Auction-Based Spectrum Sharing*

Jianwei Huang, Randall Berry and Michael L. Honig

Department of Electrical and Computer Engineering
Northwestern University, Evanston IL 60208, USA
jianwei@northwestern.edu, {rberry,mh}@ece.northwestern.edu

Abstract. We study auction-based mechanisms for sharing spectrum among a group of users, subject to a constraint on the interference temperature at collocated receivers. The users access the channel using spread spectrum signaling and thus generate interference with each other. Each user receives a utility that is a function of the received signal-to-interference plus noise ratio. We propose two auction mechanisms for allocating the received power. The first is an SINR-based auction, which, when combined with logarithmic utilities, leads to a weighted max-min fair SINR allocation. The second is a power-based auction that maximizes the total utility when the bandwidth is large enough. Both auction mechanisms achieve social optimality in a large system limit where bandwidth, power and the number of users are increased in a fixed proportion. We also give sufficient conditions for global convergence of a distributed updating algorithm and discuss the convergence speed.

1 Introduction

There has been growing interest in making more efficient use of spectrum by shifting from the conventional "command-and-control" spectrum usage models to more flexible "Exclusive Use" and "Commons" models (e.g., see [1]). In the Exclusive Use model, the licensee has exclusive rights to the spectrum, but can allow other users access as long as they keep the *interference temperature* under some threshold. Here the interference temperature is defined as the RF power available at the receiving antenna per unit bandwidth. In the Commons model, an unlimited number of users can share spectrum with usage rights governed by technical standards, but with no explicit protection from interference. In either model, a basic question is how to share the available spectrum efficiently and fairly.

In this paper we focus on a spectrum allocation problem for the Exclusive Use model. A group of spread spectrum users transmitting to collocated receivers want to share a fixed bandwidth. A manager (owner or regulator) must allocate the spectrum subject to a constraint on the interference temperature at the receivers. We model this as a constraint on the total received power. The manager can then be viewed as allocating the received power to the users. Each user has a utility, which is a function of the received Signal-to-Interference plus Noise Ratio (SINR), reflecting his desired Quality of Service (QoS). The interference a user receives is the total received power of all other users scaled by the bandwidth.

We consider auction mechanisms to allocate the received power as a function of bids the users submit. We model the resulting problem as a noncooperative game [2], and characterize the Nash equilibria and related properties for two different auction mechanisms. Our approach is similar to a *share auction* (see [3–7] and the references therein), or *divisible auction*, where a perfectly divisible good is split among bidders whose payments depend solely on the bids. A common form of bids in a share auction is for each user to submit his demand curve (e.g., [3–5]), i.e., the amount of goods a user desires as a function of the price. The auctioneer can then compute a market clearing price based on the set of demand curves. However, in our problem, a user's utility depends on his SINR level, which in turn depends on the power assigned to other users, making the users' demand curves dependent on each other. Instead, we adopt a signaling system similar to [6, 7], where users submit one dimensional bids for the resource.

We assume a weighted proportional allocation rule in which a user's power allocation is proportional to his bid. This type of allocation rule has been studied in a wide range of applications (e.g., see [8, 9]), including network resource allocation (e.g., [6, 7]). Given this allocation, the users participate in a game with the objective of maximizing their own benefit. It is well known that the Nash Equilibrium (NE) of

* This work was supported by the Northwestern-Motorola Center for Communications, and by NSF CAREER award CCR-0238382.

a game typically does not maximize the total system utility [10]. This has been referred to as the *price of anarchy* (e.g., [6]). In order to achieve a more desirable system operating point, we allow the manager to announce a unit price (e.g., [11, 12]) either for received SINR or received power. SINR pricing with logarithmic utilities leads to a weighted max-min fair SINR allocation. Power pricing maximizes the total utility for a large enough bandwidth. Both pricing schemes maximize the total utility in a large enough system if the total power and bandwidth are increased in fixed proportion to the number of users.

Related work on uplink power control for CDMA has appeared in [12–15]. A key difference here is that there is a constraint on the total received power at all times[1]. Because of this, a user's interference depends on his own power allocation, which can make the problem non-convex. This also allows us to view the received power as a divisible good, which leads naturally to the preceding share auction mechanisms.

We assume the user population is stationary, i.e., the users and their corresponding utilities stay unchanged during the time period of interest. On a larger time-scale one can view time divided into periods, during which the number of users and each user's utility are fixed and the proposed auction algorithm is used. When a new period begins, users may join or leave the system. Also remaining users may update their utility functions to reflect changes in their QoS requirements. For example, a user with data that must be delivered by a deadline might increase his utility (as a function of SINR) as the deadline approaches. Here we do not consider the associated mechanisms and the dynamics of the auctions over multiple periods.

The remainder of the paper is organized as follows. After introducing the auction mechanisms in Sect. 2, we analyze the performance for a finite system and for a limiting "large system" in Sect. 3 and 4, respectively. In Sect. 5 we give sufficient conditions for global convergence of a myopic bid updating algorithm and analyze the convergence speed. Numerical results are given in Sect. 6 and conclusions in Sect. 7. Several of the main proofs are given in the Appendix.

2 Auction Mechanisms

A spectrum with bandwidth B is to be shared among M spread spectrum users. User $i's$ valuation of the spectrum is characterized by a utility function $U_i(\theta_i, \gamma_i)$, where γ_i is the user's received SINR and θ_i is a user-dependent parameter. As a particular example, we consider the *logarithmic utility* $U_i(\theta_i, \gamma_i) = \theta_i \ln(\gamma_i)$. To simplify the notation, we let $U_i'(\theta_i, \gamma_i) = \partial U_i(\theta_i, \gamma_i)/\partial \gamma_i$ and $U_i''(\theta_i, \gamma_i) = \partial^2 U_i(\theta_i, \gamma_i)/\partial \gamma_i^2$, and at times we omit the user index i. Each user's utility satisfies the following assumption:

Assumption 1. *$U_i(\theta_i, \gamma_i)$ is increasing, strictly concave and twice continuously differentiable in γ_i.*

For each i, the SINR is given by

$$\gamma_i = \gamma_i(p_i) = \frac{p_i}{n_0 + p_{-i}/B},$$

where n_0 is the background noise power density, p_i is user i's received power allocation, and $p_{-i} = \sum_{j \neq i}^{M} p_j$. The total power allocation must satisfy:

$$\sum_{i=1}^{M} p_i = p_i + p_{-i} \leq P. \tag{1}$$

Notice that the preceding constraint is on the total *received* power at the collocated receivers, thus the link attenuations need not be considered in the model. However, once allocated the received power, the users must adjust their transmission power to compensate for the link attenuation between the transmitter and the receiver. A power allocation is *Pareto optimal* if no user's utility can be increased without decreasing another user's utility.

Proposition 1. *Assume for each i, $U_i(\theta_i, \gamma_i)$ is strictly increasing in γ_i. A power allocation scheme is Pareto optimal if and only if the total power constraint is tight, i.e., $\sum_{i=1}^{M} p_i = P$.*

This follows because if the power constraint is not tight, then each user can increase their power by $P/\sum_{i=1}^{M} p_i$, which increases the SINR for every user.

[1] We assume that any transmission power constraint for each user is large enough so that it can be ignored.

Assume the manager is interested in a *socially optimal* allocation, where the total utility over all users is maximized. Note that social optimality implies Pareto optimality, but the reverse is not true. Thus, the manager should always allocate the received power up to the constraint. A socially optimal solution can be viewed as one type of "fair" allocation. The manager may also consider other fairness objectives such as a max-min fair allocation.

We consider the following auction mechanisms, which operates in discrete time-slots $n = 1, 2, \ldots$

Auction Algorithm.

1. Initialization (during time slot 0):
   (a) The manager announces a constant reserve bid $\beta \geq 0$ and a positive unit price π^s (in a SINR-based auction) or π^p (in a power-based auction).
   (b) User $i \in \{1, ..., M\}$ submits an initial bid $b_i^{(0)} \geq 0$.
2. Iteration (during time slot $n \geq 1$):
   (a) After observing bids $b_i^{(n-1)}$, for $i = 1, \ldots, M$, the manager allocates each user i power $p_i^{(n)}$, resulting in a SINR $\gamma_i^{(n)}$ with

$$p_i^{(n)} = \frac{b_i^{(n-1)}}{b_i^{(n-1)} + b_{-i}^{(n-1)}} P, \text{ and } \gamma_i^{(n)} = \frac{p_i^{(n)}}{n_0 + \frac{P - p_i^{(n)}}{B}}. \tag{2}$$

   where $b_{-i}^{(n-1)} = \beta + \sum_{j \neq i}^{M} b_j^{(n-1)}$. If $b_i^{(n-1)} + b_{-i}^{(n-1)} = 0$, then $p_i^{(n)} = 0$.
   (b) In an SINR-based (power-based) auction, user i pays $C_i = \pi^s \gamma_i^{(n)}$ ($C_i = \pi^p p_i^{(n)}$).
   (c) Each user i submits a bid $b_i^{(n)} \geq 0$ to maximize his *surplus function* $S_i\left(b_i^{(n)}; b_{-i}^{(n-1)}\right) = U_i\left(\theta_i, \gamma_i\right) - C_i$.

The auction process stops when the bids do not change in two consecutive time slots. This means the system reaches a *Nash Equilibrium (NE)*, which is defined as a set of bids $\{b_i^*\}_{1 \leq i \leq M}$ such that $S_i(b_i^*; b_{-i}^*) \geq S_i(\hat{b}_i; b_{-i}^*)$ for any $\hat{b}_i$ and any i. Define user i's *best response bid* as the b_i that maximizes $S_i(b_i; b_{-i})$, assuming b_{-i} is fixed. At the NE every user submits his best response bid, i.e., no one has the incentive to deviate unilaterally. The existence and uniqueness of a NE depends on the choice of β and π. If the system does not converge for given values of β and π, the manager should change these values and restart the auction process. We show later that the only requirement on β is that it is positive, while the condition on π is more complex.

Our auction mechanisms differ from previous auction-based network resource allocation schemes (e.g., [6, 7]) in that the bids here are not the same as the payments. Instead, the bids are signals of willingness to pay, and the manager can reach the desired NE by setting the right β and π. This alleviates the typical inefficiency of the NE, and allows us to reach Pareto optimal or even socially optimal solutions.

3 Finite System Analysis

3.1 SINR-based Auction

In this case $C_i\left(\gamma_i\right) = \pi^s \gamma_i = \pi^s p_i / \left(n_0 + \frac{P - p_i}{B}\right)$, so that each user's payment depends on the received power, as well as the interference he receives. Define

$$k_i = \frac{\theta_i \left(P + B n_0\right)}{B\left(\pi^s P - \theta_i n_0\right)}. \tag{3}$$

Theorem 1. *In an SINR-based auction with logarithmic utility,*

1. *For $\beta > 0$, a unique Nash Equilibrium exists if $k_i > 0$ for all i and $\sum_{i=1}^{M} \frac{k_i}{1+k_i} < 1$, otherwise no Nash Equilibrium exists.*
2. *For $\beta = 0$, an infinite number of Nash Equilibria exist if $k_i > 0$ for all i and $\sum_{i=1}^{M} \frac{k_i}{1+k_i} = 1$, otherwise no Nash Equilibrium exists.*

The proof is given in the appendix; as shown there, user $i's$ best response bid is $b_i = k_i b_{-i}$. The bidding and power profiles at the NE are:

$$b_i^* = \frac{\frac{k_i}{1+k_i}}{1 - \sum_{l=1}^{M} \frac{k_l}{1+k_l}} \beta \text{ and } p_i^* = \frac{k_i}{1+k_i} P \quad \text{for } 1 \leq i \leq M. \tag{4}$$

In order to have a unique Nash Equilibrium, the manager has to announce a positive reserve bid ($\beta > 0$). Otherwise, there either exists no NE or an infinite number of NEs. However, since the users' bids at the NE are proportional to β, the power allocation p_i^* is independent of β. Thus, the manager only needs to announce an arbitrary positive constant at the initial stage of the auction.

To have $k_i > 0$ requires $\pi^s > \theta_i n_0/P$. Also, π^s should be set high enough so that $\sum_{i=1}^{M} \frac{k_i}{1+k_i} < 1$. Note that such π^s can always be found, because $k_i \to 0$ as $\pi^s \to \infty$. It may seem from Theorem 1 that the manager needs to know the utility parameters $\{\theta_i\}_{1\le i\le M}$ in order to set the right price. However, this requirement may be neither practical nor necessary. It is not practical when the utility functions are private information of the users. It is not necessary because the manager can adaptively find the right price by observing the users' bidding behaviors: if $\pi^s \le \theta_i \frac{n_0}{P}$ for any i or $\sum_{i=1}^{M} \frac{k_i}{1+k_i} \ge 1$, at least one user's bid will quickly increase towards infinity, indicating that the price needs to be increased.

An allocation $\{x_i\}_{1\le i\le M}$ is *weighted max-min fair* with weights $\{w_i\}_{1\le i\le M}$ if no x_i can be increased without decreasing some x_j such that $x_j/w_j \le x_i/w_i$. The SINR allocation at the NE is

$$\gamma_i^* = \frac{p_i^*}{n_0 + (P - p_i^*)/B} = \frac{\theta_i}{\pi^s}, \tag{5}$$

and user i pays $C(\gamma_i^*) = \pi^s \gamma_i^* = \theta_i$. It follows that both the SINR allocation and payments are weighted max-min fair with the weights $\{\theta_i\}_{1\le i\le M}$. In [16], Kelly et al. showed that logarithmic utility functions lead to a *weighted proportional fair* rate allocation in a network rate control problem. Their problem is convex and uncoupled across users since there is no externality effect (i.e. interference) among different users. Here, due to the interference among users, the problem may not be convex and the relation between the utility and the constrained resource (received power) is quite different from [16]. Indeed, in this case, a socially optimal solution is typically not proportional fair. Nevertheless, we achieve a weighted max-min fair allocation.

The information exchange during the auction is minimal, i.e., each user submits a bid to the manager and observes only his own power allocation. There is no need for the user to know the bids or power allocations of any other users; the only information a user needs to update his bid is the summation of all the other bids, which can be easily calculated from his own power allocation and bid. Also, since the power/SINR allocation at the NE only depends on a user's local variable and global system variables, it is easy for the user to check that he receives the correct allocation, which may prevent the manager from cheating.

We call a system *stable* if there exists a unique NE. In a stable system, define the *system usage efficiency* as

$$\eta = \frac{\sum_{i=1}^{M} p_i^*}{P} = \sum_{i=1}^{M} \frac{k_i}{1+k_i}. \tag{6}$$

For Pareto optimality, $\eta = 1$, but the necessary condition for stability is $\eta < 1$. Thus Pareto optimality and stability are conflicting objectives.

We define an *ε-system* as one with parameters $(P^\varepsilon, B^\varepsilon, M^\varepsilon, n_0^\varepsilon) = (P(1-\varepsilon), B, M, n_0 + \varepsilon P/B)$, where $\varepsilon \in (0,1)$. An *ε-Pareto optimal* allocation is defined as a Pareto optimal solution for the ε-system.

Proposition 2. *In an SINR-based auction with logarithmic utility, for any $\varepsilon \in (0,1)$, there exists a unique price $\pi^{s\varepsilon}$, such that the system is stable and achieves an ε-Pareto optimal solution (i.e., $\eta = 1-\varepsilon$ in the original system).*

Proof. From (3), it can be seen that as π^s increases from $\max_i \{\theta_i n_0/P\}$ to ∞, $\eta = \sum_{i=1}^{M} \frac{k_i}{1+k_i}$ continuously and monotonically decreases from $\upsilon > 1$ to 0. Thus, there must exists a unique price $\pi^{s\varepsilon} \in (\max_i \{\theta_i n_0/P\}, \infty)$ that achieves any $\eta = 1-\varepsilon \in (0,1)$.

In practice, the manager can achieve a target η^* by adjusting π^s after observing the usage efficiency at the current NE: if it is too low, the price should be decreased. Note if the price is decreased too much, the stability conditions in Theorem 1 may be violated.

Next we consider the revenue collected by the manager. Compared with linear power-based pricing, where the payment for user i is $\alpha_i p_i$, and α_i is a user-dependent constant, we have the following result:

Proposition 3. *As $M \to \infty$, the revenue collected in the SINR-based auction with logarithmic utility, $\sum_{i=1}^{M} \theta_i$, is the maximum revenue achieved by any power-based, user-dependent pricing scheme.*

The proof is given in the appendix. When there are enough users in the system, each user does not expect to affect the received interference by changing his own power. Thus each user maximizes his surplus function assuming that the received interference is fixed. In this large population scenario, the SINR-based auction collects as much revenue as any other linear power-based pricing scheme.

3.2 Power-based Auction

In this case $C_i(p_i) = \pi^p p_i$. For users with logarithmic utility functions, Theorem 1 still holds with a more complicated expression for k_i. The bidding and power profiles at the NE are again given by (4). For a more general class of utility functions, we show that in certain cases the power-based auction can achieve an ε-*socially optimal* allocation, which maximizes the total utility of the ε-system.

Theorem 2. *Assume for each $i \in \{1, ..., M\}$, $U_i(\theta_i, \gamma_i)$ satisfies Assumption 1 and*

$$\frac{|U_i''(\theta_i, \gamma_i)|}{U_i'(\theta_i, \gamma_i)} (\gamma_i + B) > 2, \tag{7}$$

for any $\gamma_i \in [0, P/n_0]$. Then there exists a price $\pi^{p\varepsilon}$ such that the system is stable and the NE achieves ε-social optimality for any $\varepsilon \in (0, 1)$.

Condition (7) guarantees that $U_i(\theta_i, \gamma_i(p_i))$ is concave in p_i, where $\gamma_i(p_i)$ is given in (2). This condition will be satisfied if the bandwidth is large enough for many utility functions, some of which are shown in Table 1.

Table 1. Condition (7) for various utility functions

$U(\theta, \gamma)$	(7) is true for any $\gamma \in [0, P/n_0]$ if
$\theta \ln(\gamma)$	$B > P/n_0$
$\theta \ln(1+\gamma)$	$B > P/n_0 + 2$
$\theta\gamma^\alpha$ $(\alpha \in (0,1))$	$B > \left(\frac{2}{1-\alpha} - 1\right) P/n_0$
$1 - e^{-\theta\gamma}$	$B > 2/\theta$

4 Large System Analysis

In this section we consider the asymptotic behavior as P, B, M and β go to infinity, while keeping P/M, P/B, M/B and β/M fixed. We assume that each user i's utility parameter θ_i is independently chosen according to a continuous probability density $f(\theta)$ over $\left[\underline{\theta}, \bar{\theta}\right]$, where $0 \leq \underline{\theta} < \bar{\theta} < \infty$. The expected value of θ is $E[\theta]$.

Proposition 4. *For the SINR-based auction with logarithmic utility, a unique NE exists in the limiting system if and only if*

$$\pi^s > E[\theta](n_0 + P/B)\frac{M}{P}. \tag{8}$$

In this case, the power and SINR allocations at the NE are weighted max-min fair with weights $\{\theta_i\}_{1 \leq i \leq M}$, and user i pays θ_i. Otherwise, no NE exists.

The proof is given in the Appendix. The system usage efficiency at the NE is $\eta = \frac{E[\theta](n_0+P/B)}{\pi^s P/M}$. As $\eta \to 1$, the price π^s converges to the right-hand side of (8), which is proportional to the system load M/P. This coincides with the *congestion pricing scheme* proposed in [15], where the equilibrium price reflects the congestion degree of the system.

At the NE of the limiting system, all users receive the same fixed noise plus interference level $(n_0 + P/B)$. This is because each user only gets a negligible proportion of the total power. This makes the SINR-based and power-based auctions equivalent if $\pi^s = (n_0 + P/B)\pi^p$.

In the limiting system, we define the socially optimal solution to be the allocation that maximizes the average utility per user, instead of the users' total utility, which is infinite in this case.

Assumption 2. *The utility function $U(\theta, \gamma)$ is asymptotically sublinear with respect to γ, i.e.,*

$$\lim_{\gamma \to \infty} \frac{1}{\gamma} U(\theta, \gamma) = 0.$$

Theorem 3. *In the limiting system, if $U(\theta, \gamma)$ satisfies Assumption 1 and 2, then both the SINR- and power-based auctions can achieve ε-social optimality for any $\varepsilon \in (0, 1)$.*

A sketch of the proof is given in the appendix. Assumption 2 is valid for common utility functions, e.g. $\theta \ln(\gamma)$, $\theta \ln(1+\gamma)$, $\theta\gamma^{\alpha}$ ($\alpha \in (0,1)$), and any upper-bounded utility. Under this assumption, even if a finite number of users are allocated non-negligible proportions of the total power, their contributions to the average utility becomes negligible as the number of users increases. Because of this, at the socially optimal solution every user is allocated a finite amount of power, and so faces the same interference level $(n_0 + P/B)$.

5 Myopic Bid Updating Algorithm

In this section, we consider how users update their bids to reach the NE. We use the SINR-based auction with logarithmic utilities as an example. User i can calculate the sum of other bids given only P and his own power p_i, by $b_{-i} = b_i \frac{P - p_i}{p_i}$. We assume that each user updates the bid using a myopic algorithm, i.e., he submits the best response bid assuming all the other bids are fixed:

$$b_i^{(n+1)} = k_i b_{-i}^{(n)} \tag{9}$$

This is similar to the PUA algorithm used in [17].

Proposition 5. *In an SINR-based auction with logarithmic utilities, the myopic bid updating algorithm in* (9) *globally and geometrically converges to the unique NE in a stable system if* $\max_{1 \le i \le M} k_i < \frac{1}{M-1}$ *or* $\sum_{i=1}^{M} k_i < 1$. *Furthermore, if all users start bidding from zero (the origin), the bids monotonically converge to the unique NE.*

The conditions in Proposition 5 will be satisfied if the manager announces a high enough unit price. Meanwhile, the price should be set low enough to achieve a target η^*. Thus the manager needs to adaptively search for the right price. In our simulations, we use the following search method:

1. Initialization: Set $(\underline{\pi}, \overline{\pi}) = (0, \infty)$, and choose an arbitrary initial price $\pi^{(0)} > 0$. Also, set a maximum iteration time T.
2. For $n = 1, 2, ...,$
   - (a) If the auction does not converge within T iterations, then stop the process. Let $\underline{\pi} = \pi^{(n-1)}$. Moreover, $\pi^{(n)} = 2\pi^{(n-1)}$ if $\overline{\pi} = \infty$, otherwise $\pi^{(n)} = (\underline{\pi} + \overline{\pi})/2$. Restart the auction.
   - (b) If the auction converges within T iterations at a system efficiency $\eta < \eta^*$, then let $\overline{\pi} = \pi^{(n-1)}$ and $\pi^{(n)} = (\underline{\pi} + \overline{\pi})/2$. Restart the auction.
   - (c) If the auction converges within T iterations with $\eta \ge \eta^*$, then the search process terminates.

Unlike the example in [17], here the sequence of the users' bids does not oscillate if users start from the origin. This is because the users' best response bids satisfy "strategic complementarities"– roughly, this means when one user submits a higher bid, the others want to do the same. Thus if all users start from the lowest bids (the original), the bids monotonically converge to the unique NE in a stable system. In this case, taking smaller updating steps or updating randomly with some probability less than 1 (e.g., the RUA and GUA algorithms in [17]) will not help convergence.

Although we only consider SINR-based auctions with logarithmic utilities, the myopic bid updating algorithm also works for the power-based auction with logarithmic utilities, as well as some other utility functions such as $U(\theta, \gamma) = \theta \log(1+\gamma)$. However, we note that in some cases, a target η^* may not be achievable, due to the non-convexity of the problem.

6 Numerical Results

In all simulations shown here, the $\{\theta_i\}_{1\leq i\leq M}$ are independently and uniformly distributed in $[1, 100]$. Each graph represents one realization of the parameters; similar observations were obtained for other realizations and different distributions of the parameters.

Fig. 1 shows a comparison of average utility per user for the two auctions as well as an upper bound on the socially optimal solution with logarithmic utilities. In both auctions, we set the prices so that η is close to 1. From Theorem 2, the power-based auction achieves social optimality for $P/(Bn_0) < 0$ dB. Fig. 1(a) shows that this is also true for the SINR-based auction. For $P/(Bn_0) > 0$ dB, the utility is not concave with power; in that case, we use a dual formulation to upper bound the average utility per user. Note that the two auctions still achieve a utility close to the maximum in this regime. In Fig. 1(b), we scale the system as in Sect. 4, and choose $P/(Bn_0) = 20$dB so that the utility is not concave in power. When $M \leq 14$, the auctions do not achieve the upper bound on the maximum average utility. For large M, the utilities associated with both auctions and the socially optimal solution converge to a constant. For this example, the asymptotic behavior is accurate with $M \geq 14$.

Fig. 2 shows the performance of the myopic bid updating algorithm for users with logarithmic utility functions. In Fig. 2(a), users start bidding from the origin and the bids monotonically converge to the unique NE. In Fig. 2(b), the performance of the updating algorithm as the system is scaled is shown. The target system usage efficiency η^* is chosen to be 0.90, 0.95 and 0.98 respectively. We can see that the number of iterations needed for convergence increases with M and approaches a constant when M is large (i.e., $M > 20$). This shows that the algorithm scales well with the system size. The figure also shows that the number of iterations needed for convergence increases with η^*, implying that fast convergence and high system usage efficiency are generally conflicting objectives.

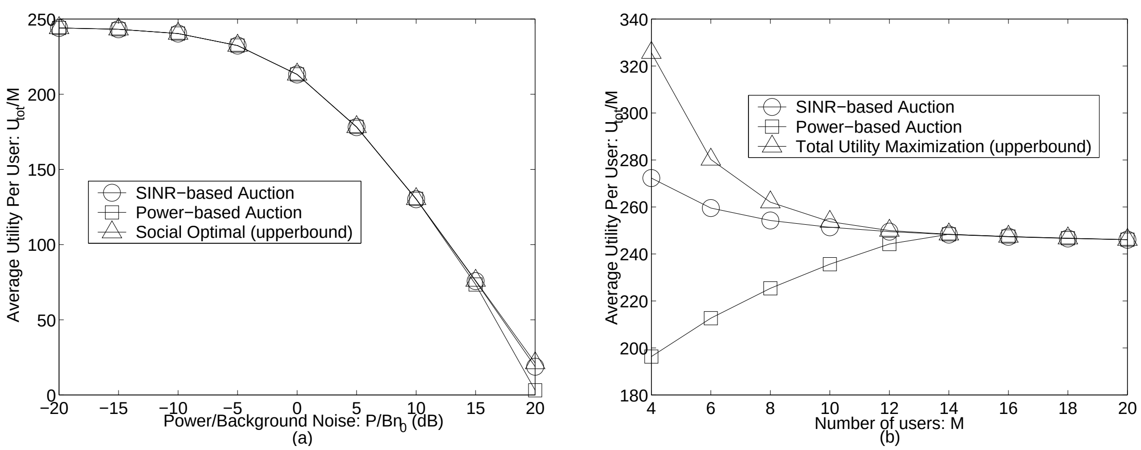

Figure 1. Average utility for the two auctions and the maximum achievable utility for the logarithmic utility function: (a) finite system with $(P, M, n_0) = (10^2, 10, 1)$ for different B; (b) system with $(P, B, n_0) = (10^4 M, 10^2 M, 1)$ for different M.

7 Conclusion

We presented two auction mechanisms (SINR-based and power-based) for sharing spectrum among a group of users subject to a constraint on the interference temperature at collocated receivers. When combined with logarithmic utilities, the SINR-based auction leads to a weighted max-min fair SINR allocation. The power-based auction maximizes the total utility for a large enough bandwidth. Both auction mechanisms are shown to achieve social optimality in a large system limit where bandwidth and power are increased in fixed proportion. We also gave sufficient conditions for global convergence of a myopic bid updating algorithm, and discussed the convergence speed both analytically and numerically.

This work is preliminary in that we only consider the interference temperature at a single point and assume that all receivers are collocated. Relaxing these assumptions is a possible direction for future

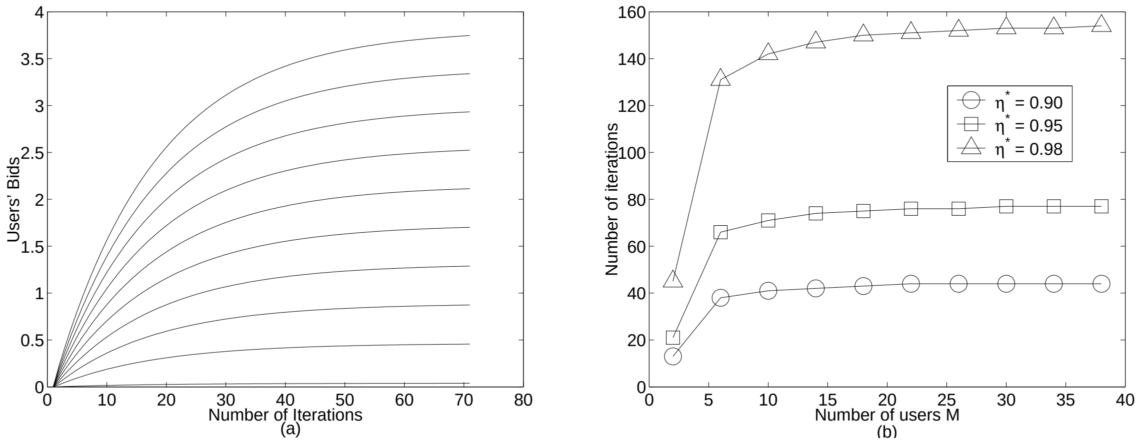

Figure 2. Performance of the myopic bid updating algorithm with logarithmic utility functions: (a) bids for each user over time for a finite system with $(P, B, M, n_0, \beta) = \left(10^2, 10^3, 10, 1, 1\right)$ and $\eta^* = 0.95$; (b) Number of iterations required for a system with $(P, B, n_0) = \left(10^4 M, 10^2 M, 1\right)$ for different values of M and target η^* .

research. We are also considering a Commons spectrum usage model, where there is no interference temperature constraint and each user is constrained only through a technical standard (e.g., which imposes a constraint on transmission power). The problem then is how to avoid the "tragedy of commons". Another extension is to consider a dynamic environment, where the number of active users varies with time.

References

1. Spectrum policy task force report. Federal Communications Commission, US (2002)
2. Mas-Colell, A., Whinston, M., Green, J.: Microeconomic Theory. Oxford University Press (1995)
3. Sunnevag, K.J.: Auction design for the allocation of emission permits. working paper (2001)
4. Wang, J.J.D., Zender, J.F.: Auctioning divisible goods. Economic Theory (2002) 673–705
5. Back, K., Zender, J.F.: Auctions of divisible goods: On the rationale for the treasury experiment. Review of Financial Studies **6** (1993) 733–764
6. Johari, R., Tsitsiklis, J.N.: Network resource allocation and a congestion game. to appear in Mathematics of Operations Research (2003)
7. Maheswaran, R., Basar, T.: Nash equilibrium and decentralized negotiation in auctioning divisible resources. Group Decision and Negotiation, to appear (2003)
8. Tijdeman, R.: The chairman assignment problem. Discrete Mathematics **32** (1980) 323–330
9. Waldspurger, C.A., Weihl, W.E.: Stride scheduling: Deterministic proportional share resource management. Techcial Memorandum MIT/LCS/TM-528, Laboratory for CS, MIT (1995)
10. Dubey, P.: Inefficiency of Nash equilibria. Mathematics of Operations Research **11** (1986) 1–8
11. D. Famolari, N. B. Mandayam, D.G., Shah, V. In: A New Framework for Power Control in Wireless Data Networks: Games, Utility and Pricing. Kluwer Academic Publishers (1999) 289–310
12. Saraydar, C., Mandayam, N.B., Goodman, D.J.: Efficient power control via pricing in wireless data networks. IEEE Trans. on Communications **50** (2002) 291–303
13. Alpcan, T., Basar, T., Srikant, R., Altman, E.: CDMA uplink power control as a noncooperative game. Wireless Networks **8** (2002) 659–670
14. Shroff, N., Xiao, M., Chong, E.: Utility based power control in cellular radio systems. In: Proceeding Inforcom 2001, Anchorage, USA (2001)
15. Heikkinen, T.M.: On congestion pricing in a wireless network. Wireless Network **8** (2002) 347–357
16. Kelly, F.P., Maulloo, A., Tan, D.: Rate control for communication networks: Shadow prices, proportional fairness and stability. Journal of Operations Research Society **49** (1998) 237–252
17. Alpcan, T., Basar, T.: Distributed algorithms for nash equilibria of flow control games. Annals of Dynamic Games **7** (2003) to appear.

Appendix

Proof of Theorem 1

Proof. First assume $\beta > 0$, thus $b_{-i} = \beta + \sum_{j \neq i}^{M} b_j > 0$. Using (2) , user i's surplus function can be written as:

$$S_i\left(b_i; b_{-i}\right) = \theta_i \log\left(\frac{b_i P}{\left(b_i + b_{-i}\right) n_0 + b_{-i}\frac{P}{B}}\right) - \pi^s \frac{b_i P}{\left(b_i + b_{-i}\right) n_0 + b_{-i}\frac{P}{B}}. \tag{10}$$

Notice that $b_i = 0$ cannot be the best response bid, since it leads to a surplus of $-\infty$ regardless of b_{-i}.

Differentiating (10) with respect to b_i yields

$$\frac{\partial S_i\left(b_i; b_{-i}\right)}{\partial b_i} = \frac{\left(\theta_i\left(Bn_0 + P\right)b_{-i} + B\left(\theta_i n_0 - \pi^s P\right)b_i\right)\left(Bn_0 + P\right)b_{-i}}{b_i\left(\left(Bn_0 + P\right)b_{-i} + Bn_0 b_i\right)^2}. \tag{11}$$

Since $b_{-i} > 0$ and $b_i > 0$, the sign of (11) only depends on the sign of the expression

$$\theta_i\left(Bn_0 + P\right)b_{-i} + B\left(\theta_i n_0 - \pi^s P\right)b_i, \tag{12}$$

which is monotonic in b_i. Setting (12) equal to 0 and solving for b_i yields

$$b_i = k_i b_{-i}, \tag{13}$$

where k_i is given in (3). For $k_i > 0$, it can be shown that (13) is the global maximum of (10), and so is user i's best response bid. Alternatively, if $k_i < 0$, then user i's best response bid is $b_i = \infty$, and there is no NE for the system.

If the system has an NE $\{b_i^*\}_{1\le i\le M}$, it must satisfy the following set of linear equations:

$$b_i^* = k_i b_{-i}^* = k_i\left(\sum_{j=1}^{M} b_j^* + \beta - b_i^*\right), \text{ and } b_i^* > 0 \text{ for } 1 \le i \le M. \tag{14}$$

Solving this using (2), we get the unique solution given in (4) if and only if $\sum_{i=1}^{M}\frac{k_i}{1+k_i} < 1$. Otherwise, these equations have no solution, and so no NE can exist.

If $\beta = 0$, then (14) can be simplified as

$$\left(1 - \sum_{i=1}^{M}\frac{k_i}{1+k_i}\right)\left(\sum_{i=1}^{M} b_i^*\right) = 0, \text{ and } b_i^* = \frac{k_i}{1+k_i}\sum_{i=1}^{M} b_i^* > 0 \text{ for } 1 \le i \le M. \tag{15}$$

There are an infinite number of solutions to (15) if and only if $k_i > 0$ for all i and $\sum_{i=1}^{M}\frac{k_i}{1+k_i} = 1$. Once again, if this is not the case, then there are no solutions to (15) and so no NE exists. □

Proof of Proposition 3

Proof. User i's surplus function under a power-based, user-dependent pricing scheme is:

$$S_i\left(p_i\right) = \theta_i \log\left(\frac{p_i}{n_0 + \frac{p_{-i}}{B}}\right) - \alpha_i p_i. \tag{16}$$

As $M \to \infty$, each user i will maximize his surplus assuming p_{-i} is fixed, yielding

$$p_i^* = \arg\max_{p_i \in [0,P]} S_i\left(p_i\right) = \begin{cases} \frac{\theta_i}{\alpha_i}, & \alpha_i \ge \frac{\theta_i}{P} \\ P, & \alpha_i < \frac{\theta_i}{P} \end{cases}. \tag{17}$$

The revenue collected by the manager is then

$$\sum_{i=1}^{M}\alpha_i p_i^* \le \sum_{i=1}^{M}\alpha_i\frac{\theta_i}{\alpha_i} = \sum_{i=1}^{M}\theta_i. \tag{18}$$

□

Proof of Proposition 4

Proof. We obtain (8) by taking the limit of the conditions in Theorem (1), under the assumed scaling. Let Lim denote $\lim_{P,B,M\to\infty}$ with $P/B, P/M, \beta/M$ fixed. Thus,

$$Lim\sum_{i=1}^{M}\frac{k_i}{1+k_i} = Lim\sum_{i=1}^{M}\frac{\theta_i\left(P/B + n_0\right)}{P\left(\pi^s + \theta_i/B\right)} = \frac{1}{M}Lim\sum_{i=1}^{M}\frac{M\theta_i\left(P/B + n_0\right)}{P\pi^s} = \frac{P/B + n_0}{P/M\pi^s}E\left[\theta\right]. \tag{19}$$

The first equality is from the definition of k_i in (3). The second equality is due to the fact that $B \to \infty$. The third equality is because of the weak law of large numbers. Condition (8) then follows directly. The weighted max-min fair SINR allocation and payments stay unchanged during the limiting process. Since every user faces the same noise plus interference $n_0 + P/B$ at the NE, then $p_i^* = \gamma_i^*(n_0 + P/B)$ for all i. This leads to a weighted max-min fair power allocation at the NE. □

Proof of Theorem 3

Proof. Due to space considerations we only give a sketch of the complete proof. In the limiting system, the maximum average utility per user is the solution to:

$$\begin{aligned} \underset{p(\theta)\geq 0}{\text{Max}} \quad & E_\theta \left[U \left(\theta, \frac{p(\theta)}{n_0 + (P - p(\theta))/B} \right) \right] \\ \text{subject to:} \quad & E_\theta \left[p(\theta) \right] = \frac{P}{M} \end{aligned} \tag{20}$$

The objective is the average utility per user in the limiting system and the constraint corresponds to the total received power constraint. In both cases we have used the law of large numbers to express these in terms of expectations over θ. The optimization is over all power allocations, $p(\theta)$, which can be viewed as functions from $[\underline{\theta}, \overline{\theta}]$ to the nonnegative real numbers. Let $U_{avg}(P/M, B/M, n_0)$ denote the solution to (20) for given values of P/M, B/M, and n_0. We first prove the following lemma:

Lemma 1. *There exists a power allocation $p(\theta)$ that solves (20), which is finite everywhere, i.e.,*

$$\lim_{P\to\infty} \frac{p(\theta)}{P} = 0, \quad \textit{for all } \theta \in \left[\underline{\theta}, \overline{\theta}\right]. \tag{21}$$

This lemma implies that each user receives a negligible fraction of the total power as the system scales. An outline of the proof follows. If the lemma is not true, then some user must be allocated infinite power as the system scales. The key idea is to show that because the utility is sublinear, this user will contribute a negligible amount to the average utility. Thus we can reallocate the user's power among the remaining users so that (21) is satisfied. This reallocation can only increase the average utility, which gives a contradiction, proving the lemma.

This lemma ensures that at a solution to (20), each user faces the same interference plus noise $n_0 + P/B$. This makes (20) a concave optimization problem. By using calculus of variations, we can solve for $p(\theta)$ in closed form, as well as the corresponding positive Lagrange multiplier λ for the average power constraint. Letting $\pi^p = \lambda$ or $\pi^s = (n_0 + P/B)\lambda$, results in the same power allocation at the NE for the power- and SINR-based auctions, respectively. □