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On Avoiding Vertexization of Robustness Problems: The Approximate Feasibility Concept

B. Ross Barmish and Pavel S. Shcherbakov

***Abstract*—For a large class of robustness problems with uncertain parameter vector q confined to a box Q, there are many papers providing results along the following lines. The desired performance specification is robustly satisfied for all $q \in Q$ if and only if it is satisfied at each vertex q^i of Q. Since the number of vertices of Q explodes combinatorically with the dimension of q, the computation associated with the implementation of such results is often intractable. The main point of this note is to introduce a new approach to such problems aimed at alleviation of this computational complexity problem. To this end, the notion of *approximate feasibility* is introduced, and the theory which follows from this definition is vertex-free.**

***Index Terms*—Computational complexity, convex optimization, Monte Carlo methods, robustness analysis and design.**

I. Introduction

In this note, we consider robustness problems for systems described in terms of a design vector $x \in X \subseteq \mathbf{R}^n$ and a real uncertain parameter vector $q \in Q \subset \mathbf{R}^\ell$, where Q is a box. For such systems, the objective is to select $x \in X$ such that a given continuous performance specification

$$f(x, q) \le 0$$

is satisfied for all $q \in Q$. When such a design vector x exists, the triple (f, X, Q) is said to be *robustly feasible*. For the case when a design vector $x \in X$ exists leading to strict inequality, this triple is said to be *strictly robustly feasible*. Equivalently, there exists some $\sigma > 0$ such that

$$f(x, q) \le -\sigma$$

for all $q \in Q$. There are a large number of papers in the literature with robust feasibility formulations which fit into this framework, e.g., see [1]–[5], and the preliminary conference version of this note [6].

A. Vertexization and Overbounding

In many papers, it is shown that the robust feasibility of $f(x, q) \le 0$ is guaranteed if an only if $f(x, q^i) \le 0$ for each of the vertices q^i of the ℓ-dimensional box Q. Henceforth, we use the word *vertexization* to describe a large number of such results in this literature. The takeoff point for this note is the fact that as the dimension ℓ of q increases, the number of vertices, $N = 2^\ell$, undergoes a so-called *combinatoric explosion*. Consequently, the computational requirements associated with a vertexization result may be excessive. One well-known example illustrating this situation involves the failure of Matlab's linear matrix inequality (LMI) toolbox which can result; i.e., for an LMI involving even a modest number of uncertain parameters, the vertexization which is typically used can lead to a computational burden which cannot be handled with the existing code. As an alternative to the computational

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burden associated with vertexization, it is often possible to introduce overbounding function in such a way as to enable convex programming in order to test for robust feasibility, e.g., see [4] for further discussion of this issue.

B. Approximate Feasibility

The main objective of this note is to describe a new approach to robust design problems which is aimed at overcoming the computational intractability problems associated with vertexization or the potential conservatism associated with overbounding. Central to this new approach is the notion of *approximate feasibility*. This new concept, introduced in [3] in the restricted context of an LMI, involves softening the robustness formulation so as to allow an arbitrarily small volume $\epsilon > 0$ of performance violation in the space of uncertain parameters.

The main result is given in Section V; that is, for a large class of so-called homogenizable robustness problems described by (f, X, Q), it is shown that their approximate feasibility counterparts are solvable via minimization of an appropriately constructed convex function Φ. The numerical results presented are of two types. The most straightforward type involves evaluation of Φ which is performed in closed form. The second type involves an evaluation of Φ which is facilitated via the Monte Carlo literature, i.e., using methods and associated sampling theory as in [7]–[11], we estimate the requisite integrals defining Φ.

II. Three Motivating Examples

To illustrate the issues addressed by the theory to follow, we now provide three motivating examples which will be revisited later in the note.

A. Example (Vertexization of Robust Quadratic Stability)

Consider the famous quadratic stability problem with uncertain parameter vector $q \in Q$, uncertain state-space matrix $A(q) = A_0 + \sum_{i=1}^{\ell} A_i q_i$ being the affine linear combination of fixed matrices and symmetric candidate Lyapunov matrix $P = P(x)$ with entries $x_i \in \mathbf{R}$ viewed as the design variables. Then, the problem of robust quadratic stability is to select a design vector $x \in X = \mathbf{R}^n$ such that $P(x) > 0$ and $A^T(q)P(x) + P(x)A(q) < 0$ for all $q \in Q$. Hence, with

$$f(x, q) = \lambda_{\max}\left(A^T(q)P(x) + P(x)A(q)\right)$$

it is well known (for example, see [1]) that this strict feasibility design problem in x is reducible to the vertices q^i of Q. That is, the satisfaction of the Lyapunov inequality above for all $q \in Q$ is equivalent to $A^T(q^i)P(x) + P(x)A(q^i) < 0$ for $i = 1, 2, \ldots, N$. This result and an analogous result for a more general linear matrix inequality, is the basis for numerical solution of the problem. That is, one considers a "large LMI" by stacking the individual vertex LMIs. However, since $N = 2^{\ell}$, we see that the computational task can easily get out of hand. For example, with five states and ten uncertain parameters, the resulting LMI is of size greater than 5000×5000.

B. Example (Vertexization of Robust Least Squares)

Many robustness problems reduce to least squares problems. Indeed, with uncertain parameter vector $q \in Q$, uncertain $m \times n$ matrix $A(q)$, uncertain $m \times 1$ vector $b(q)$ and prespecified error tolerance $\delta > 0$, the robust least squares problem (for example, see [5]) is to find a design vector $x \in X = \mathbf{R}^n$ such that $\|A(q)x - b(q)\|^2 \leq \delta^2$ for all $q \in Q$. Letting

$$f(x, q) = \|A(q)x - b(q)\|^2 - \delta^2$$

to make a connection with the notation in this note, the key observation to make is that if $A(q)$ and $b(q)$ depend affine linearly on q, the right-hand side above is convex in q. This implies the inequality above is satisfied for all $q \in Q$ if and only if $\|A(q^i)x - b(q^i)\|^2 \leq \delta^2$ at each vertex q^i of Q. Analogous to the case of the quadratic stability above, a computational scheme based on this vertexization may be impractical to carry out.

C. Example (Vertexization of Uncertain Linear Inequalities)

With $q \in Q$, $A(q)$ and $b(q)$ as defined above, many robustness problems can be reduced to finding a robustly feasible solution for the set of uncertain linear inequalities. More specifically, with constraint set X being a polyhedron, the robust feasibility problem for linear inequalities is to find a design vector $x \in X$ such that $A(q)x \leq b(q)$ for all $q \in Q$. Note that this problem is described in terms of the formulation in this note by taking

$$f(x, q) = \max_i \eta_i^T (A(q)x - b(q))$$

where η_i denotes a unit vector in the ith coordinate direction. Moreover, analogous to the robust least squares problem above, it is readily shown that if $A(q)$ and $b(q)$ depend affine linearly on q, the desired set of linear inequalities is satisfied for all $q \in Q$ if and only if $A(q^i)x \leq b(q^i)$ at each vertex q^i of Q.

III. Approximate Feasibility

As indicated in Section I, our approach to computational intractability associated with vertexization involves softening the robustness formulation so as to allow an arbitrarily small volume $\epsilon > 0$ of performance violation in the space of uncertain parameters. We now formalize this idea.

A. Approximate Feasibility

The triple (f, X, Q) is said to be *approximately feasible* if the following condition holds. Given any $\epsilon > 0$, there exists some $x^{\epsilon} \in X$ such that

$$\mathbf{Vol}\left(\{q \in Q : \ f(x^{\epsilon}, q) > 0\}\right) < \epsilon$$

where $\mathbf{Vol}(\cdot)$ denotes the volume operation. For such ϵ, x^{ϵ} is called an *ϵ-approximate solver*. As indicated above, instead of guaranteeing satisfaction of $f(x, q) \leq 0$ for all $q \in Q$, we seek solution vectors x with associated *violation set*

$$Q_{bad}(x) \doteq \{q \in Q : \ f(x, q) > 0\}$$

having volume less than any arbitrarily small prespecified level $\epsilon > 0$. Analogous to the case of robustness, we say that (f, X, Q) is *strictly approximately feasible* if there exists some $\sigma > 0$ such that the following condition holds: Given any $\epsilon > 0$, there exists some $x^{\epsilon} \in X$ such that

$$\mathbf{Vol}\left(\{q \in Q : \ f(x^{\epsilon}, q) > -\sigma\}\right) < \epsilon.$$

One of the main objectives of this note is the generation of ϵ-approximate solvers.

B. Approximate Feasibility Versus Robust Feasibility

Although robust feasibility trivially implies approximate feasibility (if x^{feas} is feasible, take $x^{\epsilon} = x^{feas}$ for all $\epsilon > 0$), there are simple examples to show that the converse is false. To illustrate, for the LMI-type scalar problem of [3] described by $f(x, q) = 1 - xq^2$, $X = \mathbf{R}$ and $|q| \leq r$ defining Q, a straightforward calculation leads to $\mathbf{Vol}\left(Q_{bad}(x)\right) = 2r$ for $x \leq 0$ and $\mathbf{Vol}\left(Q_{bad}(x)\right) = 2\min\{r, \ 1/\sqrt{x}\}$ for $x > 0$. Hence, (f, X, Q) is approximately feasible but not robustly feasible. On the other hand, under the strengthened hypothesis that either X is compact or the triple (f, X, Q) satisfies a so-called *compactifiability condition* (see

[12] for details), it is readily shown that robust and approximate feasibility are equivalent.

C. Motivation of Theory to Follow

To motivate the more formal technical exposition to follow in Sections IV and V, we first illustrate our method on a simple scalar example. To this end, we now compare the common sense solution method with the formal solution given in this note. Namely, we consider a simple scalar LMI $f(x,q) = 1 + xg(q)$ with $g(q)$ being a continuous, possibly nonlinear, function and Q defined by $|q| \leq r$. By inspection, with $X = \mathbf{R}$, the triple (f, X, Q) is strictly robustly feasible if and only if $g(q)$ has one sign. On the other hand, letting

$$\Phi(x) \doteq \int_{-r}^{r} e^{1+xg(q)} dq$$

we motivate the formalism to follow by making three key observations, which can be readily verified. First, $\Phi(x)$ is a convex function of x. Second, in view of the simply derived inequality

$$\begin{aligned}\mathbf{Vol}\,(Q_{bad}(x)) =& \mathbf{Vol}\,(\{q \in [-r,r] : 1 + xg(q) \geq 0\}) \\ \leq& \int_{-r}^{r} e^{1+xg(q)} dq = \Phi(x)\end{aligned}$$

it follows that (f, X, Q) is approximately feasible if $\Phi(x)$ can be made arbitrarily small by choice of x. Third, if the minimum of $\Phi(x)$ is zero, we can use any iteration sequence x_k leading to the minimum value of the convex function $\Phi(x)$ to obtain an ϵ-approximate solver. That is, given any $\epsilon > 0$, by picking k suitably large so as to guarantee

$$\Phi(x_k) = \int_{-r}^{r} e^{1+x_k g(q)} dq < \epsilon$$

and by taking $x^{\epsilon} = x_k$, we have obtained an ϵ-approximate solver.

IV. Approximate Feasibility Indicators and Homogenization

Motivated by the observations in the previous section, we first introduce the class of test functions which play the key role in establishing approximate feasibility of robustness problems.

A. Approximate Feasibility Indicator (AFI)

A continuous function $\phi : \mathbf{R} \rightarrow \mathbf{R}$ is said to be an *AFI* if it has the following properties:

1) $\phi(\zeta) \geq 0$ for all $\zeta \in \mathbf{R}$;
2) $\phi(\zeta) < 1$ if and only if $\zeta < 0$;
3) $\phi(\zeta) \rightarrow 0$ as $\zeta \rightarrow -\infty$.

B. Remark

Note that the aforementioned definition does not depend on (f, X, Q). However, as explained in Section V-B, there are a number of reasons associated with numerical computation why it is advantageous to tailor the choice of approximate feasibility indicator $\phi(\zeta)$ to the specification $f(x,q)$.

C. Types of AFIs

The first type of AFI, exponential, was already introduced in Section III-C; it has the form $\phi(\zeta) = e^{\zeta}$. Indeed, Conditions 1)–3) above hold. Clearly, such modifications as $\phi(\zeta) = e^{\alpha\zeta}$ with $\alpha > 0$ also satisfy the definition above and can be used as AFIs. Various piecewise linear functions can also be taken as AFIs. For instance, the second type of AFI is described by

$$\phi(\zeta) = \begin{cases} 0 & \text{for } \zeta \leq -\beta; \\ \frac{\beta+\zeta}{\beta} & \text{for } \zeta > -\beta \end{cases}$$

where $\beta > 0$ is an adjustable parameter. There exist other types of AFIs.

D. Remark

The main results of this note apply to the homogenizable performance specification functions $f(x,q)$ described below. As seen via examples in the sequel, this homogenizability requirement is satisfied in many of the common robustness formulations.

E. Homogenization

The function $f(x,q)$ is said to be *homogenizable* in x if there exists a continuous function $f^+ : (0,\infty) \times \mathbf{R}^n \times \mathbf{R}^{\ell} \rightarrow \mathbf{R}$ and a positive integer k such that

$$f^+(\gamma x_0, \gamma x, q) = \gamma^k f^+(x_0, x, q)$$

and

$$f^+(x_0, x, q) < 0, \text{ if and only if } f\left(\frac{x}{x_0}, q\right) < 0$$

for all $\gamma > 0$, $x_0 > 0$, $x \in \mathbf{R}^n$ and $q \in Q$. In this setting, the pair (x_0, x) is called the *extended design vector*.

F. Example (LMI)

To illustrate the homogenization concept, we consider the LMI

$$F_0(q) + \sum_{i=1}^{n} x_i F_i(q) < 0$$

where $F_i(q), i = 0, \ldots, n$, are known continuous symmetric matrix functions of $q \in Q$. To assure negative–definiteness above, let

$$f(x,q) \doteq \lambda_{\max}\left(F_0(q) + \sum_{i=1}^{n} x_i F_i(q)\right).$$

For the homogenization of $f(x,q)$, we take $k = 1$ and

$$f^+(x_0, x, q) = \lambda_{\max}\left(x_0 F_0(q) + \sum_{i=1}^{n} x_i F_i(q)\right).$$

Note that for some special cases, no homogenization may be needed because $f(x,q)$ may already be homogeneous. For example, the quadratic stability problem (see Section II-A), a special case of an LMI, corresponds to $F_0(q) \equiv 0$ above. In this case, one can take

$$f^+(x_0, x, q) = f(x,q).$$

G. Example (Least Squares)

In the least-squares setup (see Section II-B) with

$$f(x,q) \doteq \|A(q)x - b(q)\|^2 - \delta^2$$

a homogenization is obtained with $k = 2$ and

$$f^+(x_0, x, q) \doteq \|A(q)x - b(q)x_0\|^2 - \delta^2 x_0^2.$$

H. Example (Linear Inequalities)

For the problem in Section II-C with the performance specification

$$f(x,q) = \max_i \eta_i^T (A(q)x - b(q))$$

the natural homogenization

$$f^+(x_0, x, q) = \max_i \eta_i^T (A(q)x - b(q)x_0)$$

with $k = 1$ can be used. In the numerical example in Section VI-B, it is seen that other homogenizations are possible; a so-called extended AFI is obtained which proves to be quite convenient for computation.

V. Main Result

In the theorem to follow, the AFI $\phi(\zeta)$ is used with argument $\zeta = f^+(x_0, x, q)$ in the determination of approximate feasibility.

Theorem: Given the continuous homogenizable performance specification function $f(x,q)$, $X = \mathbf{R}^n$ and an approximate feasibility indicator $\phi(\cdot)$, define

$$\Phi(x_0, x) \doteq \int_Q \phi(f^+(x_0, x, q))dq$$

and

$$\Phi^* \doteq \inf_{x_0>0, x} \Phi(x_0, x).$$

Then, the following holds:

i) strict robust feasibility of (f, X, Q) implies $\Phi^* = 0$;
ii) $\Phi^* = 0$ implies approximate feasibility of (f, X, Q);
iii) for any $x_0 > 0$ and $x \in \mathbf{R}^n$

$$\mathbf{Vol}\left(Q_{\rm bad}\left(\frac{x}{x_0}\right)\right) \leq \Phi(x_0, x).$$

Proof: To prove i), it suffices to show that for any $\varepsilon > 0$ there exist $x_0^\varepsilon > 0$ and $x^\varepsilon \in \mathbf{R}^n$ such that $\Phi(x_0^\varepsilon, x^\varepsilon) < \varepsilon$. Indeed, by strict robust feasibility there exists some $x = x^{feas} \in \mathbf{R}^n$ such that $f(x^{feas}, q) < 0$ for all $q \in Q$. Letting $f^+(x_0, x, q)$ be the function obtained from $f(x,q)$ via the homogenizability assumption and in view of Condition 3) defining an approximate feasibility indicator ϕ, it follows that with γ suitably large, $x_0^\varepsilon = \gamma$ and $x^\varepsilon = \gamma x^{feas}$, the inequality

$$\phi\left(f^+\left(x_0^\varepsilon, x^\varepsilon, q\right)\right) < \frac{\varepsilon}{\mathbf{Vol}(Q)}$$

holds for all $q \in Q$. It now follows that:

$$\Phi\left(x_0^\varepsilon, x^\varepsilon\right) = \int_Q \phi\left(f^+\left(x_0^\varepsilon, x^\varepsilon, q\right)\right)dq < \int_Q \frac{\varepsilon}{\mathbf{Vol}(Q)}dq = \varepsilon.$$

To prove ii) and iii), we fix arbitrary $x_0 > 0$ and $x \in \mathbf{R}^n$. Using the definition of $f^+(x_0, x, q)$ and basic facts defining the AFI, it follows that:

$$\begin{aligned}\mathbf{Vol}\left(Q_{bad}\left(\frac{x}{x_0}\right)\right) &= \mathbf{Vol}\left(\left\{q \in Q : f\left(\frac{x}{x_0}, q\right) > 0\right\}\right)\\ &= \mathbf{Vol}\left(\left\{q \in Q :\ f^+(x_0, x, q) > 0\right\}\right)\\ &= \int_{f^+(x_0,x,q)\geq 0} dq\\ &\leq \int_{f^+(x_0,x,q)\geq 0} \phi(f^+(x_0, x, q))dq\\ &\leq \int_Q \phi\left(f^+\left(x_0, x, q\right)\right)dq = \Phi(x_0, x).\end{aligned}$$

A. Remarks

The theorem above indicates that the approximate feasibility question can be recast as an optimization problem and it is important to note that this optimization can often be accomplished via convex programming. Indeed, it can be readily shown that this is the case if $f^+(x_0, x, q)$ is convex in (x_0, x) and $\phi(\zeta)$ is nondecreasing convex. Whereas the conditions in the theorem of Section V for approximate feasibility do not depend on the choice of AFI ϕ, the behavior of a numerical algorithm is a different matter. This is particularly true for many cases when the integral above is not computable in closed form and Monte Carlo integration is used.

VI. Numerical Examples

In this section, two numerical examples are considered in correspondence with those given in Section II. The function $\Phi(x_0, x)$ is computed using the standard Monte Carlo technique for approximate calculation of integrals. Namely

$$\Phi(x_0, x) \approx \frac{1}{N}\sum_{j=1}^{N} e^{f^+(x_0, x, q^j)}$$

where $q^j = (q_1^j, \ldots, q_\ell^j) \in Q$, $j = 1, \ldots, N$, are samples for the uncertainty obtained via the uniform distribution. In this first example, no homogenization is needed and we use $f^+(x_0, x, q) = f(x, q)$; see Sections II-B and IV-F. In the second example, we see that it is sometimes possible to work with a function which might appropriately be called an *extended AFI*. The extended AFI has the advantage that it is tailored to the specific problem and the requisite optimization does not require Monte Carlo integration.

A. Quadratic Stability

This first example is taken from [3] where the special case of AFI theory was provided in the context of LMIs. Indeed, we consider quadratic stability of the $n \times n$ interval matrix

$$\begin{aligned}A =& A_0 + \Delta A, \quad \Delta A \doteq ((\Delta A_{ij})),\\ |\Delta A_{ij}| \leq& rS_{ij}; \quad i, j = 1, \ldots, n, \quad S \doteq ((S_{ij}))\end{aligned}$$

of [3] described by

$$A_0 = \begin{pmatrix} -2 & -2 & 0\\ 1 & 0 & 0\\ 1 & 0 & -2\end{pmatrix}$$
$$S = \begin{pmatrix} 0.1651 & 0.9394 & 0.5691\\ 0.2451 & 0.4727 & 0.1457\\ 0.7004 & 0.4014 & 0.3141\end{pmatrix}$$

and radius $r > 0$. The goal is to determine if a positive-definite matrix P exists such that $A^TP + PA < 0$ for all admissible A.

To formulate using the notation in Sections I–III, any perturbation matrix ΔA is associated with a nine-dimensional uncertainty vector q and the associated box Q is defined by the shaping matrix S. In addition, the optimization variable $x \in \mathbf{R}^6$ is composed of the six distinct entries of the symmetric positive–definite matrix P. Since this example involves nine uncertain parameters, the standard LMI technique requires solving an optimization problem described by an $M \times M$ matrix with $M = 3\times 2^9 = 1536$. Even such a moderate 3×3 problem pushes the limits of standard LMI solvers such as LMI Toolbox in Matlab in the sense that the overall size of the system matrix in the internal LMI Toolbox representation is beyond the allowable limits.

Now, with $r = 1$, we demonstrate use of the method prescribed by the theorem. Note that $\mathbf{Vol}(Q) = 512$ in this case. The convex minimization of $\Phi(x)$ was carried out with an exponential AFI $\phi(\zeta) = e^\zeta$ and using $N = 400$ samples for each integration; we obtained $\Phi^* \approx 11.6107$; Matlab execution time was about 35 s on a PC running at 488 MHz. Here, approximate feasibility is not guaranteed and it is concluded that there is no common stabilizing $P > 0$ for the interval family (LMI is infeasible). This minimum value Φ^* was achieved with the positive–definite matrix

$$P_\varepsilon = \begin{pmatrix} 19.5989 & 16.1542 & -4.6553\\ 16.1542 & 30.4427 & -2.1223\\ -4.6553 & -2.1223 & 9.4100\end{pmatrix}.$$

In order to validate this result, we carried a large-scale Monte Carlo test with $N = 100{,}000$ samples and obtained $\mathbf{Vol}\left(Q_{bad}(P_\varepsilon)\right) \approx 1.1315$. As predicted by the theory, this quantity is less than Φ^*. In the second part of this experiment, radius $r = 0.5$ was taken; this time, optimization resulted in $\Phi^* \approx 0$ and

$$P_0 = \begin{pmatrix} 2419.6 & 1228.7 & 19.3\\ 1228.7 & 5572.3 & -686.7\\ 19.3 & -686.7 & 1403.9\end{pmatrix}.$$

In accordance with the theorem, this LMI was deemed to be approximately feasible and the subsequent Monte Carlo test yielded $\mathbf{Vol}(Q_{bad}(P_\varepsilon)) = 0$.

B. Extended AFI for Robust Linear Inequalities

We consider $A(q)x \le b(q)$ with $x \in \mathbf{R}^n$ being the design vector, $A(q)$ being an $m \times n$ affine linear matrix function and $b(q)$ being an $m \times 1$ affine linear vector function of the uncertainty $q \in \mathbf{R}^\ell$, $|q_i| \le r_i$, i.e.,

$$A(q) = A_0 + \sum_{i=1}^{\ell} A_i q_i \quad b(q) = b_0 + \sum_{i=1}^{\ell} b_i q_i$$

with each A_i being a fixed $m \times n$ matrix and each b_i being a fixed $m \times 1$ vector.

1) The Function Φ: The calculations for this example are carried out using a so-called *extended AFI* which is tailored to the structure at hand. To this end, we construct a function $\Phi(x_0, x)$ which majorizes the volume of $Q_{bad}(x/x_0)$ and has the properties required in the theorem. Specifically, letting η_i denote a unit vector in the ith coordinate direction, for x as above and $x_0 > 0$, we introduce the function

$$\varphi(x_0, x, q) \doteq \sum_{i=1}^{m} e^{\eta_i^T (A(q)x - b(q)x_0)}$$

which plays the role of $\phi(f^+(x_0, x, q))$. This allows for the computation of the corresponding integral

$$\Phi(x_0, x) = \int_Q \varphi(x_0, x, q) dq$$

in closed form given by

$$\Phi(x_0, x) = 2^\ell \sum_{i=1}^{m} \prod_{j=1}^{\ell} \frac{\sinh(\eta_i^T \beta_j r_j)}{\eta_i^T \beta_j} e^{\frac{\eta_i^T \beta_0}{\ell}}$$

where $\beta_j = \beta_j(x) \doteq A_j x - b_j x_0$; $j = 0, 1, \ldots, \ell$. First, it can readily be established that strict robust feasibility implies that Φ can be driven to zero. Next, it is assumed that the infimum of Φ is zero and noted that with

$$Q_{bad}(x) \doteq \left\{ q \in Q : \eta_i^T (A(q)x - b(q)) \ge 0 \text{ for some } i \right\}$$

a lengthy but straightforward manipulation of volume and integral inequalities leads to

$$\mathbf{Vol}\left(Q_{\rm bad}\left(\frac{x}{x_0}\right)\right) \le \Phi(x_0, x)$$

for all pairs (x_0, x) with $x_0 > 0$.

2) Numerical Example: In this example, $\ell = 2$ is used and we initialized computation of the extended design parameter vector by taking $x_0 = x_0^{\rm init}$ randomly generated in $(0, 1]$ and $x = x^{\rm init}$ as a feasible point for the nominal pair (A_0, b_0). That is

$$x^{\rm init} \in X_{\rm nom}^{\rm feas} \doteq \{x \in \mathbf{R}^n : A_0 x < b_0\}.$$

In addition, an experiment was conducted with $n = 2$, $m = 3$ and the following randomly generated data:

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix} \quad A_1 = \begin{pmatrix} 0.9376 & 0.5107 \\ -0.2886 & 0.7896 \\ -0.9019 & -0.4277 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -0.4976 & 0.8816 \\ 0.8655 & 0.4037 \\ -0.7380 & 0.6955 \end{pmatrix}$$

and

$$b_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad b_1 = \begin{pmatrix} 0.0441 \\ 0.8658 \\ 0.4267 \end{pmatrix} \quad b_2 = \begin{pmatrix} -0.5439 \\ -0.1007 \\ -0.6556 \end{pmatrix}.$$

Noting that the feasibility set for the nominal (A_0, b_0) is the interior of the triangle with vertices $[-2 \quad 1]^T$, $[1 \quad 1]^T$ and $[1 \quad -2]^T$, we ran the algorithm with many different values of the uncertainty radius r to find the largest radius $r = r_{\max} \approx 1.12$ for which (f, X, Q) is strictly robustly feasible.

For $r = 1.07$ (below $r_{\max}$) and randomly generated initial conditions, the method was seen to converge leading to $\Phi^* \approx 0$ (with CPU time, dependent on initial condition, ranging from 1 to 2.5 s on a PC running at 488 MHz) and the subsequent Monte Carlo test, with $N = 1\,000\,000$ uniformly spaced samples, gives $\mathbf{Vol}(Q_{bad}) \approx 0$. When we took $r = 1.17$ (above $r_{\max}$), a variety of random initial conditions resulted in $x^\varepsilon \approx [0.1131 - 0.5125]^T$ with $\Phi^* \approx 0.0725$ and $\mathbf{Vol}(Q_{bad}) \approx 0.0259 < \Phi^*$. This result is consistent with the theory. The number of iterations is 90–140 depending on the initial conditions picked. Further increase of r gives higher values of Φ^* and respectively, greater Q_{bad} areas. Experiments were conducted with various data, e.g., unbounded X_{nom}^{feas}, higher dimensions of x and b, etc. The conclusions are of the same flavor.

VII. Future Research

The use of an *extended AFI* in Section VI resulted in the elimination of Monte Carlo integration in favor of a closed form for Φ. It is felt that further research along these lines would be worthwhile. To illustrate possible directions of research we recall that in the least squares problem with AFI $\phi(\zeta) = e^\zeta$, multivariable integration was required. To avoid such integration, one might consider other measures of feasibility; e.g., for positive integer k, let

$$\Phi_k(x) \doteq \int_Q \left(\frac{\|A(q)x - b(q)\|}{\delta} \right)^k dq$$

and note that a Chebyshev–Markov analysis leads to the following: First, for any candidate design vector x, the inequality $\mathbf{Vol}(Q_{bad}(x)) \le \Phi_k(x)$ is satisfied. Second, if the triple (f, X, Q) is strictly robustly feasible, then $\lim_{k \to \infty} \inf_x \Phi_k(x) = 0$. In view of the above, it can be argued that robust feasibility can be studied via the sequence of convex optimizations $\Phi_k^* \doteq \inf_x \Phi_k(x)$. There are now two key points to note: First, for each fixed even value of k, the requisite integral defining $\Phi_k(x)$ can be computed in closed form, i.e., Monte Carlo simulation is not needed. Second, as k gets large, the number of terms comprising the integral for $\Phi_k(x)$ becomes too large to handle.

Motivated by the computational complexity problem associated with large k above, we sketch a new direction of research which we believe will lead to low k values when the problem is suitably well *conditioned*. Indeed, let $\rho \in (0, 1)$ be an acceptable *computational threshold* for the relative volume of violation. That is, we deem (f, X, Q) to be *approximately feasible at tolerance level* ρ if

$$\frac{\mathbf{Vol}\,(Q_{bad}(x))}{\mathbf{Vol}(Q)} < \rho$$

for some $x \in X$. This leads us to consider the extent to which low ρ values are achievable using $\Phi_k(x)$ with low values of k.

Our claim is that by defining various *conditioning numbers* for a robust least squares problem, the required threshold ρ is attainable with a correspondingly low value of k if the underlying problem is well conditioned. To see this, suppose $x = \overline{x}$ achieves strict robust feasibility and consider the conditioner $\theta \in (0, 1)$ given by

$$\theta \doteq \max_{q \in Q} \frac{\|A(q)\overline{x} - b(q)\|}{\delta}.$$

That is, the closer we get to the constraint violation, the more ill-conditioned we consider the problem. Now, in view of the easily derived inequality

$$\Phi_k^* \le \theta^k \mathbf{Vol}(Q)$$

it is apparent that

$$k \geq \frac{\log \rho}{\log \theta}$$

will suffice in order to achieve the desired specification.

The above is only intended to be one example of various conditioners. To further illustrate, if we instead use the conditioner

$$\sigma_2 \doteq \frac{1}{\mathbf{Vol}(Q)} \int_Q \left(\frac{\|A(q)\overline{x} - b(q)\|}{\theta} \right)^2 dq$$

synonymous with expected behavior, the analysis yields

$$k \geq \frac{\log \left(\frac{\rho}{\sigma_2} \right)}{\log \theta}.$$

Our point of view in this new line of research is that θ and σ_2 are unknown but that we can "expect" to succeed with low k values if the problem is well conditioned. There is an obvious analogy between this point of view and the one taken in the study of mathematical programming algorithms, i.e., efficacy of an algorithm is often judged on the basis of unknown underlying conditioners related to the objective function and constraints.

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On Functional Approximation of the Equivalent Control Using Learning Variable Structure Control

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***Abstract*—A learning variable structure control (LVSC) approach is originated to obtain the *equivalent control* of a general class of multiple-input–multiple-output (MIMO) variable structure systems under repeatable control tasks. LVSC synthesizes variable structure control (VSC) as the robust part which stabilizes the system, and learning control (LC) as the "plug-in" intelligent part which completely nullifies the effects of the matched uncertainties on tracking error. Rigorous proof based on energy function and functional analysis shows that the tracking error sequence converges *uniformly* to zero, and that the bounded LC sequence converges to the *equivalent control almost everywhere*.**

***Index Terms*—Equivalent control, learning control (LC), sliding mode, variable structure control (VSC).**

I. INTRODUCTION

Variable structure control (VSC) is often used to handle the worst-case control environment: where system perturbations can be structured, unstructured, deterministic, stochastic, and persistent, and only upper bounds of system perturbations are available [1]. In practice, the inevitable switching nonidealities, such as delays, incur the chattering phenomenon. Replacing the signum function with a continuous function eliminates chattering, but degrades perfect tracking.

If the control environment is less severe than the worst case, we may come up with more appropriate control approaches, such as incorporating VSC with adaptive techniques, with time-delay control [2], etc. Each of them caters to a particular control environment with more *a priori* knowledge available than the worst case.

In practice, we often encounter the repeatable system or periodic reference/disturbance where iterative learning control [3]–[6] or repetitive control [7], [8] is well suited. In this note, we consider the tracking control tasks under a repeatable control environment, where the control system will repeat itself over iterations for a finite interval with respect to a given tracking reference. Under the repeatable tasks, we propose a learning variable structure control (LVSC) approach which has a very simple structure consisting of two components in additive form: a standard VSC based on the known upper bounds using a continuous smoothing function and a learning control (LC) which simply adds up past VSC sequences.

The LVSC approach originated in this note differs from the related existing LC schemes (e.g., [5], [9], etc.) and makes contributions in a number of respects. 1) A general class of sliding surfaces for multiple-input–multiple-output (MIMO) variable structure systems under tracking control tasks is considered instead of a linear combination of the tracking errors only. 2) Generating the *equivalent control* profile is highly desirable which assures perfect tracking and complete disturbance rejection. By virtue of repeatability, the past VSC sequences do reflect the dynamic characteristics of the uncertain system. The purpose of learning in LVSC is to extract useful control knowledge from the past VSC sequences so as to approximate the *equivalent control*. 3) The LC uses the past VSC for updating and overcomes the difficulty

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