the detection of quantum structures in other domains than the micro-world.

Keywords: quantum, language, Bell’s inequalities, context, concept

1. Introduction

From the start the physics community felt the advent of quantum mechanics to be an extraordinary event. In the beginning this was due to the introduction of unconventional concepts into physics, such as wave-particle duality and quantum jumps. Rather quickly, however, the physics community’s intuition about the extraordinary nature of quantum

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mechanics was confirmed in many ways, and even today quantum mechanics has not lost any of its ‘potential to amaze’ – at the time we were writing this article (June 2004), Nature published articles by two teams of physicists who both managed to quantum teleport atoms, teleportation being an effect whose feasibility is theoretically predicted by quantum mechanics (Riebe et al. 2004; Barrett et al. 2004). Wave-particle duality, quantum jumps, quantum teleportation, and many other quantum properties are effects that are essentially connected to the nature of the micro-world, *i.e.* to what microphysical entities ‘are’ and ‘how they interact with each other and with the rest of the universe’. However, many of the new quantum properties are not essentially confined to the nature of the micro-world. They are properties that are connected with fundamental structures of the world at large, independently of whether micro- or macro-entities of this world are involved, and often also with structures of our everyday world. Quantum mechanics has demanded scientists’ attention for these properties, but many of them are present without being acknowledged, also in other domains of the world. The following section discusses the property of ‘quantum indeterminism’ to explain our objectives in further detail.

2. Quantum indeterminism and classical indeterminism

Quantum mechanics is an indeterministic theory. A theory is indeterministic if for a specific entity S in a specific state p there is at least one experiment e such that the outcome of this experiment e is not predictable when it is executed on the entity S in state p . Concretely, if repeated (equivalent) experiments e are executed on (equivalent) entities S prepared each time over again in (equivalent) states p , then these experiments yield different outcomes. It turns out that, in the case of quantum mechanics, for a quantum entity S in state p a majority of experiments will always have no determined outcomes. Of course, indeterminism of a specific set of experimental setups is not strictly reserved to the micro-world, since indeterminism as a phenomenon related to a specific experiment or set of experiments occurs regularly in the macro-world as well. Why was, and is, indeterminism never mentioned as a characteristic property of classical mechanics? The reason is that the ‘structure of indeterminism’ encountered in quantum mechanics is of a completely different and new nature, and that it has, at first sight at least, no equivalence in the macro-world.

2.1. The nature of classical indeterminism

Any indeterminism in classical mechanics is due to a ‘lack of knowledge of the observer about the specific state of an underlying deterministic world’. By way of illustration, assuming that the entire world is governed by classical mechanics, we will consider a specific situation in which the outcome of a certain experiment cannot be predicted. Let us suppose that our experiment aims at predicting the weather at 4 p.m. on August 15, 2004 (we are now at 1 a.m. on June 17, 2004). Of course, in practice we are not able to predict the weather in August from data that we collect in June. But, under the hypothesis that the world is governed by classical mechanics, in principle we would be able to make this prediction. Classical mechanics being a deterministic theory, the states of all the objects in the world at 4 p.m. on August 15, 2004 would thus be wholly determined by the states of all the objects in the world at 1 a.m. on June 17, 2004. It is ‘only’ because we lack knowledge about the states of all the objects at 1 a.m. on June 17, 2004 that we cannot predict the states of all the objects at 4 p.m. on August 15, 2004. If our experiment consists in gathering the necessary knowledge about the states of all the relevant objects at 1 a.m. on June 17, 2004, and if we have at our disposal the time and power to make the necessary calculations using the formulas of classical mechanics, we would be able to predict the weather at 4 p.m. on August 15, 2004 with certainty, in all its details. In a classical world, no indeterminism is involved on this deepest level of reality.

2.2. Hidden variable theories for quantum mechanics

In the early years of quantum mechanics, physicists were interested in knowing whether the probabilities that appear in quantum mechanics could also be explained from a 'lack of knowledge about an underlying deterministic world' as in the case of classical mechanics. John von Neumann, one of the founding fathers of quantum mechanics, was the first to come up with a 'no-go-theorem' (von Neumann 1932). Von Neumann proved that a hypothetical classical theory yielding the probabilities of quantum mechanics in the same way as outlined above – hence the probabilities due to a lack of knowledge of the state a lack of knowledge – would never be able to produce the correct numerical probabilities as the ones encountered in quantum mechanics itself. The type of hypothetical classical theories considered by von Neumann have been called 'hidden variable theories', because they introduce extra variables such that when all these variables are known every outcome of every experiment can be predicted, and hence the theory is classical deterministic. However, these extra variables are 'hidden', which means that we do lack knowledge on their values, and it is this lack of knowledge that gives rise to probabilities, and hence a classical type of indeterminism. John von Neumann proved in 1932, only 6 years after the birth of quantum mechanics, that the quantum mechanical probabilities could not be the probabilities of a classical underlying hidden variable theory.

It is well known that a no-go theorem always contains other assumptions than the one on which it focuses, so that also any of these other assumptions could be in error and the origin of the contradiction encountered in the no-go theorem. It was John Bell who remarked that one of the assumptions made by von Neumann – the technical assumption that 'the expectation value of a linear combination of two observable quantities equals the linear combination of the expectation values' – does not need to be satisfied for any hidden variable theory, and Bell built on the spot a hidden variable model for the spin of a spin-1/2 quantum entity (Bell 1966). The fact that John Bell was able to explicitly build a hidden variable model for the spin of a spin-1/2 quantum entity proved that von Neumann's no-go theorem had to contain at least one 'overlooked' assumption. But, Bell's explicitly built hidden variable model for the spin of a spin-1/2 quantum entity, being a very theoretical model, it was impossible to establish the physical meaning of the hidden variable and hence to determine the 'overlooked' assumption in von Neumann's no-go theorem. Further discussions on the issue ensued, with some physicists taking sides with John von Neumann 'against the possibility for a hidden variable theory for quantum mechanics' and producing new and more sophisticated no-go theorems, and others supporting John Bell 'in favour of the inconclusive nature of the different no-go-theorems' from a deeply-felt drive to formulate a classical theory that would substitute quantum mechanics, so that the overall picture of 'what was the matter' gradually crystallised.

2.3. Classical and quantum probabilities: the structural difference

Although the scientists aligning with Bell against the different no-go theorems produced models and revalidated the old hidden variable theory of Louis de Broglie (Broglie de 1927) of the pilot wave in its modern form as elaborated by David Bohm (Bohm 1952), they were not able to convince the scientists who shared von Neumann's denial of the possibility of a hidden variable theory for quantum mechanics. The reason for this was that by then it had become clear that there is a deep structural difference of a mathematical nature between the probability model of classical mechanics and that of quantum mechanics. We will now look into this deep structural difference in some detail.

Probability theory had been axiomatised and elaborated into a formal mathematical theory by Kolmogorov in 1933 (Kolmogorov 1950). The probability model that appears in situations described by classical mechanics – in which the probability is due to a lack of knowledge about the state of the physical entity under consideration – satisfies these

axioms of Kolmogorov, so that it is a Kolmogorovian probability model (Kolmogorov developed his axioms for this type of probability model). The probability model that appears in quantum mechanics does not satisfy the axioms of Kolmogorov (Foulis and Randall 1972; Accardi 1982, 1984; Gudder 1988; Pitowsky 1989). The more evident this deep structural difference between the two types of probability model became, the more the disbelief in the possibility of a hidden variable theory for quantum mechanics grew among that group of physicists that were aware of the structural difference. In hindsight we can now see that it was this non-Kolmogorovian nature of the quantum probability model that was already at the core of the proof of John von Neumann's very first no-go theorem. This would appear with greater emphasis yet in later corroborations of von Neumann's no-go theorem (Gleason 1957; Jauch and Piron 1963; Kochen and Specker 1967; Gudder 1968).

2.4. Bell's inequalities

Two years before John Bell formulated his critique of John von Neumann's no-go theorem (Bell 1966) he published an article in an unknown – and very short-lived – journal. This article contained a much more important result than the article in which he criticised von Neumann: the first formulation of what we now know as 'Bell's inequalities' (Bell 1964). Bell analyses the situation presented in the Einstein Podolsky Rosen paper (Einstein, Podolsky and Rosen 1935), more specifically the example of two spin-1/2 quantum entities in a singlet spin state that was introduced by David Bohm (Bohm 1959). Bell's inequalities are a set of mathematical inequalities formulated by means of expectation values of outcomes of experiments, specifically derived for the situation of the two spin-1/2 quantum entities in a singlet spin state in Bell (1964). Bell proves that when there exists a 'local' hidden variable model for this specific quantum entity of two spin-1/2 quantum entities, then the expectation values that are predicted for the considered experiments and that appear in Bell's inequalities, will be such that the inequalities are 'not violated'. By contrast, quantum mechanics predicts that the inequalities *will* be violated. John Bell's article implied two great steps forward. Firstly, Bell operationalised the EPR argumentation, *i.e.* he provided a test that allowed to experimentally establish whether the data predicted by the theory of quantum mechanics for the Bohm example of two spin-1/2 quantum entities in a singlet spin state corresponded to the data recorded by conducting a real experiment on this quantum entity in a laboratory. Secondly, by introducing the violation versus non-violation of Bell's inequalities principle, Bell offered an experimental manner of establishing whether nature functions in classical terms, *i.e.* according to a hidden variable model, or in non-classical terms, *i.e.* not according to a hidden variable model. At the time he wrote the article, Bell himself was quite convinced that the experiments would 'not' violate his inequalities, and hence that the quantum theory would be proved faulty (Bell 1981). It turned out differently, however, for all of the many experiments that were executed yielded highly convincing evidence of the inequalities being violated in the very way predicted by quantum mechanics (Clauser and Shimony 1978; Aspect 1999). Although the experiments contain several most improbable loopholes (Aspect 1999), the physics community working on the foundations of quantum mechanics are now largely convinced that experiments on micro-entities do violate Bell's inequalities as predicted by quantum mechanics. We have chosen not to go into further detail on the nature of this specific violation of Bell's inequalities for the case of the Bohm example of two spin-1/2 quantum entities, because, as we have already said, we are not focusing here on quantum properties purely restricted to the nature of the micro-world and its entities. The question then arises why we should have referred to Bell's inequalities at all, because, as far as we have set out above, their violation seems to be linked to the specific quantum mechanical nature of micro-entities, such as spin-1/2 quantum entities in the singlet spin

state. The answer is that there is a deep structural aspect to the violation of Bell's inequalities too, although not at all obvious in Bell's days. The violation of Bell's inequalities is strictly linked to the Kolmogorovian nature of the probability model of the relevant situation, as we have discussed above. To give an example, let us consider a specific situation in which a physical entity is measured and expectation values linked to probabilities as limits of relative frequencies are gathered. Pitowsky proved that the probability model that gathers all these probabilities in a structured way is Kolmogorovian if and only if none of the Bell-type inequalities that can be defined by means of expectation values of correlations between different joint measurements of the considered set of measurements are violated. This means that the detection of a single Bell's inequality being violated suffices to render the collection of probabilities incompatible with a Kolmogorovian structure (Pitowsky 1989). When we refer to Bell-type inequalities we should add that Pitowsky does derive a generalisation of Bell's inequalities for additional measurements and their correlations, but it would be beyond the scope of this paper to discuss this aspect of Pitowsky's proof in detail.

2.4. The violation of Bell's inequalities in the macro-world

Pitowsky was not the first to show that the violation of Bell's inequalities was connected to the presence of a non-Kolmogorovian structure of the probability model in the situation concerned. Before him, Luigi Accardi had proved a similar result in a different setting that was less easy to generalise (Accardi 1982, 1984). It was about the same time that one of the authors of this article put forward a macroscopic physical situation giving rise to the violation of Bell's inequalities (Aerts 1982). It was during a conference in the eighties, in which both Accardi and Aerts participated, that both results were confronted. If it was possible to violate Bell's inequalities with measurements on a macroscopic physical entity (Aerts 1982), and if violating Bell's inequalities implied the presence of a non-Kolmogorovian structure of the probability model involved (Accardi 1982), then the macroscopic model proposed in Aerts (1982) should contain a non-Kolmogorovian probability model. Discussions during the conference did not lead to any deeper understanding of the situation and both Accardi and Aerts went home convinced that there should be a deep and probably subtle mistake involved in one of the results. Indeed, the idea of a macroscopic situation entailing a non-Kolmogorovian probability model ran very much counter to the prevailing belief of those days. In the end Aerts' example was indeed proven to entail a non-Kolmogorovian probability model, albeit that its non-Kolmogorovian aspect derived from the introduction of a 'classical lack of knowledge probability' into the measuring apparatuses used in the example. If this probability was described in an attempt to model the measuring apparatuses, it would fit into a Kolmogorovian model, but that is not the purpose of a physical theory. What a physical theory aims to do is describe the physical entity and not the measurement apparatuses. In other words, if the lack of knowledge involved in the measuring apparatuses is described in terms of the model of the physical entity under consideration, this lack of knowledge acquires a non-Kolmogorovian structure.

2.5. The hidden measurement approach

This insight was worked out in an approach called 'the hidden measurement approach', the main idea being that the origin of the quantum probability is due, not to the presence of 'hidden variables of the physical entity under study', but (1) to the presence of 'hidden variables of the measurement apparatuses involved in gathering the experimental data for the physical entity under study', and (2) to the effect that 'the measurement changes the state of the physical entity under consideration' (Aerts 1985, 1986, 1987, 1991, 1993; Aerts, S. 1996, 1998, 2002ab, in press; Czachor 1992; Aerts and Aerts 1997a; Aerts, Aerts, Coecke et al. 1997). Whether the 'hidden measurement approach' offers a genuine explanation for the

probabilities of the micro-world is still under investigation (Durt 1992; Aerts 1998, 1999; Aerts, S. 2002ab, in press; Aerts, Coecke and Smets 1999; Durt, Baudon, Mathevet et al. 2002). What is certain, however, is that the mechanism that has been identified as causing the probability model to acquire a non-Kolmogorovian structure, namely the existence of ‘lack of knowledge on measurements that change the state of the entity under study’, is abundantly present in many macroscopic situations. The examples worked out in Aerts (1982) and Aerts (1986) refer to macroscopic mechanical systems that involve measurements showing the two above effects: (1) they change the state of the macroscopic mechanical system; and (2) there is a lack of knowledge on the way these measurements influence the state of the mechanical system, and hence on the interaction between the measurement apparatuses and the mechanical system. Much more so than in cases of macroscopic mechanical systems involving measurements that show these two effects, this type of situation will occur naturally with entities that are the common subjects of scientific disciplines in what we know as the human sciences, such as psychology and sociology, since both these effects are typical of the majority of measurements carried out in these fields.

3. Quantum structures and the human mind

This section discusses our approach to using quantum-like structures for the description of entities in situations in which the influence of the (measurement) context is crucial with attention for situations connected to the human mind.

3.1. Modelling decision processes

The first situation that we analysed in this way is that of psychological decision processes, where subjects are influenced by, and form part of their opinions during the testing process (Aerts and Aerts 1994, 1997b; Aerts 1995). We have set out to prove that the probability model is non-Kolmogorovian. Let us consider the simple situation of an opinion poll, because here the cause of the non-Kolmogorovian structure can be seen intuitively. Let us suppose the survey contains the following question: “Are you in favour or against the legalisation of soft drugs?” The respondents will comprise (1) those that were already in favour before being asked the question; (2) those that were already against before they are asked; and (3) those that were neither in favour nor against before being asked, and make up their opinion in the course of the survey. It is the state of mind of the respondents in the latter group that will be affected by the measurement context (the manner in which the survey is conducted, but also all the details about the environmental situation during the survey), while moreover we lack knowledge on the exact nature of this change of state. These are the two effects that turn the probability model into a non-Kolmogorovian model (Aerts and Aerts 1994, 1997b; Aerts 1995).

We should point out the following here. Social scientists studying situations that involve opinion polls are of course very much aware of the existence of a subgroup of respondents that have ‘no opinion’ before they are asked to answer the questions. This is usually resolved by including the answer ‘no opinion’ or ‘don’t know’, next to ‘in favour’ and ‘against’. A quantum model behaves differently. In a quantum model, the introduction of such a third outcome – comparable to the outcome ‘no opinion’ – does not offer an adequate solution. There will still be a group that does not fit in with any of the three alternatives offered, because they do not allow classification along these lines prior to the survey. These respondents did not decide to vote for the third ‘no opinion’ outcome (which would make them have an opinion after all). Their mind is literally ‘made up’ during the survey itself, and hence classifying them as having ‘no opinion’ prior to the survey would be equally erroneous. There is another aspect we wish to point out. Although ad hoc models are made in the field of social sciences for these situations, all these models extract the statistical analysis that they use from ‘classical statistics’. Stating that the probability

model involved is non-Kolmogorovian is equal to stating that ‘classical statistics does not apply’ in this situation.

3.2. *Modelling concepts*

Using the insights put forward in the foregoing sections we have proposed a solution for one of the most fundamental problems in ‘concept research’, namely the description of the ‘combination of concepts’ (Aerts and Gabora in press/ab). Below is an explanation of our solution to this ‘combination problem’.

Following the ‘classical theory’ of semantic concepts, there is for each concept a set of defining properties that are necessary and sufficient for category membership (Sutcliffe 1993). Essential shortcomings of this theory have been identified in many ways (Smith and Medin 1981; Komatsu 1992). A fundamental step forward was taken in the seventies, under the influence of the work of Eleanor Rosch and her collaborators at Berkeley. They showed that the ‘typicality’ of exemplars of a concept and the ‘application values’ of properties vary. Rosch formulated the prototype theory: each concept has a prototype and the typicality of an exemplar depends on how similar it is to the prototype (Rosch 1975, 1978, 1983; Rosch and Mervis 1975). The breakthrough was that a concept appears as a fuzzy structure, where similarity becomes the measure. A variety of models were proposed, and next to the ones inspired by the prototype idea, the exemplar theories became a second important class. In this category it is not the prototype but salient exemplars that serve as reference; typicality and application value are measured as the distance to these exemplars (Medin, Altom and Murphy 1984; Nosofsky 1988, 1992; Heit and Barsalou 1996). These theories yield satisfactory predictions in the case of experiments with single concepts for many dependent variables, including typicality ratings, latency of category decision, exemplar generation frequencies, and category naming frequencies. However, problems arise when it comes to ‘combinations of concepts’, for these theories do not allow to account for phenomena such as the so-called ‘guppy effect’, where guppy is not rated as a good example of ‘pet’ nor of ‘fish’, but it is rated as a good example of ‘pet-fish’ (Osherson and Smith 1981). General fuzzy set theory has been tried in vain to deliver a description of the ‘guppy effect’ (Osherson and Smith 1982; Zadeh 1982), and this peculiarity can also be understood intuitively: if (1) activation of ‘pet’ causes a small activation of guppy, and (2) activation of ‘fish’ causes a small activation of guppy, how is it that (3) activation of ‘pet-fish’ causes a large activation of guppy?

The combination problem is considered so serious that it has been said that not much progress will be possible in the field as long as no light is shed on this problem (Fodor 1994; Kamp and Partee 1995; Rips 1995; Hampton 1997). As a consequence, existing theories concentrate on attempts to model the combination of at most two concepts (‘red apple’, ‘fake diamond’, ‘car-seat’, ‘brain-storm’, ‘pet-fish’, etc.), while the real challenge consists in modelling a sentence, or a set of sentences. Our ‘quantum mechanical’ theory for concepts models an arbitrary combination of concepts, including combinations consisting of more than two concepts, and it also describes the ‘guppy effect’ by making use of the standard quantum mechanical procedure to describe the combinations of quantum entities. We show the quantum effect called entanglement to be at the origin of the guppy effect (Aerts and Gabora in press/ab; Gabora and Aerts 2002a,b).

3.3. *Concepts and sentences versus sources and data*

The main impact of Bell’s work was that it shifted the analysis of hidden variables from philosophical speculations to experiment. Bell’s inequalities deliver a criterion that allows investigating the probabilistic and *logical* structure of a *source* on the basis of the data it produces. The data are represented by sequences of symbols and are as ‘ordinary’ or ‘classical’ as the characters used to write this article. Bell’s inequalities constitute a statistical test investigating correlations between different groups of symbols, and the probability model employed in the analysis is the usual model based on the frequencies of

occurrence of certain results. However, and this is one of the ingenious elements of the Bell analysis, certain frequencies are impossible if the source is characterised by a Kolmogorovian model of probability, or – which is roughly equivalent – by a Boolean logic. Again, it should be borne in mind that the goal of the Bell statistical test is to reveal a non-Kolmogorovian *hidden* structure of a source.

The above situation is strikingly similar to that encountered in the investigation of logical structures behind language. A speaker or author is a source, the structures that are the subject of investigation refer to the conceptual level and are as hidden as von Neumann's hidden variables, but the data produced by the source are collections of symbols contained in a computer memory or on a piece of paper. Quantitative linguistics is the field to look for theoretical tools.

When working on the modelling of concepts, and with our approach having allowed us to model entire sentences (Aerts and Gabora in press/ab), we stumbled rather by accident on articles on the World Wide Web proposing mathematical models for text fragments, more specifically, articles on Latent Semantic Analysis (LSA), many of which have been made available through the web by Thomas Landauer's team at the university of Boulder (<http://lsa.colorado.edu/>). We saw that the mathematical models used were vector space models, which was quite a surprise, since our quantum model is also a vector space model (a Hilbert space, the principal mathematical structure used in quantum mechanics, is a vector space).

Meanwhile, we know that there are different approaches to modelling language on mathematical models, and that most of them use vector spaces as principal mathematical entities. The prominent examples of such approaches are Latent Semantic Analysis (LSA) (Deerwester Dumais Furnas et al. 1990; Landauer Foltz and Laham 1998), Hyperspace Analogue to Language (HAL) (Lund and Burgess 1996), Probabilistic Latent Semantic Analysis (pLSA) (Hofmann 1999), Latent Dirichlet Allocation (Blei, Ng and Jordan Michael 2003), Topic Model (Griffiths and Steyvers 2002) or Word Association Space (WAS) (Steyvers, Shiffrin and Nelson submitted). These methods are based on text co-occurrence matrices and data-analysis techniques employing singular value decomposition. For example, LSA provides a powerful method for determining similarity of meaning of words and passages by analysis of large text corpora. Quite impressive results have been obtained from experiments simulating human performance. For example, LSA-programmed machines proved capable of passing multiple-choice exams such as a Test of English as a Foreign Language (TOEFL) – (after receiving training in general English) (Landauer and Dumais 1997) and, after learning from an introductory psychology textbook, a final exam for psychology students (Landauer 2002).

In Aerts and Czachor (2004) we explored the structural similarities between these vector space models for language and the Hilbert space model of quantum mechanics. Similarities of mathematical structures between quantum mechanics, and quantum logic and quantum information theory in particular, on the one hand, and those employed in LSA on the other, opened new perspectives for both LSA and the traditional quantum fields of interest. Certain structures quite natural in the quantum domains might help to clarify the difficulties of LSA. One of the first links we noticed was the 'bag-of-words problem' of LSA. In LSA a text passage is treated as a 'bag of words', a set where order is irrelevant (Landauer, Laham and Foltz 1998). This is considered to be a serious problem because even on the intuitive level it will be clear that syntax is important to evaluating the meaning of text. The sentences 'Mary hits John' and 'John hits Mary' cannot be distinguished by LSA. The problem is basically how to *order* words in using vector models. In quantum theories, the problem was solved a long time ago in terms of tensor structures. Quantum experiments confirming the violation of the Bell inequality in fact

aimed to test the presence of tensor structures in the non-Kolmogorovian model of quantum mechanical probability. In principle, analogous analyses performed on text corpora might reveal non-classical structures in their sources, *i.e.* the minds of their authors. The idea may seem exotic, but it is not that far from other mathematical techniques employed in quantitative linguistics, and whose roots are in physics. One can mention here the Zipf-Mandelbrot criterion for natural languages and its generalisations resulting from non-extensive thermodynamics and applied to Shakespeare's writings in Montemurro (2001) and Czachor and Naudts (2002).

Looking at LSA from the quantum perspective, one may notice other aspects that are known in quantum theories but which are not very natural from the standard semantic viewpoint. For example, there exists a kind of duality between words and sentences that resembles the so-called supersymmetry. Singular value decomposition, the basic mathematical tool of LSA, may be replaced by other kinds of 'denoising' that become natural if one relates text co-occurrence matrices to quantum density matrices, and so on.

Still, the possible influences go in both directions. The tools and experiences developed within LSA might open new perspectives on semantics of quantum programming languages, the languages necessary for programming quantum computers. Moreover, once we realised that tensor structures might form a common basis for quantum information and logic on the one hand, and semantic analysis on the other, we found links to still another field that had been developing independently during the past 15 years or so: Distributed representations of cognitive structures in symbolic AI and neural networks. To our surprise, we realised that tensor products were already employed in certain AI investigations (Smolensky 1990), and that certain alternatives to tensors had been extensively investigated in this context (Plate 2003). These structures seem unknown to the quantum information community and might prove to be of crucial importance in quantum memory models. We also mention that Widdows and Peters (2003) have investigated connections between LSA and quantum logical structures.

At the moment we are at a crossroads of at least three highly and independently developed fields of knowledge: Quantum information, semantic analysis, and symbolic AI. We are convinced that quantum probabilistic and logical structures will prove essential in the latter two domains, and it is even difficult to imagine in what direction further development will continue.

4. Violating Bell's inequalities in language

In this section we will present an example of how Bell's inequalities are violated if we apply the model developed for the representation of concepts to language (Aerts and Gabora, in press/ab). From what we explained in chapter 2, this proves that language contains genuine quantum structure.

4.1. Concepts as entities in different states under different contexts

According to Rosch, the typicality of different exemplars of one and the same concept varies (Rosch 1975, 1978, 1983; Rosch and Mervis 1975). Subjects rate different typicalities for exemplars of the concept 'fruit', for example, resulting in a classification with decreasing typicality, such as 'apple', 'strawberry', 'plum', 'pineapple', 'fig', 'olive'. For the concept 'sports' the exemplars 'football', 'hockey', 'gymnastics', 'wrestling', 'archery', 'weightlifting' are classified for decreasing typicality, and for the concept 'vegetable' this happens with 'carrot', 'celery', 'asparagus', 'onion', 'pickle', 'parsley'. According to our theory of concepts, 'for each exemplar alone' the typicality varies with the context that influences it (Aerts and Gabora, in press/ab). By analogy, 'for each property alone', the application value varies with the context. We performed an experiment to point out and measure our typicality and application value effect. Participants in the

experiment classified exemplars of the concept ‘pet’ under different contexts. The context ‘the pet is chewing a bone’ thus resulted in a classification with the following decreasing typicality: ‘dog’, ‘cat’, ‘rabbit’, ‘hamster’, ‘guinea pig’, ‘mouse’, ‘hedgehog’, ‘bird’, ‘parrot’, ‘snake’, ‘canary’, ‘goldfish’, ‘spider’, ‘guppy’. The classification of the same exemplars changed when placed in a different context: ‘the pet is being taught’. This yielded the following decreasing typicality: ‘dog’, ‘parrot’, ‘cat’, ‘bird’, ‘hamster’, ‘canary’, ‘guinea pig’, ‘rabbit’, ‘mouse’, ‘hedgehog’, ‘snake’, ‘goldfish’, ‘guppy’, ‘spider’. It changed again for the context ‘look what a pet he has, I knew he was a weird person’: ‘spider’, ‘snake’, ‘hedgehog’, ‘mouse’, ‘rabbit’, ‘guinea pig’, ‘hamster’, ‘parrot’, ‘bird’, ‘cat’, ‘dog’, ‘canary’, ‘goldfish’, ‘guppy’. The effect was also measured for the application value of a property. The main idea of our concepts theory is to describe this typicality and application value effect by introducing the notion of ‘state of a concept’, and hence to consider a concept as an entity that can be in different states, such that a context will provoke a change in the state of the concept (Aerts and Gabora, in press/ab). Concretely, the state of the concept ‘pet’ in the context ‘the pet is chewing a bone’ is different from the state of ‘pet’ in the context ‘look what kind of pet he has; I knew he was a weird person’. It is this ‘being in different states’ that gives rise to the differences in values for the typicalities of different exemplars and applications of different properties. It is the set of these states and the dynamics of change of state under the influence of context corresponding to experimental data that is modelled by our quantum mechanical formalism in Hilbert space. The problem of the ‘combination of concepts’ is resolved in our theory because in ‘combination’, the concepts are in different states; for example, in the combination ‘pet-fish’, the concept ‘pet’ is in a state under the context ‘the pet is a fish’, while the concept ‘fish’ is in a state under the context ‘the fish is a pet’. The states of ‘pet’ and ‘fish’ under these contexts have different typicalities, which explains the guppy effect (Aerts and Gabora, in press/ab).

4.2. *Two pets that eat their favourite food and violate Bell’s inequalities*

Rather than begin by presenting a theoretical exposition of Bell’s inequalities, we will introduce the inequalities along with the example discussed, and point out where they are violated. Again, our aim is to prove the presence of quantum structure in language.

Let us assume the following overall context. Amy and Carol, two sisters, both have a pet. Carol has a cat called Felix, and Amy has a dog with the name Roller. The cat and the dog live together with the two sisters, and get along well. They even do not mind to eat together in the same room. But of course, they eat different food, and they both are somewhat special, in that they both have one unique food brand that they prefer above all the rest. For Felix this is ‘Eukanuba’, a well-known brand of cat food, while for Roller this is ‘Royal Canin’, a famous brand of dog food. This means that in practice whenever Felix eats, she eats Eukanuba food, and whenever Roller eats, he eats Royal Canin. For a reason that is not completely clear to Amy and Carol they never touch each others’ food. One more aspect, Amy and Carol can distinguish very well the food that is served in the room where the eating happens, because Eukanuba and Royal Canin have completely different smells.

Amy and Carol are both playing outside in the garden. The feeding room for the cat and dog is a room that opens onto the garden, but they are playing in a part of the garden where they cannot really see what is happening inside the room. They are however aware that one of the pets is being fed by their mom.

We will now take the sentence ‘The pet eats the food’ as our conceptual entity, and denote it by p . The sentence is a ‘combination of concepts’, *i.e.* the three concepts ‘pet’, ‘eat’ and ‘food’. The contexts considered are the following:

e : Hey, I think it is Roller who is eating, because I saw him just get in. (pronounced by Amy)

f : I believe that the food that mom served is Eukanuba, because I think I smell it. (pronounced by Carol)

g : One of our pets is eating one of the foods. (thought by both)

The contexts e and f are genuine context, *i.e.* they affect the state of the conceptual entity 'The pet eats the food' if applied to it. More specifically, context e affects the concept 'pet', for if we write (e, p) or, in words: 'Hey, I think it is Roller who is eating, because I saw him just get in. The pet eats the food', the concept 'pet' of the sentence 'The pet eats the food', is changed into 'Roller', and the sentence becomes 'Roller eats the food'. Because of the overall contexts, the concept 'food' in the sentence 'The pet eats the food' will be affected too. Indeed, because Roller only eats 'Canine Royal', the concept 'food' changes to 'Canine Royal', and the sentence 'The pet eats the food' changes to 'Roller eats Canine Royal'.

In a similar way, we can consider (p, f) or, in words: 'The pet eats the food. I believe that the food that mom served is Eukanuba, because I think I smell it'. The context f affects the concept 'food' of the sentence 'The pet eats the food', in the sense that 'food' changes to 'Eukanuba'. Hence the sentence 'The pet eats the food' changes to 'The pet eats Eukanuba'. Because of the overall context, the concept 'pet' is also affected by the context f . The concept 'pet' changes to 'Felix', because it is only Felix who eats Eukanuba. Hence the sentence 'The pet eats the food' changes to 'Felix eats Eukanuba', under the influence of context f .

Context g is rather a trivial context. Both girls know that 'one of the pets is being fed with one of the foods'. This means that it affects neither the concept 'pet' nor the concept 'food' nor the sentence 'the pet eats the food'.

To introduce Bell's inequalities into our discussion, we will have to associate numbers with effects of different contexts. For this purpose, we will consider the effects of the contexts e, f and g on the sentence p , and define $E(e, p) = +1$, if it is Roller who is eating, while $E(e, p) = -1$, if it is Felix who is eating. Furthermore, we define $E(p, f) = +1$, if the food eaten is Eukanuba, and $E(p, f) = -1$, if the food eaten is Royal Canin. Similarly, $E(g, p) = +1$, if one of the foods is eaten by one of the pets, and $E(g, p) = -1$, if it is not so that one of the foods is eaten by one of the pets. Lastly, $E(p, g) = +1$, if one of the foods is eaten by one of the pets, and $E(p, g) = -1$, if it is not so that one of the foods is eaten by one of the pets.

Bell's inequalities come into play when we consider situations that vary with changing pairs of contexts for the sentence 'The pet eats the food', with one of the pairs of contexts affecting the concept 'pet' in sentence p , and the other affecting the concept 'food' in sentence p . More specifically, it is the four following combinations of pairs of contexts together with sentence p that is considered in Bell's inequalities.

(1) The pair of contexts e and f with p , such that e affects 'pet' and f affects 'food'. This is represented in symbols as (e, p, f) and in words as 'Hey, I think it is Roller who is eating, because I saw him just get in. The pet eats the food. I believe that the food that mom served is Eukanuba, because I think I smell it'.

(2) The pair of contexts e and g with p , such that e affects 'pet' and g affects 'food'. This is represented in symbols as (e, p, g) and in words as 'Hey, I think it is Roller who is eating, because I saw him just get in. The pet eats the food. One of our pets is eating one of the foods'.

(3) The pair of context g and f with p , such that g affects 'pet' and f affects 'food'. This is represented in symbols as (g, p, f) and in words as 'One of our pets is eating one of the foods. The pet eats the food. I believe that the food that mom served is Eukanuba, because I think I smell it'.

(4) The pair of contexts g and g on p , such that g affects 'pet' and g affects 'food'. This is represented in symbols as (g, p, g) and in words as 'One of our pets is eating one of the foods. The pet eats the food. One of our pets is eating one of the foods'.

We will now have to associate numbers with the effects of the contexts on these four combinations. Thus, we define $E(e, p, f) = +1$ if it is Roller who eats Eukanuba, or if it is Felix who eats Royal Canin. Similarly, $E(e, p, f) = -1$ if it is Roller who eats Royal Canin or if it is Felix who eats Eukanuba. We define $E(e, p, g) = +1$ if it is Roller who eats one of the foods, and $E(e, p, g) = -1$ if it is Felix who eats one of the foods. $E(g, p, f) = +1$ if it is one of the pets who eats Eukanuba, and $E(g, p, f) = -1$ if it is one of the pets who eats Royal Canin. Lastly, we have $E(g, p, g) = +1$ if one of the pets eats one of the foods, and $E(g, p, g) = -1$ if it is not so that one of the pets eats one of the foods.

Bell's inequalities are the following:

$$|E(e, p, f) - E(e, p, g)| + |E(g, p, f) + E(g, p, g)| \leq +2$$

Bell's inequalities are violated whenever we have:

$$|E(e, p, f) - E(e, p, g)| + |E(g, p, f) + E(g, p, g)| > +2$$

Let us see what this gives in our situation.

(1) Case (e, p, f)

Amy seems to believe that it is Roller who is eating, but Carol believes that the food is Eukanuba. This means that there are three possibilities. (A) Roller is eating Royal Canin and hence Carol is mistaken about the food. (B) Felix is eating Eukanuba, and hence Amy is mistaken about which pet is eating. (C) Perhaps a new uncommon event occurred, and Roller is eating the cat food. In case A and case B, we have $E(e, p, f) = -1$. In case C, we have $E(e, p, f) = +1$.

(2) Case (e, p, g)

Amy believes that Roller is eating, and since the context is g , there is no reason to doubt her. Certainly one of the pets is eating one of the foods. Hence $E(e, p, g) = (+1)$

(3) Case (g, p, f)

Carol believes that the food is Eukanuba. In this case too there is no reason to doubt her. Certainly one of the pets is eating one of the foods. Hence $E(g, p, f) = +1$.

(4) Case (g, p, g)

One of the pets is eating one of the foods, hence $E(g, p, g) = +1$.

Let us suppose that it is certain that we are dealing with either case A or case B (thus ruling out the option of Roller's having made a move to eat cat food). In this case we have:

$$|E(e, p, f) - E(e, p, g)| + |E(g, p, f) + E(g, p, g)| = |-1 -1| + |+1 +1| = +4$$

i.e. an extreme violation of Bell's inequalities (indeed, it is the maximum violation).

The more we allow the possibility of case C happening, the less will be the extent of the violation of Bell's inequalities. Yet a real experiment in which the participants are presented with a situation in which the context is explained to them and they are asked to rate their assessments of what is really happening by giving numbers +1 and -1, would still result in a violation of Bell's, albeit not to the maximum extent of +4. The reason for this is that some of the participants would believe case C to be taking place, rather than suppose that one of the girls is mistaken.

4.3. *The violation of Bell's inequalities in language*

Before we start to analyze the violation of Bell's inequalities we wish to make the following remark. Suppose that we have the following product equalities:

$$E(e, p, f) = E(e, p) E(p, f)$$

$$\begin{aligned}
E(e, p, g) &= E(e, p) E(p, g) \\
E(g, p, f) &= E(g, p) E(p, f) \\
E(g, p, g) &= E(g, p) E(p, g)
\end{aligned}$$

In this case we have:

$$\begin{aligned}
&|E(e, p, f) - E(e, p, g)| + |E(g, p, f) + E(g, p, g)| \\
&= |E(e, p) E(p, f) - E(e, p) E(p, g)| + |E(g, p) E(p, f) + E(g, p) E(p, g)| \\
&= |E(e, p) (E(p, f) - E(p, g))| + |E(g, p) (E(p, f) + E(p, g))| \\
&= |E(e, p)| |E(p, f) - E(p, g)| + |E(g, p)| |E(p, f) + E(p, g)|
\end{aligned}$$

The fact that $E(e, p) = +1$ or $E(e, p) = -1$ implies that $|E(e, p)| = +1$, and similarly $|E(g, p)| = +1$, hence:

$$= |E(p, f) - E(p, g)| + |E(p, f) + E(p, g)|$$

The fact that $E(p, f) = +1$ or $E(p, f) = -1$, and $E(p, g) = +1$ or $E(p, g) = -1$, yields $|E(p, f) - E(p, g)| = 0$ or $|E(p, f) - E(p, g)| = +2$, and also $|E(p, f) + E(p, g)| = 0$ or $|E(p, f) + E(p, g)| = +2$, and clearly if $|E(p, f) - E(p, g)| = 0$ then $|E(p, f) + E(p, g)| = +2$, while if $|E(p, f) - E(p, g)| = +2$ then $|E(p, f) + E(p, g)| = 0$, which proves that

$$|E(p, f) - E(p, g)| + |E(p, f) + E(p, g)| = +2$$

This proves that in this case Bell's inequalities are never violated. This means that any violation of Bell's inequalities requires a violation of at least one of the product equalities. In our example of the two pets eating their favorite food, we have

$$E(e, p, f) \neq E(e, p) E(p, f)$$

Indeed, $E(e, p) = +1$, because Roller is the one who is eating (because of what Amy has seen), and $E(p, f) = +1$, because it is Eukanuba that is being eaten (because of what Carol smells), but $E(e, p, f) = -1$, because the overall context makes it very probable that one of the girls is mistaken.

What does all this show?

(i) The origin of the violation is the fact that in the sentence 'The pet eats the food' the concepts 'pet' and 'food' are entangled. More concretely, this means that when the concept 'cat' collapses to 'Felix', the concept 'food' collapses to 'the food that Felix is eating'. We formulated the overall contexts in such a way that this collapse is well defined, so that if 'pet' collapses to 'Felix', then 'food' collapses to 'Eukanuba'. In the overall contexts of the world at large, this entanglement will not be so neatly defined. Even so, the fact is that if 'pet' collapses to a specific 'pet', the 'food' collapses to a specific food, namely the food that this specific pet eats. This underlying mechanism of entanglement is the mechanism that carries the meaning of the sentence 'The pet eats the food', and it is this which makes this sentence different from 'just the bag of words' {pet, eat, food}. That the tensor product can be used to describe this entanglement, exactly as it is done in quantum mechanics, is shown explicitly in

Aerts and Gabora (in press/ab), where the full quantum mathematical description in Hilbert space is worked out for the sentence ‘The cat eats the food’.

(ii) The violation also requires aspects of language that are not purely logical. It does play a role that we have constructed a situation where the meaning of a particular combination of sentences allows concluding that one of the two girls must be mistaken. This would not be the case if we reduced the situation to a set of logical propositions. In this respect it is worth noting the following: Bell’s inequalities are violated because in case (e, p, f) it is plausible that one of the two girls is mistaken, and that it is not Roller who is eating cat food. Of course, also in cases (e, p, g) and (g, p, f) it is quite possible that one of the girls is mistaken. But language functions in such a way that this possibility will be much less taken into account, because there is no reason to believe that one of the girls is mistaken in these situations, whereas there is in situation (e, p, f) , which is contradictory, given the assumption that Roller never eats cat food.

(iii) We might be tempted to believe, taking into account our remark in (ii), that violations of Bell’s inequalities in language are strictly linked to contradictory situations. This is not true, however. The contradiction per se is of no importance. What is important is that the effect of context e on sentence p (and more specifically on the concept ‘pet’ as part of sentence p) depends on whether it is context f that affects sentence p (and more specifically the concept ‘food’ as part of sentence p) or context g that affects sentence p (and more specifically the concept ‘food’ as part of sentence p). If this is the case, we have

$$E(e, p, f) \neq E(e, p) E(p, f)$$

with a violation of Bell’s inequalities being possible. Our example contains a contradiction (one of the two girls is mistaken) only for the sake of making a clear case without having to perform real experiments on subjects. Although it is obvious that the effect may well be less strong in other contexts, it will still be there, so that we can say that any such other contexts will still give rise to a violation of Bell’s inequalities.

We had presented in earlier work a situation where Bell inequalities are violated in cognition (Aerts, Aerts, Broekaert and Gabora 2000). However the situation considered there introduced conceptual as well as physical contexts, and hence cannot be reduced easily to a purely linguistic situation, as the one considered in the present article.

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