Drawing Graphs on Two and Three Lines[⋆]

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Abstract. We give a linear-time algorithm to decide whether a graph has a planar LL-drawing, i.e. a planar drawing on two parallel lines. This has previously been known only for trees. We utilize this result to obtain planar drawings on three lines for a generalization of bipartite graphs, also in linear time.

1 Introduction

Let $G = (A \cup B, E)$ be a *partitioned graph*, i.e. a graph with a partition of its vertex set into two disjoint sets A and B. We will refer to the vertices of A as A-vertices and to the vertices of B as B-vertices, respectively. The question of drawing partitioned graphs on parallel lines arises from drawing bipartite graphs, i.e. partitioned graphs such that A and B are independent sets. A natural way to draw such graphs is to draw all vertices of A on one – say horizontal – line, all vertices of B on a parallel line, and all edges as straight-line segments between their end-vertices. Such a drawing will be denoted by BA-*drawing*.

If G is planar, a drawing without edge crossings would be desirable. Harary and Schwenk [6] and Eades et al. [3] showed that a bipartite graph G has a planar BA-drawing if and only of G is a caterpillar, i.e. a tree such that the set of all vertices of degree larger than one induces a path.

To obtain planar drawings of a larger class of bipartite graphs, Fößmeier and Kaufmann [5] proposed BAB-*drawings*. Again, every edge is a straight-line segment between its end-vertices and all vertices of A are drawn on one horizontal line, but the vertices of B may be drawn on two parallel lines – one above the A-vertices and one below. Fößmeier and Kaufmann [5] gave a linear-time algorithm to test whether a bipartite graph has a planar BAB-drawing. Another generalization of planar BA-drawings of bipartite graphs are planar drawings of leveled graphs which are considered by Jünger et al. [7].

Planar drawings for non-bipartite partitioned graphs are considered by Biedl et al. [1,2]. A complete characterization of graphs that have a planar BA-drawing is given in [1]. Felsner et al. [4] considered line-drawings of unpartitioned graphs. They gave a linear time algorithm that decides whether a tree has a planar straight line drawing on a fixed number of lines.

We will show how to decide in linear time whether an arbitrary graph has a planar *LL-drawing*, i.e. a straight line drawing on two parallel lines without

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edge crossings. Our algorithm works even if the end-vertices of some edges are constrained to be on different lines. For planar BAB-drawings, we relax the condition of bipartiteness to partitioned graphs with the only constraint that the neighbor of each vertex of B with degree one is in A. Actually, even this restriction is only necessary in very special cases, which are discussed in Section 2.2. Note that LL-drawings of unpartitioned graphs are the special case of BAB-drawings with $A = \emptyset$.

This paper is organized as follows. In Section 2, we consider planar BAB-drawings. First, we decompose the input graph such that the A-vertices of each component induce a path. We then show that we can substitute each of these components by a graph that contains only B-vertices, but simulates the possible planar BAB-drawings of the component. Finally, in Section 3, we show how to decide whether an unpartitioned graph has a planar drawing on two parallel lines.

2 BAB-Drawings

Let $G = (A \cup B, E)$ be a partitioned graph such that every B-vertex of degree one is adjacent to an A-vertex. Since we want to test planarity, we can reject every graph with more than $3n - 6$ edges. So we can assume that the number of edges is linear in the number $n := |A \cup B|$ of vertices. For a subset $S \subset A \cup B$ we denote by $G(S)$ the graph that is induced by S.

2.1 Decomposition

If G has a planar BAB-drawing, the connected components of $G(A)$ have to be paths. Therefore, the vertex set of a connected component of $G(A)$ will be called an *A-path* of G. By $\mathcal{P}(A)$, we denote the set of A-paths. $\mathcal{P}(A)$ can be determined in linear time. Next, we want to decompose G into components. A *subdivision B-path between two vertices b_1 and b_k* is a set $\{b_2, \ldots, b_{k-1}\} \subset B$ such that

- $\{b_i, b_{i+1}\} \in E$ for $i = 1 \ldots k - 1$ and
- degree $b_i = 2$ for $i = 2 \ldots k - 1$.

For an A-path $P \in \mathcal{P}(A)$, the *A-path component G_P* is the graph induced by the union of the following sets of vertices

- set P,
- set B_P of all B-vertices that are incident to P,
- all *subdivision B-paths* between two vertices of B_P that are not *subdivision B-paths* between two vertices of $B_{P'}$ for any $P' \in \mathcal{P}(A) \setminus \{P\}$.

Similar to the subdivision paths, we also omit edges that would be contained in several A-path components in any of these components. Figure 1 shows a graph with three A-path components. A vertex of an A-path component G_P that is adjacent to a vertex of $G - G_P$ is called a *connection vertex* of G_P. Given a BAB-drawing of G, we call the first and last vertex of G_P on each of the
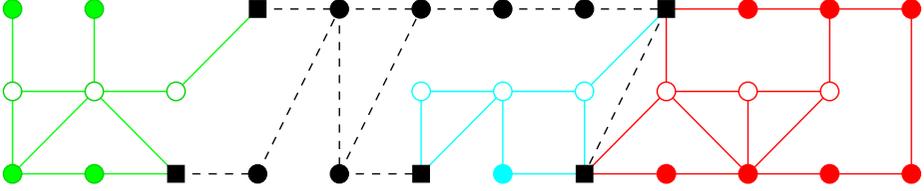
Fig. 1. Decomposition of a graph into three A-path components. White vertices are A-vertices, all other vertices are B-vertices. Rectangularly shaped vertices are connection vertices. Dashed edges are not contained in any of the three A-path components.

three lines a *terminal* of G_P. By the restriction on B-vertices of degree one, the following lemma is immediate.

Lemma 1. *Let P be an A-path of G.*

1. *All connection vertices of G_P are in B_P.*
2. *In any planar BAB-drawing of G, every connection vertex of G_P is a terminal of G_P .*
3. *If G has a planar BAB-drawing, G_P has at most four connection vertices.*

Let $P, P' \in \mathcal{P}(A)$ be distinct and $b_1, b_2 \in B_P \cap B_{P'}$ be two connection vertices of both G_P and $G_{P'}$. Then b_1 and b_2 are drawn on different lines in any planar BAB-drawing. In this case, we add edge $\{b_1, b_2\}$ to G. We will refer to such an edge as a *reminder edge*.

Lemma 2. *1. The sum of the sizes of all A-path components is in $\mathcal{O}(n)$.*
2. *There is a linear-time algorithm that either computes all A-path components and reminder edges or returns an A-path component that has more than four connection vertices.*

Proof. 1. By definition, each edge is contained in at most one A-path component. The number of A-path components is at most $|A|$ and each A-path component is connected. Thus the sum of the number of vertices in all A-path components is at most $|E| + |A| \in \mathcal{O}(n)$.
2. First, each subdivision B-path is substituted by a single edge between its two end-vertices and all sets B_P are computed. This can be done in linear time, e.g. by depth first search. Then for each $P \in \mathcal{P}(A)$, we examine the incident edges e of all $b \in B_P$. If both end vertices of e are contained in $B_P \cup P$, we add e to G_P, if not, we classify b as a connection vertex and stop the examination of this vertex b. This guarantees that for each vertex, at most one edge that is not in G_P is touched. If the number of connection vertices of G_P is greater than 4, we can return G_P, else we add a 'pre-reminder' edge labeled G_P between all six pairs of connection vertices of G_P to G.

In a final walk through the adjacency list of each connection vertex, we can use the pre-reminder edges to determine the reminder edges and those edges of G between connection vertices that have to be added to an A-path component. Finally, subdivision vertices are reinserted into the replacement edges.   □

2.2   Substitution

In this section, we show how to substitute an A-path component G_P by a graph that contains only B-vertices, but simulates all possible planar BAB-drawings of G_P in which the connection vertices are terminals. We say that two terminals of G_P are on the *same side* of G_P in a BAB-drawing if they are both first or both last vertices on their lines. They are on *different sides* if one is a first and the other one a last vertex. Essentially, there are six questions of interest:

#: How many connection vertices are contained in G_P?
η: Does the number of connection-vertices equal the number of B-vertices of G_P?
τ: Does G_P have a planar BA-drawing?
τ_v: For each connection vertex v of G_P, is there a planar BAB-drawing of G_P that induces a BA-drawing of $G_P - v$?
σ_{vw}: For each pair v, w of connection-vertices of G_P, is there a planar BAB-drawing of G_P, such that v and w are on the same side of G_P?
δ_{vw}: For each pair v, w of connection-vertices of G_P, is there a planar BAB-drawing of G_P, such that v and w are on different sides of G_P?

Note that τ implies τ_v and τ_v implies σ_{vw} and δ_{vw} for any pair v, w of connection vertices of G_P. Thus, provided that there exists some planar BAB-drawing of G_P, these six questions lead to the cases listed in Table 1. Note that the listed substitutes contain different kind of edges. There is one case with a parallel edge and one case with a dotted edge. Let G' be the graph constructed from G by substituting each A-path component in the way described above. We say that an edge in an LL-drawing is *vertical* if the end vertices of e are drawn on different lines.

Lemma 3.   *1. The size of G' is linear in the size of G.*
 2. G' has a planar LL-drawing with every multiple edge and every reminder edge drawn vertically and no dotted edge having a parallel edge if and only if G has a planar BAB-drawing.

We discuss only, where the precondition on degree one vertices is needed and omit the rest of the proof. If we have a planar BAB-drawing and we want to construct a planar LL-drawing of G' by replacing each A-path component G_P by its substitute, we have to make sure, that for any two vertices v and w of G_P all vertices that are drawn between v and w do also belong to G_P. But there might be also subdivision paths S between vertices of degree one and vertices $b \in B_P$. If b is not a connection vertex, we can add S to G_P, but if b is a connection vertex, there need not exist a suitable substitute any more. There is no problem, if there are at least three subdivision paths between b and vertices of degree one, since then at least one of them has to be drawn on the other line then b and then any of them can be drawn there, thus neither has to be added to G_P.

Lemma 4. *For each component, the six questions that determine the substitute, can be answered in time linear in the size of the component.*

Table 1. Substitutes for the A-path components. Cases that correspond up to the names of the connection vertices are omitted. A $+$ means that the corresponding question is answered by yes. A $\oplus$ means that the answer to this property is implied. No entry means that the according property is not fulfilled.

1 connection vertex

η	τ	τ_v	
$+$	$\oplus$	$\oplus$	v
	$+$	$\oplus$	$v - \bullet$
		$+$	v `| \` $\bullet - \bullet$
			$v - \bullet$ `| / |` $\bullet - \bullet$

4 connection vertices

σ_{vx}	σ_{vy}	σ_{vz}	
$+$	$+$	$+$	$v - \bullet \qquad \bullet - y$ `    \   /` $w - \bullet - \bullet - \bullet - x$
$+$	$+$		$v - \bullet - \bullet - x$ `	|` $y - \bullet - \bullet - z$
		$+$	$v - \bullet - x$ `|	|` $y - \bullet - z$

2 connection vertices

η	τ	τ_v	τ_w	σ_{vw}	δ_{vw}	
$+$	$+$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$v - w$
$+$		$\oplus$	$\oplus$	$\oplus$	$\oplus$	v `||` w
	$+$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$v - \bullet - w$
		$+$	$+$	$\oplus$	$\oplus$	$v - w$ `| /` $\bullet$
			$+$	$\oplus$	$\oplus$	$v - \bullet$ `| \ |` $w - \bullet$
				$+$	$+$	$v - \bullet - \bullet$ `|   |` $w - \bullet - \bullet$
					$+$	$v - \bullet - \bullet$ `|	|` $w - \bullet - \bullet$
					$+$	$v - \bullet - \bullet$ `|	|` $\bullet - \bullet - w$

3 connection vertices

τ_v	τ_w	τ_x	σ_{vw}	σ_{vx}	σ_{wx}	
$+$	$+$	$+$	$\oplus$	$\oplus$	$\oplus$	$v - w$ `| /` x
$+$	$+$		$\oplus$	$\oplus$	$\oplus$	$v - \bullet - x$ `| /` w
		$+$	$+$	$\oplus$	$\oplus$	$v - \bullet - \bullet - w$ `	/` x
		$+$		$\oplus$	$\oplus$	$x - v$ `| \ |` $w - \bullet$
		$+$	$+$	$+$		$v - \bullet \qquad \bullet - \bullet$ `    \   /` $w - \bullet - \bullet - \bullet - x$
			$+$	$+$		$v - \bullet$ `| \ |` $x - w$
		$+$				$v - \bullet - \bullet$ `|	|` $w - \bullet - x$

Proof. It is obvious how to decide Properties $\#$ and η. To decide properties σ_{vw} and δ_{vw}, suppose, we have an A-path component G_P with an A-path P : $a_1, \ldots, a_k$. For technical reasons, we add vertices a_0 and a_{k+1}, one at each end, to P.

Let W be the set of vertices of a connected component of $G_P(B)$. Suppose first that we wouldn't allow edges between B-vertices on different lines. In this case $G(W)$ has to be a path and the vertices in W have to occur in the same order as their adjacent vertices in P. Furthermore, let a_ℓ (a_r) be the A-vertex

with lowest (highest) index that is adjacent to W. Then we say that the edges $\{a_i, a_{i+1}\}, i = \ell, \ldots, r - 1$ are *occupied* by $G(W)$. Finally, if a vertex b in $G(W)$ is a connection vertex of G_P then b has to be a terminal. Thus, if a_ℓ is an A-vertex with the lowest (highest) index that is adjacent to b, then the edges $\{a_i, a_{i+1}\}, i = 0, \ldots, \ell$ (or $i = \ell, \ldots, k + 1$) are also occupied by $G(W)$.

In general, if G_P has a planar BAB-drawing, $G(W)$ might be a path with at most two chords and the A-vertices adjacent to this path may change at most twice between increasing and decreasing order. Thus, there are three reasons for changing the line in W: a chord, a change in the order of the adjacent A-vertices, or a connection vertex. Note also that such a change of lines might occur in at most two connected components of $G_P(B)$. If there is a line change between b_1 and b_2 in W, then similar to the case of connection vertices above, $G(W)$ also occupies the edges between the adjacent A-vertices of b_1 and b_2 and one end of the A-path P. Note that in case of a line change some edges in $G(P)$ might be occupied twice.
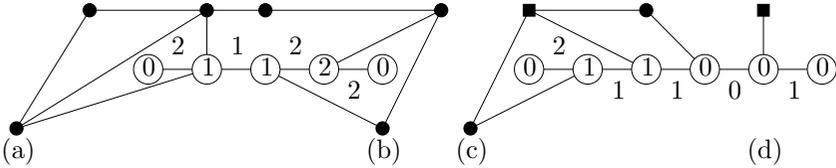

Fig. 2. Three reasons for changing a line: a) a chord, b) a change in the order of adjacent A-vertices, and c) a connection vertex. d) A connection vertex occupies a part of the A-path. The numbers indicate how many times an edge or a vertex in the A-path is occupied.

We label every edge and every vertex in $G(P)$ with the number of times it is occupied as indicated in Fig. 2, where the number of times a vertex $v \in P$ is *occupied* by $G(W)$ is defined as follows. v is occupied k times by $G(W)$, if both adjacent edges in the A-path are occupied k-times by W and v is not adjacent to an end vertex of W and $k - 1$ times if it is adjacent to such an end vertex. It is also occupied $k - 1$-times if one adjacent edge is occupied k-times and the other one $k - 1$ times. There exists a planar BAB-drawing with the corresponding choice of the direction of the connection vertices, if and only if

1. in total, every edge in $G(P)$ is occupied at most twice and
2. each vertex $v \in P$ that is adjacent to an end vertex of a connected component of $G_P(B)$ is occupied at most once.

This test works in linear time, since we can reject the graph, as soon as an edge gets a label higher than 2 and because there are at most four connection vertices and at most two components that change lines for which different orientations have to be checked. A similar approach can be used to answer questions τ and τ_v. □

The results in the next section show how to test the conditions of Lemma 3,2, which completes the algorithm for deciding whether a planar BAB-drawing exists.

3   LL-Drawings

Given a graph $G = (V, E)$ and a set of edges $U \subset E$, we show how to decide in linear time whether there exists a planar LL-drawing of G with the property that each edge in U is drawn vertically. A long version of the planar LL-drawability test can be found in [8]. We will first discuss trees and then the general case.

3.1   LL-Drawings of Trees

Felsner et al. [4] characterized trees that have a planar LL-drawing and called them strip-drawable. We give a slightly different characterization.

Lemma 5. *A tree T has a planar LL-drawing, if and only if it contains a* spine, *i.e. a path S such that $T - S$ is a collection of paths.*

Proof. $\Rightarrow$ The unique path between a leftmost and a rightmost vertex of T in a planar LL-drawing is a spine of T.
    $\Leftarrow$ An LL-drawing can be achieved by placing the spine on one of the lines and the path components in the order of their adjacency to the spine on the other line. $\square$

As indicated in [4], the inclusion minimal spine $S_{\min}$ can be computed in linear time. If a vertex v is required to be an end vertex of a spine, we only have to check if there is a path between v and an end vertex of $S_{\min}$ in $T - S_{\min}$. We will use this fact when we examine the general case. Finally, the following lemma characterizes whether the edges in a set $U \subset E$ can be drawn vertically.

Lemma 6. *An edge e can be drawn vertically if and only if there exists a spine S that contains at least one end vertex of e.*

3.2   LL-Drawings of General Graphs

For general graphs, the existence of a spine is still necessary for the existence of a planar LL-drawing, but it is not necessarily sufficient. For example, see the

Fig. 3. LL-drawings of a tree. The spine is drawn black.

graph on the right. To test whether a general graph has a
planar LL-drawing, we first split the graph into components
like paths, trees and two connected components. We give nec-
essary conditions for these components and show how to test
them in linear time. Finally, by constructing a drawing, we
show that these necessary conditions are also sufficient. First
we make the following observation.
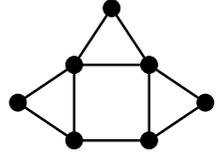
Lemma 7. *Every graph that has a planar LL-drawing is outer planar.*

Proof. Since all vertices are placed on two horizontal lines, every vertex is either
a top most or bottom most vertex and is thus incident to the outer face.   □

Suppose now without loss of generality that G is a connected graph. A vertex
v is called a *connection vertex*, if it is contained in a cycle and its removal
disconnects G. We denote the set of all connection vertices by V_c. The connection
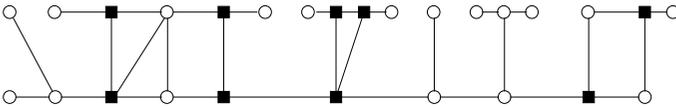vertices can be determined with depth first search in linear time.

Fig. 4. A graph with a planar LL-drawing. Connection vertices are solid.

A subgraph L of G is called a *single line component*, if it is maximal with the
property that L is an induced graph and there exists a vertex $v \in V_c$ such that
L is a connected component of $G - v$. By $\bar{L}$, we denote a single line component
L including its incident connection vertex. If $\bar{L}$ forms a path we call it a *strict
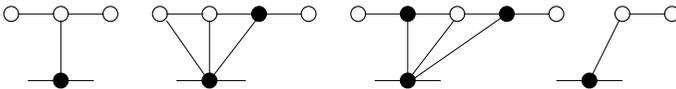single line component*, otherwise we call it a *fan*.

Fig. 5. Fans and a strict single line component, connection vertices are solid.

Simply testing for all $v \in V_c$, whether the connected components of $G(V \setminus
\{v\})$ are paths leads to a quadratic time algorithm for finding the single line
components. But we can use outer-planarity to do it in linear time. Note that
any single line component contains at most two connection vertices (see Fig. 5 for
illustration). Thus, we can find them, by storing the three last visited connection
vertices on a walk around the outer face and testing, whether the vertices between
the first and second occurrence of the same connection vertex induce a path.

Let G' be the subgraph that results from G by deleting all single line components. The *two-lined components* of G are the components that we get by splitting G' at the connection vertices of G. Two-lined components that do not contain a cycle will be called *tree components*. All other two-lined components will be denoted by *two-lined biconnected components*. Figure 6 illustrates the different components of the graph in Fig. 4.
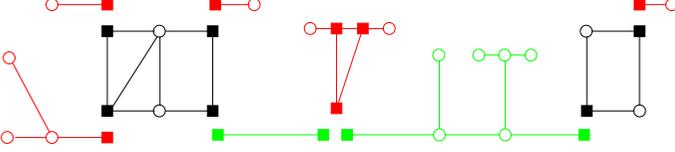

Fig. 6. Components of the graph in Fig. 4. There are two trees, two two-lined biconnected components, two fans, and three strictly single lined components.

We have now defined all components that we need to characterize those graphs that have a planar LL-drawing. Note that a graph that contains no two-lined biconnected components has a planar LL-drawing if and only if it contains a spine. Before we formulate the characterization for the general case, consider the set $\mathcal{L}$ of all two-lined components. Let $\mathcal{P}$ be the graph with vertex set $\mathcal{L}$ in which two two-lined components are adjacent if and only if they share a connection vertex. Suppose a planar LL-drawing for G is given. Then all two-lined biconnected components require at least two lines. All tree components also require at least two lines or are connected to two components that require two lines. Thus $\mathcal{P}$ has to be a path. We will refer to this property by saying that *the two-lined components induce a path*.

**Theorem 1.** *A graph G has a planar LL-drawing if and only if*

1. *G is outer planar,*
2. *the two-lined components induce a path,*
3. *each tree component T has a spine S such that the connection vertices of T are end vertices of S, and*
4. *for each two-lined biconnected component B there is a drawing with the following properties.*
   (a) *Connection vertices are leftmost or rightmost vertices of B on their line.*
   (b) *At most one two-sided component is connected to each vertex of B.*
   (c) *If a two-sided component or a fan is connected to a vertex of B, then a vertex on the same side is at most connected to a path.*

The necessity of these conditions follows from Section 3.1, Lemma 7, and the observations mentioned above. To show that these conditions are also sufficient, we sketch how to find a drawing of G in linear time.

Drawing and Sufficient Conditions. In this subsection we sketch how to construct in linear time a planar LL-drawing if the conditions in Theorem 1 are fulfilled. We also mention which edges can be drawn vertically. Since a linear order on the two-lined components is given, we only have to show how to draw each component separately.

It was already shown in Section 3.1 how to find a spine and that a tree with a spine has a planar LL-drawing. For drawing two-lined biconnected components, first note that a biconnected graph has a unique outer planar embedding. Starting with a connection vertex on any line, we add the others in the order of their appearance around the outer face and switch lines only if necessary, i.e. at a connection vertex or a chord. We do this procedure for each direction around the outer face. Those cases in which lines are switched at most twice, are the possible outer planar drawings of a two-lined biconnected component, with respect to Theorem 1,4. Hence, if a drawing exists, it can be found in linear time.

Vertical edges in two-lined biconnected components only yield another condition for switching lines. Let $\overline{L}$ be a fan on a single line component L and a connection vertex v. If $\overline{L}$ is connected to two two-lined components, only edges between v and L can be vertical edges. Else, also edges indicated by dotted lines in the Figure on the right hand side can be drawn upright, but only one of the edges e_1 and e_2 simultaneously.
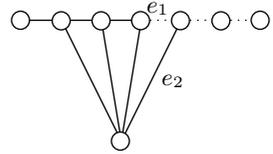
4   Conclusion

We showed how to decide in linear time, whether an arbitrary graph has a planar drawing on two parallel lines, even if the input contains edges that have to be drawn vertically. We applied this result to decide in linear time whether a partitioned graph $G = (A \cup B, E)$ with the property that every B-vertex of degree one is adjacent to an A-vertex has a planar BAB-drawing. The algorithm worked in three steps. First, the graph was decomposed into A-path components. Then, each of these components was substituted by a graph that contained only B-vertices, but simulates the possible positions of the connection vertices. Finally, we test whether the resulting graph has a planar LL-drawing. We discussed that the restriction on the vertices of degree one is only needed in the following sense: If b is a connection vertex of an A-path component, the number of subdivision B-paths between b and vertices of degree one may not be one or two.

For further research, it would be interesting to examine whether our approach could be extended to $BAAB$-drawings, that is drawings for which there are two parallel middle lines for the A-vertices.

References

1. T. C. Biedl. Drawing planar partitions I: LL-drawings and LH-drawings. In *Proceedings of the 14th Annual ACM Symposium on Computational Geometry (SCG '98)*, pages 287–296, 1998.

2. T. C. Biedl, M. Kaufmann, and P. Mutzel. Drawing planar partitions II: HH-drawings. In J. Hromkovic, editor, *Graph Theoretic Cocepts in Computer Science, 24th International Workshop, (WG '98)*, volume 1517 of *Lecture Notes in Computer Science*, pages 124–136. Springer, 1998.
3. P. Eades, B. D. McKay, and N. C. Wormald. On an edge crossing problem. In *Proceedings of the 9th Australian Computer Science Conference (ACSC 9)*, pages 327–334, 1986.
4. S. Felsner, G. Liotta, and S. K. Wismath. Straight-line drawings on restricted integer grids in two and three dimensions. In M. Jünger and P. Mutzel, editors, *Proceedings of the 9th International Symposium on Graph Drawing (GD 2001)*, volume 2265 of *Lecture Notes in Computer Science*, pages 328–342. Springer, 2002.
5. U. Fößmeier and M. Kaufmann. Nice drawings for planar bipartite graphs. In G. Bongiovanni, D. P. Bovet, and G. Di Battista, editors, *Proceedings of the 3rd Italian Conference on Algorithms and Complexity (CIAC '97)*, volume 1203 of *Lecture Notes in Computer Science*, pages 122–134. Springer, 1997.
6. F. Harary and A. Schwenk. A new crossing number for bipartite graphs. *Utilitas Mathematica*, 1:203–209, 1972.
7. M. Jünger, S. Leipert, and P. Mutzel. Level planarity testing in linear time. In S. H. Whitesides, editor, *Proceedings of the 6th International Symposium on Graph Drawing (GD '98)*, volume 1547 of *Lecture Notes in Computer Science*, pages 224–237. Springer, 1998.
8. T. Schank. Algorithmen zur Visualisierung planarer partitionierter Graphen. Master's thesis, Universität Konstanz, 2001. `http://www.inf.uni-konstanz.de/algo/lehre/theses/`.