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Consensus of Information Under Dynamically Changing Interaction Topologies

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***Abstract*— This paper considers the problem of information consensus among multiple agents in the presence of limited and unreliable information exchange with dynamically changing interaction topologies. Both discrete and continuous update schemes are proposed for consensus of information. That the union of a collection of interaction graphs across some time intervals has a spanning tree frequently enough as the system evolves is shown to be a necessary and sufficient condition for information consensus under dynamically changing interaction topologies. Simulation results show the effectiveness of our results.**

I. Introduction

The study of information flow and interaction among multiple agents in a group plays an important role in understanding the coordinated movements of these agents. Research efforts toward this direction are reported in [1], [2], [3], [4], [5], to name a few. Some applications of coordinated control require information to be shared among multiple agents in a group (c.f. [6], [4], [7], [8], [9], [10]), which in turn requires information consensus. In this paper, we extend some of the results of [4] to the case of directed graphs and show a necessary and sufficient condition for consensus of information under dynamically changing interaction topologies. Our work relies on two infrastructures including graph theory and nonnegative matrices.

Graph theory has been used to effectively model the interaction between agents (c.f. [11], [12], [1], [2], [3], [4], [13]). In [12], using graph theory, a team of nonholonomic mobile robots is controlled to navigate in a terrain with obstacles while maintaining a desired formation and changing formations when required. Ref. [1] studies information exchange techniques to improve stability margins and formation performance for vehicle formations. In [3], formation control graphs are used to analyze the input-to-state stability for leader-follower formations.

Nonnegative matrices (c.f. [14], [15]) have been studied extensively in the mathematics community. The well-known Perron-Frobenius theory for nonnegative matrices provides a useful tool in analyzing the properties of the graph Laplacian (c.f. [16]). The classical result in [17] demonstrates the property of the infinite products of certain categories of nonnegative matrices, which is proved to be useful in studying certain switched linear systems (c.f. [4]).

In this paper, directed graphs will be used to represent the interaction (information exchange) topology among multiple agents, where information can be exchanged between agents via communication or sensing. A preliminary result for information consensus is addressed in [10], where a linear update scheme is proposed but no complete answers are given for the issue of whether the linear update scheme achieves global consensus asymptotically when accounting for all known communication links. In [18], we provide a complete answer to the above issue under a time-invariant communication topology and propose strategies for consensus and evolution of the group level information in a distributed multi-vehicle coordinated control context.

In real applications, the interaction topology between agents may change dynamically. For example, in the case of interaction via communications, the communication links between vehicles may be unreliable due to disturbances and/or subject to communication range limit. In the case of interaction via sensing, the agents that can be sensed by a certain agent may change over time. In the ground breaking work by Jadbabaie et al. [4], a theoretical explanation is provided for the observed behavior of Vicsek model [6]. Ref. [4] explicitly takes into account possible changes of each agent's nearest neighbors over time, which can be thought of as an example of information consensus under dynamically changing interaction topologies. Ref. [4] proved that information (the heading of each agent in their context) can reach consensus provided that the union of a collection of graphs for all agents is connected frequently enough as the system evolves. However, the approaches in [4] are based on undirected graphs, which assume bidirectional information exchanges. It might be the case that information exchange is unidirectional, that is, consensus may need to be achieved in the presence of limited information exchanges, which will make information consensus more challenging. Also, certain constraints are set for the weighting factors in the update schemes in [4], which may be extended to more general cases. For example, it may be desirable to weigh the information from different agents differently to represent the relative confidence of each agent's information or relative reliabilities of different communication or sensing links between agents.

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The main purpose of this paper is to extend the work of Jadbabaie et al. [4] to the case of directed graphs and explore the minimum requirements to reach global consensus. As a comparison, Ref. [5] solves the *average-consensus* problems with directed graphs, which requires the graph to be strongly connected and balanced. We show that under certain assumptions consensus[1] can be achieved globally asymptotically under dynamically changing interaction topologies if and only if the union of a collection of graphs across some time intervals has a spanning tree frequently enough as the system evolves. Having a spanning tree for a union graph is a much milder condition than being connected and is therefore more suitable for practical applications. We also allow the weighting factors in our discrete and continuous update schemes to be dynamically changing, which provides additional flexibility. As a result, the convergence conditions and update schemes in [4] are proved to be a special case of a more general result. Also a byproduct of this paper is that we prove that a nonnegative matrix with the same positive row sums has its spectral radius (its row sum in this case) as a simple eigenvalue if and only if the directed graph of this matrix has a spanning tree while Perron-Frobenius theorem for nonnegative matrices only deals with irreducible matrices, that is, matrices with strongly connected graphs. Besides having a spanning tree, if this matrix also has positive diagonal entries, we show that its row sum is the unique eigenvalue of maximum modulus.

The remainder of the paper is organized as follows. In Section II, we associate directed graphs with dynamically changing interaction topologies and propose discrete and continuous update schemes accounting for dynamically changing interaction topologies and weighting factors. In Section III, we prove necessary and sufficient conditions for consensus of information under dynamically changing interaction topologies using both the discrete update scheme and continuous update scheme. Simulation results are presented in Section IV and Section V offers our conclusion.

II. Problem Statement

In this paper, we let $\mathcal{A} = \{A_i | i \in \mathcal{I}\}$ be a set of n agents in the group whose information needs to reach consensus, where $\mathcal{I} = \{1, 2, \cdots, n\}$. A directed graph $\mathcal{G}$ will be used to model the interaction topology among these agents. In $\mathcal{G}$, the ith vertex represents the ith agent A_i and a directed arc from A_i to A_j denoted as (A_i, A_j) represents a unidirectional information exchange link from A_i to A_j, that is, agent j can receive or obtain information from agent i, $(i, j) \in \mathcal{I}$. Throughout the paper, we always assume that there is a link from one vertex to itself. With regard to the fact that the interaction topology among these agents may be dynamically changing, we use $\bar{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_M\}$ to denote the set of all possible simple interaction graphs defined for these n agents. In actual applications, the possible interaction topologies may only be a subset of $\bar{\mathcal{G}}$. It is obvious to see that set $\bar{\mathcal{G}}$ has finite elements. The union of a group of simple graphs $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \cdots, \mathcal{G}_{i_m}\} \subset \bar{\mathcal{G}}$ is a simple graph with vertices given by A_i, $i \in \mathcal{I}$ and arc set given by the union of the arc sets of $\mathcal{G}_{i_j}$, $j = 1, \cdots, m$.

A directed path in graph $\mathcal{G}$ is a sequence of arcs $(A_{i_1}, A_{i_2}), (A_{i_2}, A_{i_3}), (A_{i_3}, A_{i_4}), \cdots$ in that graph. Graph $\mathcal{G}$ is called strongly connected if there is a directed path from A_i to A_j and A_j to A_i between any pair of distinct vertices A_i and A_j, $\forall (i, j) \in \mathcal{I}$. A directed tree is a directed graph, where every node, except the root, has exactly one parent. A spanning tree of a directed graph is a tree formed by graph arcs that connect all the vertices of the graph (c.f. [16]). Let $M_n(I\!R)$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(I\!R)$, the directed graph of A, denoted by $\Gamma(A)$, is the directed graph on n vertices V_i, $i \in \mathcal{I}$, such that there is a directed arc in $\Gamma(A)$ from V_j to V_i if and only if $a_{ij} \neq 0$ (c.f. [19]).

Let $\xi_i \in I\!R$, $i \in \mathcal{I}$, represent the ith information variable associated with the ith agent. The set of agents $\mathcal{A}$ is said to achieve global consensus asymptotically if for any $\xi_i(0)$, $i \in \mathcal{I}$, $\|\xi_i(t) - \xi_j(t)\| \to 0$ as $t \to \infty$ for each $(i, j) \in \mathcal{I}$ [10].

Given T as the sampling period, a discrete time consensus scheme is given by

$$\xi_i(k+1) = \frac{1}{\sum_{j=1}^n \alpha_{ij}(k) G_{ij}(k)} \sum_{j=1}^n \alpha_{ij}(k) G_{ij}(k) \xi_j(k), \tag{1}$$

where $k \in \{0, 1, 2, \cdots\}$ is the discrete time index, $(i, j) \in \mathcal{I}$, $\alpha_{ij}(k) > 0$ is a weighting factor, $G_{ii}(k) \stackrel{\triangle}{=} 1$, and $G_{ij}(k)$, $\forall j \neq i$, is 1 if there is an information exchange link from A_j to A_i at time $t = kT$ and 0 otherwise.

Eq. (1) can be written in matrix form as

$$\xi(k+1) = D(k)\xi(k), \tag{2}$$

where $\xi = [\xi_1, \cdots, \xi_n]^T$, $D = [d_{ij}]$, $(i, j) \in \mathcal{I}$, with $d_{ij} = \frac{\alpha_{ij}(k) G_{ij}(k)}{\sum_{j=1}^n \alpha_{ij}(k) G_{ij}(k)}$.

We will consider the continuous consensus scheme given by

$$\dot{\xi}_i(t) = -\sum_{j=1}^N \sigma_{ij}(t) G_{ij}(t)(\xi_i(t) - \xi_j(t)), \tag{3}$$

where $(i, j) \in \mathcal{I}$, $\sigma_{ij}(t) > 0$ is a weighting factor, $G_{ii}(t) \stackrel{\triangle}{=} 1$, and $G_{ij}(t)$, $\forall j \neq i$, is 1 if there is an information exchange link from A_j to A_i at time t and 0 otherwise.

Eq. (3) can also be written in matrix form as

$$\dot{\xi}(t) = C(t)\xi(t), \tag{4}$$

where $C = [c_{ij}]$, $(i, j) \in \mathcal{I}$, with $c_{ii} = -\sum_{j \neq i}(\sigma_{ij}(t) G_{ij}(t))$ and $c_{ij} = \sigma_{ij}(t) G_{ij}(t)$, $j \neq i$.

Note that the interaction topology $\mathcal{G}$ may be dynamically changing over time due to unreliable transmission or limited

[1] may not be average-consensus

communication/sensing range, which implies that $G_{ij}(k)$ in Eq. (1) and $G_{ij}(t)$ in Eq. (3) may be time-varying. We use $\mathcal{G}(k)$ and $\mathcal{G}(t)$ to denote the dynamically changing interaction topologies corresponding to Eq. (1) and Eq. (3) respectively. We also allow the weighting factors $\alpha_{ij}(k)$ in Eq. (1) and $\sigma_{ij}(t)$ in Eq. (3) to be dynamically changing to represent possibly time-varying relative confidence of each agent's information variable or relative reliabilities of different information exchange links between agents. As a result, both matrix $D(k)$ in Eq. (1) and matrix $C(t)$ in Eq. (3) are dynamically changing over time.

Compared to the models in [4], we do not set constraints for weighting factors $\alpha_{ij}(k)$ in Eq. (1) as long as they are positive, which provides more flexibility for some applications. The Vicsek model and simplified Vicsek model used in [4] can be thought of as special cases of our discrete time consensus scheme. If we let $\alpha_{ij}(k) \stackrel{\triangle}{=} 1$ in Eq. (1), we obtain the Vicsek model. Also the simplified Vicsek model can be acquired if we let $\alpha_{ij}(k) \stackrel{\triangle}{=} \frac{1}{g}$, $\forall j \neq i$, and $\alpha_{ii}(k) \stackrel{\triangle}{=} 1 - \sum_{j\neq i} \frac{1}{g} G_{ij}(k)$, where $g > n$ is a constant. Compared to [10], where the interaction graph is assumed to be time-invariant and weighting factors σ_{ij} are specified a priori to be constant and equal to each other, we study continuous time consensus scheme with dynamically changing interaction topoligies and weighting factors. The continuous update rule in [4] can also be regarded as a special case of our continuous update scheme by letting $\sigma_{ij} \stackrel{\triangle}{=} \frac{1}{n}$.

III. Consensus of Information Under Dynamically Changing Interaction Topologies

Let $\mathbf{1}$ denote an $n \times 1$ column vector with all the entries equal to 1. Also let I_n denote the $n \times n$ identity matrix. A matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ is nonnegative, denoted as $A \geq 0$, if all its entries are nonnegative. Furthermore, if all its row sums are $+1$, A is said to be a (row) stochastic matrix [19]. A stochastic matrix P is called indecomposable and aperiodic (SIA) if $\lim_{n\to\infty} P^n = \mathbf{1}y^T$, where y is some column vector [17]. For nonnegative matrices, $A \geq B$ implies that $A - B$ is a nonnegative matrix. It is easy to verify that if $A \geq \rho B$, $\forall \rho > 0$, and the directed graph of B has a spanning tree, then the directed graph of A has a spanning tree.

We need the following two lemmas. The first lemma is from [4] and the second lemma is originally from [17] and restated in [4].

Lemma 3.1: [4] Let $m \geq 2$ be a positive integer and let $P_1, P_2, \cdots, P_m$ be nonnegative $n \times n$ matrices with positive diagonal elements, then

$$P_1 P_2 \cdots P_m \geq \gamma (P_1 + P_2 + \cdots + P_m),$$

where $\gamma > 0$ can be specified from matrices P_i, $i = 1, \cdots, m$.

Lemma 3.2: [17] Let $S_1, S_2, \cdots, S_k$ be a finite set of SIA matrices with the property that for each sequence $S_{i_1}, S_{i_2}, \cdots, S_{i_j}$ of positive length, the matrix product $S_{i_j} S_{i_{j-1}} \cdots S_{i_1}$ is SIA. Then for each infinite sequence $S_{i_1}, S_{i_2}, \cdots$ there exists a column vector y such that

$$\lim_{j\to\infty} S_{i_j} S_{i_{j-1}} \cdots S_{i_1} = \mathbf{1}y^T.$$

We also need the following propositions and lemmas for our main results.

Proposition 3.1: Given matrix $A = [a_{ij}] \in M_n(\mathbb{R})$, where $a_{ii} \leq 0$, $a_{ij} \geq 0$, $\forall i \neq j$, and $\sum_{j=1}^{n} a_{ij} = 0$, then A has at least one zero eigenvalue and all the other non-zero eigenvalues are on the open left half plane. Furthermore, A has exactly one zero eigenvalue if and only if the directed graph associated with A has a spanning tree.

Proof: see Corollary 1 in [18]. ■

Lemma 3.3: If a nonnegative matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ has the same positive constant row sums given by $\mu > 0$, then μ is an eigenvalue of A with an associated eigenvector $\mathbf{1}$ and $\rho(A) = \mu$, where $\rho(\cdot)$ denotes the spectral radius of a matrix. Also the eigenvalue μ of A has algebraic multiplicity 1 if and only if the graph associated with A has a spanning tree. Furthermore, if the graph associated with A has a spanning tree and $a_{ii} > 0$, then μ is the unique eigenvalue of maximum modulus.

Proof: The first part of the lemma directly follows the properties of nonnegative matrices (c.f. [19]).

For the second part of the lemma, we need to show both the sufficient part and necessary part.

(Sufficiency.) If the graph associated with A has a spanning tree, then the graph associated with $B = A - \mu I_n$ also has a spanning tree. We know that $\lambda_i(A) = \lambda_i(B) + \mu$, where $i = 1, \cdots, n$, and $\lambda_i(\cdot)$ represents the ith eigenvalue of a matrix. Noting that B satisfies the conditions in Proposition 3.1, we know that zero is an eigenvalue of B with algebraic multiplicity 1, which then implies that A has algebraic multiplicity 1 for its eigenvalue μ.

(Necessity.) If the graph associated with A does not have a spanning tree, we know that B has more than one zero eigenvalue from Proposition 3.1, which in turn implies that A has more than one eigenvalue equal to μ.

For the third part of the lemma, from Gersgorin disc theorem, all the eigenvalues of A are located in the union of n discs given by

$$\bigcup_{i=1}^{n} \{z \in \mathbf{C} : |z - a_{ii}| \leq \sum_{j\neq i} |a_{ij}|\}.$$

It is easy to see that this union is included in the circle given by $\{z \in \mathbf{C} : |z| \leq \mu\}$ with only one intersection at $z = \mu$. Thus we know that $|\lambda| < \mu$ for every eigenvalue of A satisfying $\lambda \neq \mu$. Combining the second part of the lemma, we know that μ is the unique eigenvalue of maximum modulus. ■

Corollary 3.2: A stochastic matrix has algebraic multiplicity 1 for its eigenvalue $\lambda = 1$ if and only if the graph associated with this matrix has a spanning tree. Furthermore, a stochastic matrix with positive diagonal elements has the property that $|\lambda| < 1$ for every eigenvalue satisfying $\lambda \neq 1$.

Proposition 3.3: If $A \in M_n$ and $A \geq 0$, then $\rho(A)$ is an eigenvalue of A and there is a nonnegative vector $x \geq 0$, $x \neq 0$, such that $Ax = \rho(A)x$.

Proof: see Theorem 8.3.1 in [19]. ∎

Lemma 3.4: Let $A = [a_{ij}] \in M_n(\mathbb{R})$ be a stochastic matrix. If A has eigenvalue 1 with algebraic multiplicity 1 and all the other eigenvalues satisfy $|\lambda| < 1$, then A is SIA, that is, $\lim_{m\to\infty} A^m \to \mathbf{1}\nu^T$, where ν satisfies $A^T\nu = \nu$ and $\mathbf{1}^T\nu = 1$. Furthermore, each element of ν is nonnegative.

Proof: The first part of the lemma follows Lemma 8.2.7 in [19]. For the second part, it is obvious that A^T is also nonnegative and has $\rho(A^T) = 1$ as an eigenvalue with algebraic multiplicity 1. Thus the eigenspace of A^T associated with eigenvalue 1 is given by cx, where $c \in \mathbf{C}$, $c \neq 0$, and x is a nonnegative eigenvector associated with eigenvalue 1 from Proposition 3.3. Since ν is also an eigenvector of A^T associated with eigenvalue 1 and satisfies $\mathbf{1}^T\nu = 1$, it can be verified that each element of ν must be nonnegative. ∎

A. Consensus Using Discrete Time Update Scheme

Before moving to the general case, we first propose a necessary and sufficient condition for consensus of information using discrete time update scheme (1) with a time-invariant interaction topology and constant weighting factors, that is, constant matrix D.

Theorem 3.4: With a time-invariant interaction topology and constant weighting factors, the discrete time update scheme (1) achieves global consensus asymptotically for $\mathcal{A}$ if and only if the associated interaction graph $\mathcal{G}$ has a spanning tree.

Proof: (Sufficiency.) To show that ξ_i can achieve global consensus asymptotically, it is equivalent to show that $D^k \to \mathbf{1}c^T$, where c is some column vector, which implies that $\xi_i(k) \to c^T\xi(0)$, $\forall i \in \mathcal{I}$, as $k \to \infty$.

Obviously D is a stochastic matrix with positive diagonal entries. The fact that graph $\mathcal{G}$ has a spanning tree also implies that the directed graph of D has a spanning tree. Combining Corollary 3.2 and Lemma 3.4, we know that $\lim_{k\to\infty} D^k \to \mathbf{1}\nu^T$, where ν satisfies the properties defined in Lemma 3.4.

(Necessity.) If $\mathcal{G}$ does not have a spanning tree, neither does the directed graph of D, which implies that the algebraic multiplicity of eigenvalue 1 of D is greater than 1. As a result, the rank of $\lim_{k\to\infty} D^k$ is greater than 1, which implies that $\mathcal{A}$ cannot reach global consensus asymptotically. ∎

Next, we will show that under certain conditions having a spanning tree is also a necessary and sufficient condition for consensus under dynamically changing interaction topologies using the discrete update scheme. We need the following lemma.

Lemma 3.5: If the union of a set of simple graphs $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \cdots, \mathcal{G}_{i_m}\} \subset \bar{\mathcal{G}}$ has a spanning tree, then the matrix product $D_{i_m} \cdots D_{i_2} D_{i_1}$ is SIA, where D_{i_j} is a matrix corresponding to each simple graph $\mathcal{G}_{i_j}$ in Eq. (2), $j = 1, \cdots, m$.

Proof: From Lemma 3.1, we know that $D_{i_m} \cdots D_{i_2} D_{i_1} \geq \gamma \sum_{j=1}^m D_{i_j}$ for some $\gamma > 0$.

Since the union of $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \cdots, \mathcal{G}_{i_m}\}$ has a spanning tree, we know that the directed graph of matrix $\sum_{j=1}^m D_{i_j}$ has spanning tree, which in turn implies that the directed graph of the matrix product $D_{i_m} \cdots D_{i_2} D_{i_1}$ has a spanning tree. Also the matrix product $D_{i_m} \cdots D_{i_2} D_{i_1}$ is a stochastic matrix with positive diagonal entries since stochastic matrices with positive diagonal entries are closed under matrix multiplication.

Combining Corollary 3.2 and Lemma 3.4, we know that the matrix product $D_{i_1} D_{i_2} \cdots D_{i_m}$ is SIA. ∎

The following theorem extends the discrete time convergence result in [4].

Theorem 3.5: Let $\mathcal{G}(k) \in \bar{\mathcal{G}}$ be a switching interaction graph at time $t = kT$. Also let $\alpha_{ij}(k) \in \bar{\alpha}$, where $\bar{\alpha}$ is a finite set of arbitrary positive numbers. The discrete update scheme (1) achieves global consensus asymptotically for $\mathcal{A}$ if and only there exists an infinite sequence of bounded, non-overlapping time intervals $[k_jT, (k_j + l_j)T)$, $j = 1, 2, \cdots$, starting at $k_1 = 0$, with the property that each interval $[(k_j + l_j)T, k_{j+1}T)$ is bounded and the union of the graphs across each such interval has a spanning tree.

Proof: Let $\bar{D}$ denote the set of all possible matrices $D(k)$ under dynamically changing interaction topologies and weighting factors. We know that $\bar{D}$ is a finite set since both set $\bar{\mathcal{G}}$ and set $\bar{\alpha}$ are finite.

(Sufficiency.) Consider the jth time interval $[k_jT, k_{j+1}T)$, which includes the time interval $[k_jT, (k_j + l_j)T)$ and must be bounded since both $[k_jT, (k_j + l_j)T)$ and $[(k_j + l_j)T, k_{j+1}T)$ are bounded. Also the sequence of time intervals $[k_jT, k_{j+1}T)$, $j = 1, 2, \cdots$, are contiguous.

The union of the graphs across $[k_jT, k_{j+1}T)$, denoted as $\bar{\mathcal{G}}(k_j)$, has a spanning tree since the union of the graphs across $[k_jT, (k_j + l_j)T)$ has a spanning tree. Let $\{D(k_j), D(k_j + 1), \cdots, D(k_{j+1} - 1)\}$ be a set of matrices corresponding to each graph in the union $\bar{\mathcal{G}}_{i_j}$. Following Lemma 3.5, the matrix product $D(k_{j+1} - 1) \cdots D(k_j + 1)D(k_j)$, $j = 1, 2, \cdots$, is SIA. Then by applying Lemma 3.2 and mimicking a similar proof for Theorem 2 in [4], the sufficient part can be proved.

(Necessity.) If the sufficient condition of this theorem is not satified, which implies that the union of the graphs does not have a spanning tree after some finite time $\hat{t}$. Therefore, during the infinite time interval $[\hat{t}, \infty)$, there exist at least two agents such that there is no path in the union of the graphs that contains these two agents, which then implies that information of these two agents cannot reach consensus. ■

B. Consensus Using Continuous Time Update Scheme

It has been shown in [18] that having a spanning tree is also a necessary and sufficient condition for consensus of information using continuous time update scheme (3) with a time-invariant interaction topology and constant weighting factors, that is, constant matrix C.

Here we will show that like the discrete time case, under certain conditions having a spanning tree is also a necessary and sufficient condition for consensus under dynamically changing interaction topologies using the continuous time update scheme. We need the following lemma.

Lemma 3.6: If the union of the simple graphs $\{\mathcal{G}_{t_1}, \mathcal{G}_{t_2}, \cdots, \mathcal{G}_{t_m}\} \subset \bar{\mathcal{G}}$ has a spanning tree, then the matrix product $e^{C_{t_m}\Delta t_m} \cdots e^{C_{t_2}\Delta t_2} e^{C_{t_1}\Delta t_1}$ is SIA, where $\Delta t_i > 0$ is bounded and C_{t_i} is a matrix corresponding to each simple graph $\mathcal{G}_{t_i}$ in Eq. (4), $i = 1, \cdots, m$.

Proof: From Eq. (4), each matrix C_{t_i} satisfies the properties defined in Proposition 3.1. Thus each C_{t_i} can be written as the sum of a nonnegative matrix M_{t_i} and $-\eta_{t_i} I_n$, where η_{t_i} is the maximum absolute value of the diagonal entries of C_{t_i}, $i = 1, \cdots, m$.

From Lemma 1 in [18], we know that $e^{C_{t_i}\Delta t_i} = e^{-\eta_{t_i}\Delta t_i} e^{M_{t_i}\Delta t_i} \geq \rho_i M_{t_i}$ for some $\rho_i > 0$. Since the union of the simple graphs $\{\mathcal{G}_{t_1}, \mathcal{G}_{t_2}, \cdots, \mathcal{G}_{t_m}\}$ has a spanning tree, we know that the union of the directed graphs of M_{t_i} has a spanning tree, which in turn implies that the union of the directed graphs of $e^{C_{t_i}\Delta t_i}$ has a spanning tree. From Lemma 3.1, we know that $e^{C_{t_m}\Delta t_m} \cdots e^{C_{t_2}\Delta t_2} e^{C_{t_1}\Delta t_1} \geq \gamma \sum_{i=1}^{m} e^{C_{t_i}\Delta t_i}$ for some $\gamma > 0$, which implies that the above matrix product also has a spanning tree.

It can also be verified that each matrix $e^{C_{t_i}\Delta t_i}$ is a stochastic matrix with positive diagonal entries, which implies that the above matrix product is also stochastic with positive diagonal entries.

Combining Corollary 3.2 and Lemma 3.4, we know that the above matrix product is SIA. ■

In this paper, we also apply dwell time (c.f. [20], [4]) to the continuous time update scheme (4), which implies that the interaction graph and weighting factors are constrained to change only at discrete times, that is, matrix $C(t)$ is piecewise constant in this case.

Eq. (4) can be rewritten as

$$\dot{\xi}(t) = C(t_i)\xi(t), \quad t \in [t_i, t_i + \tau_i) \tag{5}$$

where t_0 is the initial time and $t_1, t_2, \cdots$ is an infinite time sequence at which the interaction graph or weighting factors change, that is, matrix $C(t)$ changes.

Let $\tau_i = t_{i+1} - t_i$ be the dwell time, $i = 0, 1, \cdots$. Note that the solution to Eq. (5) is given by $\xi(t) = e^{C(t_k)(t-t_k)} e^{C(t_{k-1})\tau_{k-1}} \cdots e^{C(t_1)\tau_1} e^{C(t_0)\tau_0} \xi(0)$, where k is the largest nonnegative integer satisfying $t_k \leq t$. Let $\bar{\tau}$ be a finite set of arbitrary positive numbers. Let Υ be an infinite set generated from set $\bar{\tau}$, which is closed under additions of its elements and multiplications by positive integers. We assume that $\tau_i \in \Upsilon$, $i = 0, 1, \cdots$. By choosing set $\bar{\tau}$ properly, dwell time can be chosen from an infinite set Υ, which somewhat simulates the case when the interaction graph $\mathcal{G}$ changes dynamically over time.

The following theorem extends the continuous time convergence result in [4].

Theorem 3.6: Let $t_1, t_2, \cdots$ be an infinite time sequence at which the interaction graph or weighting factors switch and $\tau_i = t_{i+1} - t_i \in \Upsilon$, $i = 0, 1, \cdots$. Let $\mathcal{G}(t_i) \in \bar{\mathcal{G}}$ be a switching interaction graph at time $t = t_i$ and $\sigma_{ij}(t_i) \in \bar{\sigma}$, where $\bar{\sigma}$ is a finite set of arbitrary positive numbers. The continuous time update scheme (3) achieves global consensus asymptotically for $\mathcal{A}$ if and only there exists an infinite sequence of bounded, non-overlapping time intervals $[t_{i_j}, t_{i_j+l_j})$, $j = 1, 2, \cdots$, starting at $t_{i_1} = t_0$, with the property that each interval $[t_{i_j+l_j}, t_{i_{j+1}})$ is bounded and the union of the graphs across each such interval has a spanning tree.

Proof: The set of all possible matrices $e^{C(t_i)\tau_i}$, where $\tau_i \in \Upsilon$, under dynamically changing interaction topologies and weighting factors can be chosen or constructed by matrix multiplications from a matrix set $\bar{E} = \{e^{C(t_i)\tau_i}, \tau_i \in \bar{\tau}\}$. It is easy to see that set $\bar{E}$ is finite since set $\bar{\mathcal{G}}$, $\bar{\sigma}$, and $\bar{\tau}$ are all finite.

(Sufficiency.) Consider the jth time interval $[t_{i_j}, t_{i_{j+1}})$, which includes the time interval $[t_{i_j}, t_{i_j+l_j})$ and must be bounded since both $[t_{i_j}, t_{i_j+l_j})$ and $[t_{i_j+l_j}, t_{i_{j+1}})$ are bounded. Also the sequence of time intervals $[t_{i_j}, t_{i_{j+1}})$, $j = 1, 2, \cdots$, are contiguous.

The union of the graphs across $[t_{i_j}, t_{i_{j+1}})$, denoted as $\bar{\mathcal{G}}(t_{i_j})$, has a spanning tree since the union of graphs across $[t_{i_j}, t_{i_j+l_j})$ has a spanning tree. Let $\{C(t_{i_j}), C(t_{i_j+1}), \cdots, C(t_{i_{j+1}-1})\}$ be a set of matrices corresponding to each graph in the union $\bar{\mathcal{G}}(t_{i_j})$. Following Lemma 3.6, the matrix product $e^{C(t_{i_{j+1}-1})\tau_{i_{j+1}-1}} \cdots e^{C(t_{i_j+1})\tau_{i_j+1}} e^{C(t_{i_j})\tau_{i_j}}$, $j = 1, 2, \cdots$, is SIA. Then by applying Lemma 3.2 and mimicking a similar proof for Theorem 2 in [4], the sufficient part can be proved.

(Necessity.) The necessary part is similar to that in Theorem 3.5. ■

C. Discussions

Compared to the results in [4], which are based on undirected graphs, our results are based on more general directed graphs. Therefore, unidirectional information exchange is allowed instead of requiring bidirectional information exchange all the time, which is more proper for real applications since bidirectional interaction may not always be guaranteed or available due to unreliable or limited information exchange.

Ref. [4] shows that consensus of information (the heading of each agent in their context) can be achieved if the union of a collection of graphs is connected frequently enough. Here we show that the same result can be achieved as long as the union of the graphs has a spanning tree, which is a much milder requirement than being connected and implies that one half of the information exchange links required in [4] can be lost without adversely affecting the convergence result. In this sense, the results for convergence in [4] can be thought of as a special case of a more general result. Of course, the final achieved equilibrium points will depend on the property of the directed graphs. For example, compared to strongly connected graphs, graphs that are not strongly connected will reach different final equilibrium points (see [18] for an analysis of the final equilibrium points).

The leader following scenario in [4] can also be thought of as a special case of a more general result in the following sense. If there is one agent in the group which does not have any incoming link all the time but the union of the graphs across some time intervals has a spanning tree, then this agent must be the root of the spanning tree, that is, the leader in this case. If this happens frequently enough, then all the other agents must reach consensus to this agent from the general result. Therefore, being linked to the leader frequently enough shown in [4] is just a special case of having a spanning tree with the leader as the root frequently enough.

For the continuous model used in [4], the switching times of the interaction graph is constrained to be separated by τ_D time units, where τ_D is a constant dwell time. Our continuous update scheme allows the switching times to be within an infinite set of positive numbers generated by any finite set of positive numbers, which can achieve a better result for simulating the random switching of interaction graphs. As a result, the continuous scheme in [4] can be thought of a special case of a general result by letting $\bar{\tau} = \{\tau_d\}$ and $\Upsilon = \{k\tau_d | k = 1, 2, \cdots\}$.

Unlike the update schemes in [4], there are no constraints for weighting factors in our discrete and continuous update schemes as long as they are positive, which provides more flexibility to take into account the relative confidence and relative reliability for information from different agents. It has been shown that each update scheme in [4] can be thought of as a special case of our discrete time or continuous time update scheme with certain constraints on the weighting factors.

We also add some new results to the properties of a special class of nonnegative matrices, that is, nonnegative matrices with the same positive row sums. Perron-Frobenius theorem says that if a nonnegative matrix A is irreducible, that is, the directed graph of A is strongly connected, then the spectral radius of A is a simple eigenvalue. We show that the above condition of being irreducible is too stringent for nonnegative matrices with the same positive row sums. We prove that for a nonnegative matrix A with the same positive row sums, the spectral radius of A (the row sum in this case) is a simple eigenvalue if and only if the directed graph of A has a spanning tree, that is, A may be reducible but still has its spectral radius as a simple eigenvalue. Furthermore, if A has a spanning tree and positive diagonal entries, we know that the spectral radius of A is the unique eigenvalue of maximum modulus.

IV. Simulation Results

In this section, we simulate a case of information consensus for five agents under dynamically changing interaction topologies using the discrete time update scheme (2) and the continuous time update scheme (5) respectively.

For simplicity, we constrain the possible interaction graphs for these five agents to be within the set $\mathcal{G}_s = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5\}$ as shown in Fig. 1, which is obviously a subset of $\bar{\mathcal{G}}$. For discrete time update scheme, we assume that the interaction graph switches randomly from $\mathcal{G}_s$ at each time $t = kT$, where $k = 0, 1, 2, \cdots$ and T is 0.5 seconds. For continuous update scheme, we assume that the interaction graph switches randomly from $\mathcal{G}_s$ at each random time $t = t_k$, $k = 0, 1, 2, \cdots$. The weighting factors in Eqs. (2) and (5) are chosen arbitrarily as long as $\alpha_{ij}(k) > 0$ and $\sigma_{ij}(t_k) > 0$, $(i, j) \in \mathcal{I}$ and $k = 0, 1, 2, \cdots$.

Note that each simple graph in $\mathcal{G}_s$ does not have a spanning tree but the union of these graphs do have a spanning tree as shown in Fig. 2, which satisfies the necessary and sufficient condition for consensus. It is obvious to see that the union of these graphs is by no means connected, which implies that the conditions in [4] are not satisfied. However, we will show next that these five agents can achieve global consensus asymptotically using both the discrete time update scheme and the continuous time update scheme.

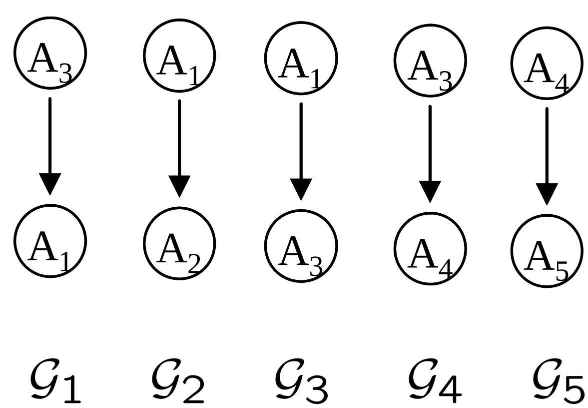

Fig. 1. Possible interaction topologies for $\mathcal{A} = \{A_i | i = 1, \cdots, 5\}$.

We arbitrarily choose initial conditions for $\xi_i(t)$ as $\xi_i = 0.2 * i$, $i = 1, \cdots, 5$. Fig. 3 shows the consensus results using both the discrete time update scheme and the continuous time update scheme. Note that $\xi_i(t)$, $i = 1, \cdots, 5$, reaches

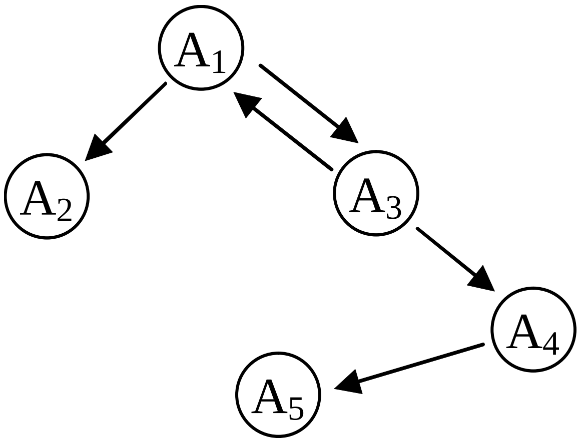

Fig. 2. The union of $\mathcal{G}_s$.

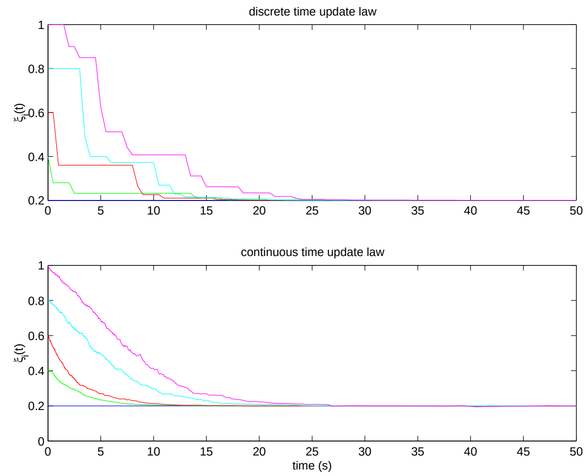

Fig. 4. Consensus with $\mathcal{G}(k)$ and $\mathcal{G}(t_k)$ randomly switching from $\mathcal{G}_s'$.

consensus for both cases even under randomly switching interaction topologies from $\mathcal{G}_s$. We then assume that there exists no information exchange link in $\mathcal{G}_1$, that is, each agent is isolated in $\mathcal{G}_1$. We denote this new set of simple graphs as $\mathcal{G}_s'$. As a result, there is no information exchange link from A_3 to A_1 in the union of $\mathcal{G}_i$, $i = 1, \cdots, 5$. In this case, the union of these simple graphs still has a spanning tree. However, unlike the previous case shown in Fig. 2, there is no incoming information exchange link to A_1. Fig. 4 shows the consensus results using both the discrete time update scheme and the continuous time update scheme. Note that $\xi_i(t)$, $i = 1, \cdots, 5$, achieves consensus to $\xi_1(0)$ in both cases. This is similar to the leader following case in [4] except that we do not need the followers to be jointly linked to the leader, that is, the union of the simple graphs is not necessarily connected.

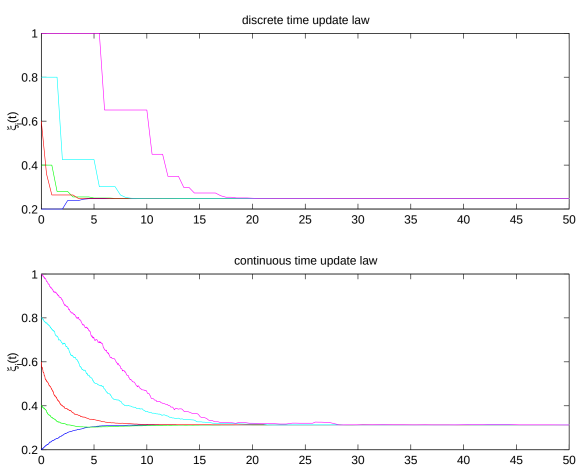

Fig. 3. Consensus with $\mathcal{G}(k)$ and $\mathcal{G}(t_k)$ randomly switching from $\mathcal{G}_s$.

V. CONCLUSION

This paper has considered the problem of information consensus under dynamically changing interaction topologies and weighting factors. We have applied directed graphs to represent information exchanges among multiple agents, which takes into account the general case when there exist only unidirectional links between agents. We also proposed discrete and continuous update schemes for information consensus and gave necessary and sufficient conditions for information consensus under dynamically changing interaction topologies and weighting factors using these update schemes. Simulation examples were presented to illustrate our results.

Acknowledgments

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