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Dynamic allocation indices for restless projects and queueing admission control: a polyhedral approach*

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Abstract. This paper develops a polyhedral approach to the design, analysis, and computation of dynamic allocation indices for scheduling binary-action (engage/rest) Markovian stochastic projects which can change state when rested (*restless bandits (RBs)*), based on *partial conservation laws (PCLs)*. This extends previous work by the author [J. Niño-Mora (2001): Restless bandits, partial conservation laws and indexability. *Adv. Appl. Probab.* **33**, 76–98], where PCLs were shown to imply the optimality of index policies *with a postulated structure* in stochastic scheduling problems, under *admissible linear objectives*, and they were deployed to obtain simple sufficient conditions for the existence of Whittle's (1988) RB index (*indexability*), along with an adaptive-greedy index algorithm. The new contributions include: (i) we develop the polyhedral foundation of the PCL framework, based on the structural and algorithmic properties of a new polytope associated with an *accessible set system* $(J, \mathcal{F})$ ($\mathcal{F}$*-extended polymatroid*); (ii) we present new dynamic allocation indices for RBs, motivated by an admission control model, which extend Whittle's and have a significantly increased scope; (iii) we deploy PCLs to obtain both sufficient conditions for the existence of the new indices (*PCL-indexability*), and a new adaptive-greedy index algorithm; (iv) we interpret PCL-indexability as a form of the classic economics law of *diminishing marginal returns*, and characterize the index as an *optimal marginal cost rate*; we further solve a related optimal *constrained* control problem; (v) we carry out a PCL-indexability analysis of the motivating admission control model, under time-discounted and long-run average criteria; this gives, under mild conditions, a new index characterization of optimal threshold policies; and (vi) we apply the latter to present new heuristic index policies for two hard queueing control problems: admission control and routing to parallel queues; and scheduling a multiclass make-to-stock queue with lost sales, both under state-dependent holding cost rates and birth-death dynamics.

Key words. Markov decision process – restless bandits – polyhedral combinatorics – extended polymatroid – adaptive-greedy algorithm – dynamic allocation index – stochastic scheduling – threshold policy – index policy – Gittins index – Klimov index – Whittle index – control of queues – admission control – routing – make-to-stock – multiclass queue – finite buffers – conservation laws – achievable performance

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1. Introduction

This paper develops a polyhedral approach to the design, analysis, and computation of dynamic allocation indices for scheduling binary-action (engage/rest) Markovian stochastic projects which can change state when rested, or *restless bandits (RBs)*. The work draws on and contributes to three research areas which have evolved with substantial autonomy: (1) index policies in stochastic scheduling; (2) monotone optimal policies in *Markov decision processes (MDPs)*; and (3) polyhedral methods in resource allocation problems. We next briefly discuss each area's relevant background.

Index policies in stochastic scheduling

Stochastic scheduling (cf. [27]) is concerned with the dynamic resource allocation to competing, randomly evolving activities. An important model class concerns the design of a *scheduling policy* for optimal dynamic *effort* allocation to a collection of Markovian *stochastic projects*, which can be either *engaged* or *rested*. Following Whittle [37], we shall call such projects *restless bandits (RBs)*, and refer to a corresponding multi-project model as a *restless bandit problem (RBP)*. The term "project" is used here in a lax sense, befitting the application at hand. Thus, a project may represent, e.g., a *queue* subject to admission control, whose evolution depends on the *policy* adopted to decide whether each arriving customer should be admitted into or rejected from the system.

Index policies are particularly appealing for such problems: an index $\nu_k(j_k)$ is attached to the *states* j_k of each project k; then, the required number of projects with larger indices are engaged at each time. The quest for models with optimal index policies drew major research efforts in the 1960s and 1970s, yielding a classic body of work. This includes the celebrated *$c\mu$-rule* [8] for scheduling a multiclass $M/G/1$ queue, *Klimov's index rule* [21] for the corresponding model with feedback, and the *Gittins index rule* [14, 15] for the *multiarmed bandit problem (MBP)*.

The MBP is a paradigm among such well-solved models, yielding unifying insights. In it, rested projects do not change state, one project is engaged at each time, and a discounted criterion is employed. The optimal *Gittins index* $\nu_k(j_k)$ has an insightful interpretation. It was introduced in [14] via a *single-project subproblem*, where at each time one can continue or abandon operation, earning in the latter case a pension at constant rate ν; $\nu_k(j_k)$ is then the *fair passivity subsidy* in state j_k, i.e., the minimum value of ν one should be willing to accept to rest project k. Gittins [15] further characterized his index as the maximal rate of expected discounted reward per unit of expected discounted time, or *maximal reward rate*, starting at each state.

As for the general RBP, its increased modeling power comes at the expense of tractability: it is *P-SPACE HARD* [28]. The research focus must hence shift to the design of well-grounded, tractable *heuristic index policies*. Whittle [37] first proposed such a policy, which recovers Gittins' in the MBP case, and enjoys a form of asymptotic optimality. See [36]. The *Whittle index* is also defined as a *fair passivity subsidy* via a single-project subproblem, precisely as outlined above for the Gittins index. The *Whittle index policy* prescribes to assign higher priority to projects with larger indices.

In contrast to the Gittins index, passive transitions cause the Whittle index to have a *limited scope*: it is defined only for an *indexable* project, whose *optimal active set* (states where it should be engaged) decreases as the passive subsidy grows. The lack of simple sufficient conditions for *indexability* hindered the application of such index in the 1990s. An alternative index, free from such scope limitation, was proposed in [3]. Regarding indexability, we first presented in [26] a tractable set of sufficient conditions, based on the notion of *partial conservation laws*, along with a one-pass index algorithm.

Monotone optimal policies in MDPs

In MDP applications intuition often leads to *postulate* qualitative properties on optimal policies. The optimal action, e.g., may be *monotone* on the state. Thus, in a model for control of admission to a queue, one might postulate that arriving customers should be accepted iff the queue length exceeds a critical *threshold*. Establishing the optimality of such policies can lead to efficient special algorithms.

The most developed approach for such purpose is grounded on the theory of submodular functions on lattices. See [32]. One must establish *submodularity* properties on the problem's value function, exploiting the *dynamic programming (DP)* equations, by induction on the *finite* horizon. Infinite-horizon models inherit such properties. See, e.g., [17, Ch. 8] and [31, 1].

Related yet distinct qualitative properties are suggested by the *indexability* analysis of RB models, given in terms of the *monotonicity of the index on the state*. Consider a queueing admission control RB model, where the active action corresponds to *shutting* the entry gate, and the passive action to *opening* it. The Whittle index would then represent a fair subsidy for keeping the gate open *per unit time*; or, equivalently, a fair charge for keeping the gate shut per unit time, in each state. To be consistent with the optimality of *threshold policies* (see above), the index should increase monotonically on the queue length. Such requirement is critical when the index is used to define a heuristic policy for related RBPs, such as those discussed in Section 8.

Yet we have found that, when such model incorporates state-dependent arrival rates, the Whittle index can fail to possess the required monotonicity. See Appendix C. Such considerations motivate us in this paper to develop extensions of the Whittle index which are consistent with a postulated structure on optimal policies.

Polyhedral methods in resource allocation problems

The application of polyhedral methods to resource allocation originated in *combinatorial optimization*, within the area of *polyhedral combinatorics*. See, e.g., [25]. Edmonds

[11, 12] first explained the optimality of the classic *greedy algorithm*—the simplest *index rule* for resource allocation—from properties of underlying polyhedra, termed *polymatroids*, arising in the problem's *linear programming (LP)* formulation.

The application of LP to MDPs started with the LP formulation of a general finite-state and -action MDP in [10, 23]. The seminal application of LP to stochastic scheduling is due to Klimov [21]. He formulated the problem of optimal scheduling of a multiclass $M/G/1$ queue with feedback as an LP, whose constraints represent *flow conservation laws*. He solved such LP by an *adaptive-greedy algorithm*, giving an optimal index rule.

Coffman and Mitrani [7] formulated a simpler model—*without* feedback—as an LP, whose constraints formulate *work conservation laws*. These characterize the *region of achievable (expected delay) performance* as a *polymatroid*, thus giving a polyhedral account for the optimality of the classic *$c\mu$ rule*. The relation between conservation laws and polymatroids was clarified in [13, 30].

Tsoucas [33] applied work conservation laws to Klimov's model, obtaining a new LP formulation over an *extended polymatroid* (cf. [4]). His analysis was extended into the *generalized conservation laws (GCLs)* framework in [2], giving a polyhedral account of the optimality of Gittins' index rule for the MBP and extensions. *Approximate GCLs* were deployed in [16] to establish the near-optimality of Klimov's rule in the parallel-server case. See [9] for an overview of such *achievable region approach*.

The theory of conservation laws was extended in [26], through the notion of *partial conservation laws (PCLs)*, which were brought to bear on the analysis of Whittle's RB index. PCLs imply the optimality of index policies *with a postulated structure* under *admissible linear objectives*. Their application yielded the class of *PCL-indexable* RBs, where the Whittle index exists and is calculated by an extension of Klimov's algorithm.

Goals, contributions, and structure

The prime goal of this paper is the development, analysis, and application of well-grounded extensions of Whittle's RB index, which significantly increase its scope. For such purpose, we shall deepen the understanding of the PCL framework and its polyhedral foundation, which is the paper's second goal.

The contributions include: (i) we develop the polyhedral foundation of the PCLs, based on properties of a new polytope associated with a *set system* $(J, \mathcal{F})$ (*$\mathcal{F}$-extended polymatroid*); (ii) we present new dynamic allocation indices for RBs, motivated by an admission control model, which extend Whittle's and have a significantly increased scope; (iii) we deploy PCLs to obtain both sufficient conditions for the existence of the new indices (*PCL-indexability*), and a new adaptive-greedy index algorithm; (iv) we interpret PCL-indexability as a form of the classic economics law of *diminishing marginal returns*, and characterize the index as an *optimal marginal cost rate*; we further solve a related optimal *constrained* control problem; (v) we carry out a PCL-indexability analysis of the motivating admission control model, under time-discounted and time-average criteria; this gives, under mild conditions, a new index characterization of optimal threshold policies; and (vi) we apply the latter to present new heuristic index policies for two hard queueing control problems: admission control and routing to parallel queues; and scheduling a multiclass make-to-stock queue with lost sales.

The rest of the paper is organized as follows. Section 2 describes the motivating admission control model, and introduces the new solution approach. Section 3 describes a general RB model, introduces new indices, and formulates the issues to be resolved. Section 4 introduces $\mathcal{F}$-extended polymatroids, and studies their properties. Section 5 reviews the PCL framework. Section 6 applies PCLs to the analysis of RBs, yielding sufficient indexability conditions and an index algorithm. Section 7 deploys such results in the admission control model. Section 8 applies the new indices to present new policies for two complex queueing control models. Section 9 ends the paper with some concluding remarks. Three appendices contain important yet ancillary material.

2. Motivating problem: optimal control of admission to a birth-death queue

This section discusses a model for the optimal control of admission to a birth-death queue, a fundamental problem which has drawn extensive research attention. See [24, 31, 1, 19, 6]. We shall use the model to motivate our approach, by introducing a novel analysis grounded on an intuitive index characterization of optimal *threshold* policies.

2.1. Model description

Consider the system portrayed in Figure 2.1, which represents a single-server facility catering to an incoming customer stream, endowed with a finite buffer capable of holding n customers, waiting or in service. Customer flow is regulated by a *gatekeeper*, who dynamically opens or shuts an *entry gate* which customers must cross to enter the buffer; those finding a shut gate, or a full buffer, on arrival are rejected and lost.

The *state* $L(t)$, recording the number in system at times $t \geq 0$, evolves as a controlled *birth-death process* over *state space* $N = \{0, \ldots, n\}$. While in state i, customers arrive at rate λ_i (being then admitted or rejected), and the server works at rate μ_i.

We assume that *holding costs* are continuously incurred in state i at rate h_i, and a *charge* ν is incurred *per customer rejection*. Costs are discounted at rate $\alpha > 0$.

The system is governed by an *admission control policy* u, prescribing the action $a(t) \in \{0, 1\}$ to take at each time t. Policies are chosen from the class $\mathcal{U}$ of *stationary* policies, basing action choice on the state. Given policy u and state j, we denote by $u(j) \in [0, 1]$ the *probability* of taking action $a = 1$ (*shut* the entry gate), so that $1 - u(j)$ is the probability of action $a = 0$ (*open* it). We shall refer to $a = 1$ as the *active* action, and to $a = 0$ as the *passive* action; one may imagine that the gate is naturally open, unless the gatekeeper intervenes to shut it. We shall adopt the convention that $u(n) \equiv 1$, so that action choice is effectively limited to the set $N^{\{0,1\}} = \{0, \ldots, n-1\}$ of *controllable states*. The single state in $N^{\{1\}} = \{n\}$ will be termed *uncontrollable*.

Denote by $E_i^u[\cdot]$ the expectation under policy u when starting at i. Let

$$v_i^u = E_i^u \left[\int_0^\infty h_{L(t)} \, e^{-\alpha t} \, dt \right] \tag{2.1}$$

be the corresponding expected total discounted value of holding costs incurred, and let

$$b_i^u = E_i^u \left[\int_0^\infty \lambda_{L(t)} \, a(t) \, e^{-\alpha t} \, dt \right] \tag{2.2}$$

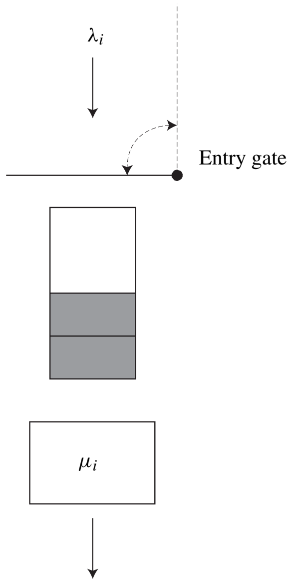

Fig. 2.1. Control of admission to a single queue

be the expected total discounted number of customer rejections. The *cost objective* is

$$v_i^u(\nu) = v_i^u + \nu\, b_i^u.$$

The *admission control problem* is to find a policy minimizing such objective:

$$v_i(\nu) = \min \left\{ v_i^u(\nu) : u \in \mathcal{U} \right\}. \tag{2.3}$$

We shall refer to (2.3) as the *ν-charge problem*. By standard MDP results, there exists an optimal policy that is both *deterministic* and independent of the initial state i.

Several variations of problem (2.3) have drawn extensive research attention, aiming to establish the optimality of *threshold* policies (which shut the entry gate iff $L(t)$ lies above a critical threshold), and to compute and optimal threshold. See [24, 31, 6, 19, 1].

2.2. *Optimal index-based threshold policy*

In contrast with previous analyses, we introduce next a novel solution approach grounded on the following observation: one would expect that, under "natural" regularity conditions, as rejection charge ν *grows* from $-\infty$ to $+\infty$, the subset $S(\nu)$ of *controllable* states where it is optimal to shut the gate in (2.3) *decreases monotonically*, from $N^{\{0,1\}}$ to $\emptyset$, dropping states in the order $n-1$, ..., 0, consistent with threshold policies.

In such case, say that the ν-charge problem is *indexable relative to threshold policies*. Then, to each state $j \in N^{\{0,1\}}$ there corresponds a unique *critical charge* ν_j under which it is optimal both to admit and to reject a customer arriving in that state. Call ν_j the *dynamic allocation index* of state j. Since

$$\nu_0 \leq \nu_1 \leq \cdots \leq \nu_{n-1},$$

such indices yield an optimal *index policy* for the ν-charge problem: shut the entry gate in state $j \in N^{\{0,1\}}$ iff $\nu \leq \nu_j$. The *optimal rejection set* is thus

$$S(\nu) = \left\{ j \in N^{\{0,1\}} : \nu \leq \nu_j \right\}, \quad \nu \in \mathbb{R}. \tag{2.4}$$

2.3. *Combinatorial optimization formulation*

The ν-charge problem (2.3) admits a natural *combinatorial optimization* formulation, which will play a key role in our solution approach. Represent each stationary deterministic policy by the subset S of controllable states where it takes the active action, and call it then the *S-active policy*, writing b_i^S, v_i^S, $v_i^S(\nu)$. This gives a reformulation of ν-charge problem (2.3) in terms of finding an optimal *active set*:

$$v_i(\nu) = \min \left\{ v_i^S(\nu) : S \in 2^{N^{\{0,1\}}} \right\}.$$

Represent now the family of threshold policies by a *set system* $(N^{\{0,1\}}, \mathcal{F})$, where $\mathcal{F} \subseteq 2^{N^{\{0,1\}}}$ is the nested family of *feasible rejection sets* given by

$$\mathcal{F} = \{S_1, \ldots, S_{n+1}\}, \tag{2.5}$$

with $S_{n+1} = \emptyset$ and

$$S_k = \{k-1, \ldots, n-1\}, \quad 1 \leq k \leq n. \tag{2.6}$$

We shall address and solve the following problems:

Problem 1: Give sufficient conditions on model parameters under which ν-charge problem (2.3) is indexable relative to threshold policies, so that, in particular,

$$v_i(\nu) = \min \left\{ v_i^S(\nu) : S \in \mathcal{F} \right\}, \quad \nu \in \mathbb{R}.$$

Problem 2: Give an efficient algorithm for finding an optimal threshold policy/optimal active set; or, equivalently, for constructing the indices ν_j.

3. RBs: optimality of index policies with a postulated structure

This section extends the approach outlined above to a general *RB* model.

3.1. The ν-charge problem for a single RB

Consider the problem of optimal dynamic effort allocation to a single stochastic project modeled as an RB, whose state $X(t)$ evolves over discrete time periods $t = 0, 1, \dots,$ through the finite state space N. Its evolution is governed by a policy u, prescribing at each period t which of two actions to take: *active* (engage the project; $a(t) = 1$) or *passive* (let it rest; $a(t) = 0$). Denote by $\mathcal{U}$ the class of state-dependent, or *stationary* policies. A policy $u \in \mathcal{U}$ is thus a mapping $u : N \to [0, 1]$, where $u(i)$ (resp. $1 - u(i)$) is the probability that action $a = 1$ (resp. $a = 0$) is taken in state i.

Taking action a in state i has two effects: first, cost h_i^a is incurred in the current period, discounted by factor $\beta \in (0, 1)$; second, the next state changes to j with probability p_{ij}^a. Write $\mathbf{h}^a = (h_i^a)_{i \in N}$ and $\mathbf{P}^a = (p_{ij}^a)_{i,j \in N}$.

We shall partition the states as $N = N^{\{0,1\}} \cup N^{\{1\}}$. Here, $N^{\{0,1\}}$ is the *controllable state space*, where active and passive actions differ in some respect; and $N^{\{1\}} = N \setminus N^{\{0,1\}}$ is the *uncontrollable state space*, where there is no effective choice. We shall assume that policies $u \in \mathcal{U}$ take the active action at uncontrollable states, i.e.,

$$u(i) \equiv 1, \quad i \in N^{\{1\}}.$$

Let v_i^u be the expected total discounted value of costs incurred over an infinite horizon under policy u, starting at i, i.e.,

$$v_i^u = E_i^u \left[\sum_{t=0}^{\infty} h_{X(t)}^{a(t)} \, \beta^t \right].$$

Besides such *cost measure*, we shall consider the *activity measure*

$$b_i^u = E_i^u \left[\sum_{t=0}^{\infty} \theta_{X(t)}^1 \, a(t) \, \beta^t \right], \tag{3.1}$$

where $\boldsymbol{\theta}^1 = (\theta_j^1)_{j \in N} > \mathbf{0}$ is a given *activity weight* vector. A convenient interpretation results by considering that the model is obtained via *uniformization* (cf. Appendix A) from a continuous-time model, as that in Section 2. Suppose in the original model there is a *distinguished event* (e.g., rejection of an arriving customer), which can only occur under the active action. Let θ_j^1 be the probability of the event happening during a period in state j; then, b_i^u is the expected total discounted number of times such event occurs.

Incorporate further into the model an *activity charge* ν, incurred each time the active action is taken *and* the distinguished event occurs. Note that ν corresponds to the *rejection charge* in the previous section. The *total cost objective* is then

$$v_i^u(\nu) = v_i^u + \nu \, b_i^u.$$

The ν-*charge problem* of concern is to find a policy minimizing such objective:

$$v_i(\nu) = \min \left\{ v_i^u(\nu) : u \in \mathcal{U} \right\}. \tag{3.2}$$

Again, there exists an optimal deterministic policy which is independent of i.

3.2. DP formulation and polynomial-time solvability

The conventional approach to tackle ν-charge problem (3.2) is based on formulating and solving its DP equations, which characterize the *optimal value function* $v_i(\nu)$:

$$v_i(\nu) = \begin{cases} \min_{a \in \{0,1\}} h_i^a + \nu\,\theta_i^1\, a + \beta \sum_{j \in N} p_{ij}^a\, v_j(\nu) & \text{if } i \in N^{\{0,1\}} \\ h_i^1 + \nu\,\theta_i^1 + \beta \sum_{j \in N} p_{ij}^1\, v_j(\nu) & \text{if } i \in N^{\{1\}}. \end{cases} \quad (3.3)$$

In theory, problem (3.2) can be solved in *polynomial time* on the state space's size $|N|$. This follows from (i) the polynomial size of the standard LP reformulation of (3.3); and (ii) the polynomial-time solvability of LP by the *ellipsoid method*.

In practice, however, solution of (3.3) through general-purpose computational techniques can lead to prohibitively long running times when $|N|$ is large. Furthermore, even if such solution is obtained, it is not clear how it could be used to design heuristics for more complex models, where RBs arise as building blocks.

3.3. Solution by index policies with a postulated structure

We next develop an index solution approach to the ν-charge problem, motivated by that outlined in Section 2.2, which extends Whittle's original approach in [37].

As in Section 2.3, ν-charge problem (3.2) admits a combinatorial optimization formulation. Associate to every $S \subset N^{\{0,1\}}$ a corresponding *S-active policy*, which is active over states in $S \cup N^{\{1\}}$ and passive over $N^{\{0,1\}} \setminus S$. Write v_i^S, b_i^S and $v_i^S(\nu)$. The ν-charge problem is thus reformulated in terms of finding an optimal *active set*:

$$v_i(\nu) = \min \left\{ v_i^S(\nu) : S \in 2^{N^{\{0,1\}}} \right\}. \quad (3.4)$$

As before, we shall be concerned with establishing the existence of optimal policies within a *postulated* family, given by a *set system* $(N^{\{0,1\}}, \mathcal{F})$. Here, $\mathcal{F} \subseteq 2^{N^{\{0,1\}}}$ is the corresponding family of *feasible active sets*. Let $S \subseteq N^{\{0,1\}}$.

Definition 3.1 ($\mathcal{F}$-policy). We say that the S-active policy is an *$\mathcal{F}$-policy* if $S \in \mathcal{F}$.

Thus, in the model of Section 2, the $\mathcal{F}$-policies corresponding to the definition of $\mathcal{F}$ in (2.5) are precisely the threshold policies. We shall require set system $(N^{\{0,1\}}, \mathcal{F})$ to be *accessible* and *augmentable*. See Assumption 4.1 in Section 4.

We next define a key property of the ν-charge problem. Let $S(\nu) \subseteq N^{\{0,1\}}$ be, as before, the corresponding set of controllable states where the active action is optimal.

Definition 3.2 (Indexability). We say that the ν-charge problem is *indexable relative to $\mathcal{F}$-policies* if, as ν increases from $-\infty$ to $+\infty$, $S(\nu)$ decreases monotonically from $N^{\{0,1\}}$ to $\emptyset$, with $S(\nu) \in \mathcal{F}$ for $\nu \in \mathbb{R}$.

Under indexability, to each state $j \in N^{\{0,1\}}$ is attached a *critical charge* ν_j, and

$$S(\nu) = \left\{ j \in N^{\{0,1\}} : \nu \le \nu_j \right\} \in \mathcal{F}, \quad \nu \in \mathbb{R}.$$

Definition 3.3 (Dynamic allocation index). We say that ν_j is the *dynamic allocation index* of controllable state $j \in N^{\{0,1\}}$ relative to activity measure b^u.

Remark 3.1. Definitions 3.2 and 3.3 extend Whittle's [37] notion of *indexability* and his index, which are recovered in the case $N^{\{0,1\}} = N$, $\mathcal{F} = 2^N$, $\theta_j^1 = 1$ for $j \in N$.

Regarding problems 1 and 2 in Section 2.1, in light of the above we shall address and solve them as special cases of the following problems:

Problem 1: Give sufficient conditions on model parameters under which the ν-charge problem is indexable relative to $\mathcal{F}$-policies, so that, in particular,

$$v_i(\nu) = \min \left\{ v_i^S(\nu) : S \in \mathcal{F} \right\}, \quad \nu \in \mathbb{R}.$$

Problem 2: Give an efficient algorithm for finding an optimal $\mathcal{F}$-policy; or, equivalently, for constructing the indices ν_j.

We shall solve such problems in Section 6, by casting them into the polyhedral framework developed in Sections 4 and 5 below.

4. $\mathcal{F}$-extended polymatroids: properties and optimization

This section introduces a new polytope associated with an accessible set system $(J, \mathcal{F})$, which generalizes classic polymatroids and the extended polymatroids in [4, 2]. As we shall see, the problems of concern in this paper can be formulated and solved as LPs over such polyhedra. Most proofs in this section will remain close to those of analogous results for extended polymatroids. The exposition will thus focus on the distinctive features of the new polyhedra. The reader is referred to [2] to fill the details.

4.1. $\mathcal{F}$-extended polymatroids

Let J be a finite ground set with $|J| = n$ elements, and let $\mathcal{F} \subseteq 2^J$ be a family of subsets of J. Given a *feasible set* $S \in \mathcal{F}$, let $\partial_{\mathcal{F}}^- S$ and $\partial_{\mathcal{F}}^+ S$ be the *inner and outer boundaries of S relative to $\mathcal{F}$*, defined by

$$\partial_{\mathcal{F}}^- S = \{j \in S : S \setminus \{j\} \in \mathcal{F}\} \quad \text{and} \quad \partial_{\mathcal{F}}^+ S = \{j \in J \setminus S : S \cup \{j\} \in \mathcal{F}\},$$

respectively. We shall require *set system* $(J, \mathcal{F})$ to satisfy the conditions stated next.

Assumption 4.1. *The following conditions hold:*
(i) $\emptyset \in \mathcal{F}$.
(ii) *Accessibility:* $\emptyset \neq S \in \mathcal{F} \Longrightarrow \partial_{\mathcal{F}}^- S \neq \emptyset$.
(iii) *Augmentability:* $J \neq S \in \mathcal{F} \Longrightarrow \partial_{\mathcal{F}}^+ S \neq \emptyset$.

We next introduce the notion of *full $\mathcal{F}$-string*. Let $\boldsymbol{\pi} = (\pi_1, \dots, \pi_n)$ be an n-vector spanning J, so that $J = \{\pi_1, \dots, \pi_n\}$. Let

$$S_k = \{\pi_k, \dots, \pi_n\}, \quad 1 \le k \le n. \tag{4.1}$$

Definition 4.1 (Full $\mathcal{F}$-string). We say that $\boldsymbol{\pi}$ is a *full $\mathcal{F}$-string* if

$$S_k \in \mathcal{F}, \quad 1 \le k \le n.$$

We shall denote by $\Pi(\mathcal{F})$ the set of all full $\mathcal{F}$-strings. Given coefficients $b^S \ge 0$ and $w_j^S > 0$ for $j \in S \in \mathcal{F}$, consider the *polytope* $P(\mathcal{F})$ on $\mathbb{R}^J$ defined by

$$\begin{aligned} &\sum_{j \in S} w_j^S x_j \ge b^S, \; S \in \mathcal{F} \setminus \{J\} \\ &\sum_{j \in J} w_j^J x_j = b^J \\ &\qquad x_j \ge 0, \quad j \in J. \end{aligned} \tag{4.2}$$

For each $\boldsymbol{\pi} \in \Pi(\mathcal{F})$, let $\mathbf{x}^{\boldsymbol{\pi}} = (x_j^{\boldsymbol{\pi}})_{j \in J}$ be the unique solution to

$$w_{\pi_k}^{S_k} x_{\pi_k} + \cdots + w_{\pi_n}^{S_k} x_{\pi_n} = b^{S_k}, \quad 1 \le k \le n, \tag{4.3}$$

Definition 4.2 ($\mathcal{F}$-extended polymatroid). We say that $P(\mathcal{F})$ is an *$\mathcal{F}$-extended polymatroid* if, for each $\boldsymbol{\pi} \in \Pi(\mathcal{F})$, $\mathbf{x}^{\boldsymbol{\pi}} \in P(\mathcal{F})$.

Remark 4.1.

1. Assumption 4.1 ensures the existence of a full $\mathcal{F}$-string, hence $P(\mathcal{F}) \neq \emptyset$.
2. The extended polymatroids in [4, 2] correspond to the case $\mathcal{F} = 2^J$. Classic polymatroids are further recovered when $w_j^S \equiv 1$ for $j \in S \in 2^J$.

4.2. LP over $\mathcal{F}$-extended polymatroids

Consider the following LP problem over $\mathcal{F}$-extended polymatroid $P(\mathcal{F})$:

$$v^{\mathrm{LP}} = \min \left\{ \sum_{j \in J} c_j x_j : \mathbf{x} \in P(\mathcal{F}) \right\}. \tag{4.4}$$

We wish to design an efficient algorithm for solving LP (4.4), for which we start by investigating the vertices of $P(\mathcal{F})$. The next result gives a *partial* characterization, which is in contrast with the *complete* one available for extended polymatroids.

Lemma 4.1. *For $\boldsymbol{\pi} \in \Pi(\mathcal{F})$, $\mathbf{x}^{\boldsymbol{\pi}}$ is a vertex of $P(\mathcal{F})$.*

Proof. The result follows from Definition 4.2, along with the standard algebraic characterization of a polyhedron's vertices.

□

Lemma 4.1 implies that, under *some* cost vectors $\mathbf{c} = (c_j)_{j \in J}$, LP (4.4) is solved by a vertex of the form $\mathbf{x}^{\boldsymbol{\pi}}$, so that

$$v^{\mathrm{LP}} = \min \left\{ \sum_{j \in J} c_j x_j^{\boldsymbol{\pi}} : \boldsymbol{\pi} \in \Pi(\mathcal{F}) \right\}. \tag{4.5}$$

We shall thus seek to solve the LP for a restricted domain of *admissible* cost vectors, for which an efficient test for property (4.5) is available.

To proceed, consider the dual LP. By associating dual variable y^S with the primal constraint for feasible set $S \in \mathcal{F}$, the latter is formulated as

$$\begin{aligned} v^{\mathrm{LP}} = \max \; & \sum_{S \in \mathcal{F}} b^S \, y^S \\ & \text{subject to} \\ & \sum_{S : j \in S \in \mathcal{F}} w_j^S \, y^S \le c_j, \quad j \in J \\ & y^S \ge 0, \quad S \in \mathcal{F} \setminus \{J\} \\ & y^J \text{ unrestricted.} \end{aligned} \tag{4.6}$$

Note that, since $P(\mathcal{F})$ is a nonempty polytope, strong duality ensures that both the primal and the dual LP have the same finite optimal value v^{LP}.

4.3. Adaptive-greedy algorithm and allocation indices

This section discusses the *adaptive-greedy algorithm* $\mathrm{AG}_1(\cdot|\mathcal{F})$, described in Figure 4.1, which we introduced in [26]. It defines a tractable domain of *admissible* cost vectors, under which it constructs an optimal *index-based* solution to dual LP (4.6).

The algorithm is fed with input cost vector $\mathbf{c}$, and produces as output a triplet $(ADMISSIBLE, \boldsymbol{\pi}, \boldsymbol{\nu})$. Here, $ADMISSIBLE \in \{TRUE, FALSE\}$ is a Boolean variable; $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n) \in \Pi(\mathcal{F})$ is a full $\mathcal{F}$-string; and $\boldsymbol{\nu} = (\nu_j)_{j \in J}$ is an *index* vector. Since it runs in n steps, the algorithm will run in *polynomial time* if, for $S \in \mathcal{F}$, calculation of w_j^S $(j \in S)$ and membership test $j \in \partial_{\mathcal{F}}^- S$ are done in polynomial time.

The algorithm has two new features relative to its counterpart for extended polymatroids (cf. [4, 2]), recovered as $\mathrm{AG}_1(\cdot|2^J)$. First, the minimization in step k is performed over the set $\partial_{\mathcal{F}}^- S_k$, which is often much smaller than $\partial_{2^J}^- S_k = S_k$. Thus, in the model of Section 2, $S_k = \{k-1, \ldots, n-1\}$ and $\partial_{\mathcal{F}}^- S_k = \{k-1\}$. Second, the algorithm ends with a *cost admissibility test*, checking whether the generated index sequence is nondecreasing. We could instead have implemented such test by checking at each step k whether $\nu_{\pi_k} < \nu_{\pi_{k-1}}$, in which case execution would be terminated.

Definition 4.3 ($\mathcal{F}$-admissible costs). We say that cost vector $\mathbf{c}$ is *$\mathcal{F}$-admissible for LP (4.4)* if algorithm $\mathrm{AG}_1(\cdot|\mathcal{F})$, when fed with input $\mathbf{c}$, returns an output satisfying

$$\nu_{\pi_1} \le \nu_{\pi_2} \le \cdots \le \nu_{\pi_n}. \tag{4.7}$$

so that $ADMISSIBLE = TRUE$.

We call the set $\mathcal{C}(\mathcal{F})$ of $\mathcal{F}$-admissible $\mathbf{c}$'s the *$\mathcal{F}$-admissible cost domain of LP (4.4).*

Remark 4.2.

1. In the extended polymatroid case, we have $\mathcal{C}(2^J) = \mathbb{R}^J$.

ALGORITHM $\mathrm{AG}_1(\cdot|\mathcal{F})$
Input: c
Output: $(ADMISSIBLE, \boldsymbol{\pi}, \boldsymbol{\nu})$

Initialization: **let** $S_1 := J$
let $y^{S_1} := \min \left\{ \frac{c_j}{w_j^{S_1}} : j \in \partial_{\mathcal{F}}^- S_1 \right\}$;
choose π_1 **attaining the minimum above**; **let** $\nu_{\pi_1} := y^{S_1}$

Loop:
for $k := 2$ **to** n **do**
  let $S_k := S_{k-1} \setminus \{\pi_{k-1}\}$
  let $y^{S_k} := \min \left\{ \frac{1}{w_j^{S_k}} \left[c_j - \sum_{l=1}^{k-1} y^{S_l} w_j^{S_l} \right] : j \in \partial_{\mathcal{F}}^- S_k \right\}$
  choose π_k **attaining the minimum above**; **let** $\nu_{\pi_k} := \nu_{\pi_{k-1}} + y^{S_k}$
end {for}

Cost admissibility test:
if $\nu_{\pi_1} \le \cdots \le \nu_{\pi_n}$ **then let** $ADMISSIBLE := TRUE$ **else let** $ADMISSIBLE := FALSE$

Fig. 4.1. Adaptive-greedy algorithm $\mathrm{AG}_1(\cdot|\mathcal{F})$ for LP over $\mathcal{F}$-extended polymatroids

2. It is readily verified that Definition 4.3 is consistent, i.e., the values of the outputs $ADMISSIBLE$ and $\boldsymbol{\nu}$ do not depend on the tie-breaking order in the algorithm.

Definition 4.4 (Allocation index). We say the ν_j's are LP (4.4)'s *allocation indices*.

In the extended polymatroid case, such indices give an optimality criterion. See [2]. We next extend such result. Let $\mathbf{c} \in \mathcal{C}(\mathcal{F})$. Suppose $\mathrm{AG}_1(\cdot|\mathcal{F})$ is run on $\mathbf{c}$, giving output $(ADMISSIBLE, \boldsymbol{\pi}, \boldsymbol{\nu})$. Let S_k be as in (4.1), and let $\mathbf{y}^{\boldsymbol{\pi}} = \left(y^{\boldsymbol{\pi},S}\right)_{S\in\mathcal{F}}$ be given by

$$y^{\boldsymbol{\pi},S} = \begin{cases} \nu_{\pi_k} - \nu_{\pi_{k-1}} & \text{if } S = S_k, \text{ for some } 2 \le k \le n \\ \nu_{\pi_1} & \text{if } S = S_1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Notice $y^{\boldsymbol{\pi},S_k}$, for $1 \le k \le n$, is characterized as the unique solution to

$$w_{\pi_k}^{S_1} y^{S_1} + \cdots + w_{\pi_k}^{S_k} y^{S_k} = c_{\pi_k}, \quad 1 \le k \le n. \tag{4.9}$$

The next result is proven as its extended polymatroid counterpart (cf. [2]).

Theorem 4.2 (Index-based objective representation and optimality criterion).
(a) LP (4.4)'s objective can be represented as

$$\sum_{j\in J} c_j x_j = \nu_{\pi_1} \sum_{j\in S_1} w_j^{S_1} x_j + \sum_{k=2}^{n} (\nu_{\pi_k} - \nu_{\pi_{k-1}}) \sum_{j\in S_k} w_j^{S_k} x_j;$$

furthermore,

$$v^{\boldsymbol{\pi}} = \sum_{j\in J} c_j x_j^{\boldsymbol{\pi}} = \nu_{\pi_1} b^{S_1} + \sum_{k=2}^{n} (\nu_{\pi_k} - \nu_{\pi_{k-1}}) b^{S_k}.$$

(b) If condition (4.7) holds, so that $\mathbf{c} \in \mathcal{C}(\mathcal{F})$*, then* $\mathbf{x}^{\boldsymbol{\pi}}$ *and* $\mathbf{y}^{\boldsymbol{\pi}}$ *is an optimal primal-dual pair for LPs (4.4) and (4.6). The optimal value is then*

$$v^{\mathrm{LP}} = \nu_{\pi_1}\, b^{S_1} + \sum_{k=2}^{n} (\nu_{\pi_k} - \nu_{\pi_{k-1}})\, b^{S_k}. \tag{4.10}$$

4.4. Allocation index and admissible cost domain decomposition

The allocation indices of extended polymatroids possess a useful *decomposition* property (cf. [2]), which we extend next to $\mathcal{F}$-extended polymatroids.

Suppose set system $(J, \mathcal{F})$ is constructed as follows. We are given m set systems $(J_k, \mathcal{F}_k)$, for $1 \le k \le m$, satisfying Assumption 4.1, where $J_1, \ldots, J_m$ are disjoint. Let

$$J = \bigcup_{k=1}^{m} J_k,$$

$$\mathcal{F} = \left\{ S = \bigcup_{k=1}^{m} S_k : S_k \in \mathcal{F}_k,\ 1 \le k \le m \right\}. \tag{4.11}$$

It is readily verified that set system $(J, \mathcal{F})$ also satisfies Assumption 4.1.

Suppose we are given $b^S \ge 0$ and $w_j^S > 0$, for $j \in S \in \mathcal{F}$, such that $P(\mathcal{F})$ defined by (4.2) is an $\mathcal{F}$-extended polymatroid. Then, Definition 4.2 implies that each $P_k(\mathcal{F}_k)$ on $\mathbb{R}^{J_k}$ (with b^{S_k} and $w_{j_k}^{S_k}$, for $j_k \in S_k \in \mathcal{F}_k$) is an $\mathcal{F}_k$-extended polymatroid.

We shall require coefficients w_j^S to satisfy the following requirement.

Assumption 4.3. *For* $1 \le k \le m$*,*

$$w_{j_k}^{S} = w_{j_k}^{S \cap J_k}, \quad S \in \mathcal{F},\ j_k \in S \cap J_k.$$

Given cost vector $\mathbf{c} = (c_j)_{j \in J}$, let $\mathbf{c}^k = (c_{j_k})_{j_k \in J_k}$ for each k. Consider the corresponding LPs given by (4.4) and

$$v^{k,\mathrm{LP}} = \min \left\{ \sum_{j_k \in J_k} c_{j_k}\, x_{j_k} : \mathbf{x}^k \in P_k(\mathcal{F}_k) \right\}, \tag{4.12}$$

having admissible cost domains $\mathcal{C}(\mathcal{F})$ and $\mathcal{C}(\mathcal{F}_k)$, respectively. Let $\boldsymbol{\nu} = (\nu_j)_{j \in J}$ (resp. $\boldsymbol{\nu}^k = (\nu_{j_k}^k)_{j_k \in J_k}$) be the index vector produced by the algorithm on input $\mathbf{c}$ (resp. $\mathbf{c}_k$).

We state next the decomposition result without proof, as this follows along the same lines as Theorem 3's in [2].

Theorem 4.4 (Admissible cost domain and index decomposition). *Under Assumption 4.3, the following holds:*

ALGORITHM $\mathrm{AG}_2(\cdot|\mathcal{F})$:
Input: c
Output: $(ADMISSIBLE, \boldsymbol{\pi}, \boldsymbol{\nu})$

Initialization: **let** $S_1 = J$; **let** $\nu_j^{S_1} := c_j / w_j^{S_1}, \quad j \in J$
choose $\pi_1 \in \operatorname{argmin} \left\{ \nu_j^{S_1} : j \in \partial_{\mathcal{F}}^- S_1 \right\}$; **let** $\nu_{\pi_1} := \nu_{\pi_1}^{S_1}$

Loop:
for = $k := 2$ **to** n **do**
  let $S_k := S_{k-1} \setminus \{\pi_{k-1}\}$
  let $\nu_j^{S_k} := \nu_j^{S_{k-1}} + \left(\dfrac{w_j^{S_{k-1}}}{w_j^{S_k}} - 1 \right) \left[\nu_j^{S_{k-1}} - \nu_{\pi_{k-1}}^{S_{k-1}} \right], \quad j \in S_k$
  choose $\pi_k \in \operatorname{argmin} \left\{ \nu_j^{S_k} : j \in \partial_{\mathcal{F}}^- S_k \right\}$; **let** $\nu_{\pi_k} := \nu_{\pi_k}^{S_k}$
end {for}

Cost admissibility test:
if $\nu_{\pi_1} \le \cdots \le \nu_{\pi_n}$ **then let** $ADMISSIBLE := TRUE$ **else let** $ADMISSIBLE := FALSE$

Fig. 4.2. Adaptive-greedy algorithm $\mathrm{AG}_2(\cdot|\mathcal{F})$ for LP over $\mathcal{F}$-extended polymatroids

(a) $\mathbf{c} \in \mathcal{C}(\mathcal{F})$ *if and only if* $\mathbf{c}^k \in \mathcal{C}(\mathcal{F}_k)$ *for* $1 \le k \le m$, *i.e.,*

$$\mathcal{C}(\mathcal{F}) = \prod_{k=1}^{m} \mathcal{C}(\mathcal{F}_k).$$

(b) For $1 \le k \le m$,

$$\nu_{j_k} = \nu_{j_k}^k, \quad j_k \in J_k.$$

Remark 4.3.

1. Theorem 4.4(a) shows that the admissible cost domain of LP (4.4) decomposes as the product of the corresponding domains of the m LPs in (4.12). The $\mathcal{F}$-admissibility test for **c** thus decomposes into m simpler tasks, which can be performed *in parallel*.
2. Theorem 4.4(b) shows that the calculation of indices ν_j for LP (4.4) can also be decomposed into m simpler parallel tasks, each involving the calculation of indices $\nu_{j_k}^k$ for the corresponding LP in (4.12).

4.5. A new version of the index algorithm

We have found that the algorithm above does not lend itself well to model *analysis*. This motivates us to develop the reformulated version $\mathrm{AG}_2(\cdot|\mathcal{F})$, shown in Figure 4.2. This represents an extension of Klimov's [21] algorithm, recovered as $\mathrm{AG}_2(\cdot|2^J)$. We shall later apply $\mathrm{AG}_2(\cdot|\mathcal{F})$ to calculate the RB indices introduced in this paper. We remark that Varaiya et al. [34] first applied Klimov's algorithm to calculate the Gittins index for classic (nonrestless) bandits.

The latter is based on the incorporation of coefficients $c_j^{S_k}$, recursively defined (relative to the full $\mathcal{F}$-string $\boldsymbol{\pi}$ being generated) by

$$\begin{aligned} c_j^{S_1} &= c_j, \quad j \in S_1 = J \\ c_j^{S_k} &= c_j^{S_{k-1}} - \frac{c_{\pi_{k-1}}^{S_{k-1}}}{w_{\pi_{k-1}}^{S_{k-1}}} \left[w_j^{S_{k-1}} - w_j^{S_k} \right], \quad j \in S_k, 2 \le k \le n, \end{aligned} \tag{4.13}$$

which allows to simplify the expressions in $\mathrm{AG}_1(\cdot|\mathcal{F})$. We shall further write

$$\nu_j^{S_k} = \frac{c_j^{S_k}}{w_j^{S_k}}, \quad j \in S_k, 1 \le k \le n. \tag{4.14}$$

From (4.13), it follows that the ratios $\nu_j^{S_k}$ are characterized by the recursion

$$\begin{aligned} \nu_j^{S_1} &= \frac{c_j}{w_j^{S_1}}, \quad j \in S_1 = J \\ \nu_j^{S_k} &= \nu_j^{S_{k-1}} + \left(\frac{w_j^{S_{k-1}}}{w_j^{S_k}} - 1 \right) \left[\nu_j^{S_{k-1}} - \nu_{\pi_{k-1}}^{S_{k-1}} \right], \ j \in S_k, 2 \le k \le n, \end{aligned} \tag{4.15}$$

The next result gives the key relations between both algorithms. Let $\boldsymbol{\pi}$ and $\boldsymbol{\nu}$ be produced by algorithm $\mathrm{AG}_1(\cdot|\mathcal{F})$ on input $\mathbf{c}$, and let S_k be given by (4.1).

Lemma 4.2. *For* $1 \le k \le n$ *and* $j \in S_k$,

$$\nu_j^{S_k} = \begin{cases} \dfrac{c_j}{w_j^{S_1}}, & \text{if } k = 1 \\ \nu_{\pi_{k-1}} + \dfrac{c_j - \nu_{\pi_1} w_j^{S_1} - \sum_{l=2}^{k-1} \left(\nu_{\pi_l} - \nu_{\pi_{l-1}} \right) w_j^{S_l}}{w_j^{S_k}}, & \text{if } k \ge 2; \end{cases}$$

furthermore,

$$\nu_{\pi_k} = \nu_{\pi_k}^{S_k}. \tag{4.16}$$

Proof. We proceed by induction on k. The case $k = 1$ follows from (4.13).

Assume now the result holds for $k - 1$, where $k \le n$, so that

$$\nu_j^{S_{k-1}} = \nu_{\pi_{k-2}} + \frac{c_j - \nu_{\pi_1} w_j^{S_1} - \sum_{l=2}^{k-2} \left(\nu_{\pi_l} - \nu_{\pi_{l-1}} \right) w_j^{S_l}}{w_j^{S_{k-1}}}, \quad j \in S_{k-1},$$

and $\nu_{\pi_{k-1}} = \nu_{\pi_{k-1}}^{S_{k-1}}$. Then, applying the induction hypothesis and (4.13), yields the following: for $j \in S_k$,

$$\begin{aligned}\nu_j^{S_k} &= \frac{c_j^{S_{k-1}} - \nu_{\pi_{k-1}} \left[w_j^{S_{k-1}} - w_j^{S_k}\right]}{w_j^{S_k}} \\ &= \frac{\nu_{\pi_{k-2}}\, w_j^{S_{k-1}} + c_j - \nu_{\pi_1}\, w_j^{S_1} - \sum_{l=2}^{k-2} \left(\nu_{\pi_l} - \nu_{\pi_{l-1}}\right) w_j^{S_l} - \nu_{\pi_{k-1}} \left[w_j^{S_{k-1}} - w_j^{S_k}\right]}{w_j^{S_k}} \\ &= \nu_{\pi_{k-1}} + \frac{c_j - \nu_{\pi_1}\, w_j^{S_1} - \sum_{l=2}^{k-1} (\nu_{\pi_l} - \nu_{\pi_{l-1}})\, w_j^{S_l}}{w_j^{S_k}}.\end{aligned}$$

Combining the last identity with (4.8)–(4.9), gives $\nu_{\pi_k} = \nu_{\pi_k}^{S_k}$, completing the proof. □

We are now ready to establish the main result of this section.

Theorem 4.5. *Algorithms* $\mathrm{AG}_1(\cdot|\mathcal{F})$ *and* $\mathrm{AG}_2(\cdot|\mathcal{F})$ *are equivalent.*

Proof. The result follows from Lemma 4.2 and the description of each algorithm. □

4.6. Properties and interpretation of coefficients $c_j^{S_k}$, $\nu_j^{S_k}$, and of indices ν_j

Given the central role that coefficients $c_j^{S_k}$, $\nu_j^{S_k}$ and indices ν_j play in this paper, it is of interest to discuss their properties and interpretation. Assume below that $\boldsymbol{\pi}$, $\boldsymbol{\nu}$ are produced by $\mathrm{AG}_2(\cdot|\mathcal{F})$ on input $\mathbf{c} \in \mathcal{C}(\mathcal{F})$.

The next result shows that the $c_j^{S_k}$'s represent *marginal*, or *reduced costs* of LP (4.4). The proof follows easily by induction, and is hence omitted.

Proposition 4.1. *For* $1 \leq m \leq n-1$,

$$v^{\mathrm{LP}} = \sum_{k=1}^{m} \nu_{\pi_k} \left[b^{S_k} - b^{S_{k+1}}\right] + \sum_{j \in S_{m+1}} c_j^{S_{m+1}}\, x_j^{\boldsymbol{\pi}}. \tag{4.17}$$

Remark 4.4. Proposition 4.1 sheds further light on $\mathrm{AG}_2(\cdot|\mathcal{F})$. Identity (4.17) shows that, once the first m elements of optimal $\mathcal{F}$-string $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_m, \cdot, \ldots, \cdot)$ have been fixed, its construction proceeds by optimizing the *reduced objective* $\sum_{j \in S_{m+1}} c_j^{S_{m+1}}\, x_j$.

We next address the following issue. In step k of algorithm $\mathrm{AG}_2(\cdot|\mathcal{F})$, the next element π_k is picked through a minimization over $j \in \partial_{\mathcal{F}}^{-} S_k$, so that

$$\nu_{\pi_k} = \min\left\{\nu_j^{S_k} : j \in \partial_{\mathcal{F}}^{-} S_k\right\}.$$

Hence, ν_{π_k} is a *locally optimal marginal cost rate* over $j \in \partial^-_{\mathcal{F}} S_k$. We shall next show that ν_{π_k} is an optimal marginal cost rate over the (typically larger) set S_k, i.e.,

$$\nu_{\pi_k} = \min \left\{ \nu_j^{S_k} : j \in S_k \right\}.$$

We shall need the following preliminary result, easily proven by induction on m.

Lemma 4.3. *For* $1 \leq m \leq n$*,*

$$\nu_{\pi_k}^{S_m} = \nu_{\pi_m} + \frac{1}{w_{\pi_k}^{S_m}} \sum_{l=m+1}^{k} (\nu_{\pi_l} - \nu_{\pi_{l-1}})\, w_{\pi_k}^{S_l}, \quad m \leq k \leq n.$$

We are now ready to establish the index characterization discussed above.

Proposition 4.2. *For* $1 \leq m \leq n$*, the index* ν_{π_m} *is characterized as*

$$\nu_{\pi_m} = \min \left\{ \nu_j^{S_m} : j \in S_m \right\}. \tag{4.18}$$

Proof. By Lemma 4.3, we have, for $m \leq k \leq n$,

$$\nu_{\pi_k}^{S_m} = \nu_{\pi_m} + \frac{1}{w_{\pi_k}^{S_m}} \sum_{l=m+1}^{k} (\nu_{\pi_l} - \nu_{\pi_{l-1}})\, w_{\pi_k}^{S_l} \geq \nu_{\pi_m},$$

where the inequality follows from the index ordering (4.7).

□

4.7. Index characterization under monotone w_j^S's

We have found that, in applications, coefficients w_j^S are often *nondecreasing on S*.

Assumption 4.6. *For* $j \in S \subset T$*,* $S, T \in \mathcal{F}$*,*

$$w_j^S \leq w_j^T.$$

This section shows that Assumption 4.6 implies interesting additional properties, including a new index characterization. Let $\boldsymbol{\pi}$, $\boldsymbol{\nu}$, S_k be as in Section 4.6.

Lemma 4.4. *Under Assumption 4.6, the following holds:*

(a) For $1 \leq k \leq n-1$*,*

$$\nu_j^{S_k} \leq \nu_j^{S_{k+1}}, \quad j \in S_{k+1}.$$

(b) For $1 \leq k < l \leq n$*,*

$$\nu_{\pi_k} = \nu_{\pi_k}^{S_k} \leq \nu_{\pi_l}^{S_k} \leq \nu_{\pi_l}^{S_{k+1}} \leq \cdots \leq \nu_{\pi_l}^{S_{l-1}} \leq \nu_{\pi_l}^{S_l} = \nu_{\pi_l}.$$

Proof. (a) From (4.15) and (4.16), together with $w_j^{S_k} - w_j^{S_{k+1}} \geq 0$, it follows that

$$\nu_{\pi_k} \leq \nu_j^{S_k} \Longrightarrow \nu_j^{S_k} \leq \nu_j^{S_{k+1}}.$$

Since the first inequality holds by Proposition 4.2, we obtain the required result.
(b) The result follows directly from part (a) and Proposition 4.2. □

We next give the new index characterization referred to above.

Theorem 4.7. *Under Assumption 4.6, the index ν_j is characterized as*

$$\nu_j = \max\left\{\nu_j^S : j \in S \in \{S_1, \ldots, S_n\}\right\}, \quad j \in J.$$

Proof. The result follows directly from Lemma 4.4(b). □

Remark 4.5. Theorem 4.7 characterizes the indices as *maximal marginal cost rates* relative to feasible sets. This is to be contrasted with the result in Proposition 4.2.

4.8. A recursion for the w_j^S's under symmetric marginal costs

Recall that the marginal costs $c_j^{S_k}$ were defined *relative to a given* $\pi \in \Pi(\mathcal{F})$. This prevents us from extending (4.13) into a definition of coefficients c_j^S, for $S \in \mathcal{F}$, since the *order* in which S is constructed might lead to different values. Yet, in certain applications, including RBs (cf. Section 6), such coefficients are *symmetric*.

Definition 4.5 (Symmetric marginal costs). We say that marginal costs are *symmetric* if the following recursion gives a consistent definition of c_j^S, for $j \in S \in \mathcal{F}$:

$$\begin{aligned} c_j^J &= c_j, \quad j \in J \\ c_j^{S\setminus\{i\}} &= c_j^S - \frac{c_i^S}{w_i^S}\left[w_j^S - w_j^{S\setminus\{i\}}\right] \end{aligned} \tag{4.19}$$

Note that, under marginal cost symmetry, we can further define *marginal cost rates* ν_j^S, for $j \in S \in \mathcal{F}$, by the natural extension of recursion (4.15). The next result shows that marginal cost symmetry, under Assumption 4.6, is equivalent to satisfaction of a *second-order recursion* by the w_j^S's, useful for their calculation.

Proposition 4.3. *Under Assumption 4.6, marginal costs are symmetric iff, for $S \in \mathcal{F}$, $i_1 \in \partial_{\mathcal{F}}^- S \cap \partial_{\mathcal{F}}^-(S \setminus \{i_2\})$, $i_2 \in \partial_{\mathcal{F}}^- S \cap \partial_{\mathcal{F}}^-(S \setminus \{i_1\})$, and $j \in S \setminus \{i_1, i_2\}$, it holds that*

$$w_j^{S\setminus\{i_1,i_2\}} = \frac{\dfrac{w_{i_1}^S}{w_{i_1}^{S\setminus\{i_2\}}} w_j^{S\setminus\{i_2\}} + \dfrac{w_{i_2}^S}{w_{i_2}^{S\setminus\{i_1\}}} w_j^{S\setminus\{i_1\}} - w_j^S}{\dfrac{w_{i_1}^S}{w_{i_1}^{S\setminus\{i_2\}}} + \dfrac{w_{i_2}^S}{w_{i_2}^{S\setminus\{i_1\}}} - 1}. \tag{4.20}$$

Proof. The result follows by recursively calculating $c_j^{S\setminus\{i_1,i_2\}}$ in two different ways, using (4.19): through the sequence $S \to S \setminus \{i_1\} \to S \setminus \{i_1, i_2\}$, and through the sequence $S \to S \setminus \{i_2\} \to S \setminus \{i_1, i_2\}$. Each gives different expressions for $c_j^{S\setminus\{i_1,i_2\}}$. Equating the coefficients of corresponding marginal cost terms yields the stated identity.

□

5. Partial conservation laws

This section reviews the *partial conservation laws (PCLs)* framework introduced in [26], emphasizing its grounding on $\mathcal{F}$-extended polymatroid theory.

Consider a scheduling model involving a finite set J of n *job classes*. Effort is allocated to competing jobs through a *scheduling policy* u, chosen from the space $\mathcal{U}$ of *admissible* policies. Policy u's performance over class j is given by *performance measure* $x_j^u \geq 0$. Write $\mathbf{x}^u = (x_j^u)_{j\in J}$. Associate to every *full string* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ spanning the n classes a corresponding $\boldsymbol{\pi}$-*priority policy*, assigning higher *priority* to class π_l over π_k if $l > k$. Write $x_j^{\boldsymbol{\pi}}$. Given $S \subseteq J$, say that a policy gives priority to S*-jobs* if it gives priority to any class $i \in S$ over any class $j \in S^c = J \setminus S$.

We shall be concerned with solving the *scheduling problem*

$$v = \min \left\{ \sum_{j \in J} c_j \, x_j^u \, : u \in \mathcal{U} \right\}, \tag{5.1}$$

which is to find an admissible policy minimizing the stated linear cost objective. Motivated by applications, we shall seek to identify conditions under which an optimal policy exists within a given family of policies with a postulated structure. As in Section 3.3, we represent the latter by a *set system* $(J, \mathcal{F})$ satisfying Assumption 4.1. Let $\boldsymbol{\pi}$ be as above. Recall the notion of *full* $\mathcal{F}$*-string* from Definition 4.1.

Definition 5.1 ($\mathcal{F}$-policy). We say that the $\boldsymbol{\pi}$-priority policy is an $\mathcal{F}$*-policy* if $\boldsymbol{\pi} \in \Pi(\mathcal{F})$, i.e., $\boldsymbol{\pi}$ is a full $\mathcal{F}$-string of set system $(J, \mathcal{F})$.

Remark 5.1. Sets $S \in \mathcal{F}$ represent *feasible high-priority class subsets* under $\mathcal{F}$-policies.

Consider the following problems:

1. Give sufficient conditions under which $\mathcal{F}$-policies are optimal, so that

$$v = \min \left\{ \sum_{j \in J} c_j \, x_j^{\boldsymbol{\pi}} \, : \boldsymbol{\pi} \in \Pi(\mathcal{F}) \right\}.$$

2. Give an efficient algorithm for finding an optimal $\mathcal{F}$-policy.

To address such problems, consider the *achievable performance region*

$$\mathcal{X} = \left\{ \mathbf{x}^u : u \in \mathcal{U} \right\},$$

which allows us to reformulate (5.1) as the *mathematical programming problem*

$$v = \min \left\{ \sum_{j \in J} c_j \, x_j \, : \mathbf{x} \in \mathcal{X} \right\}.$$

To proceed, we must assume appropriate properties on $\mathcal{X}$, as discussed next.

5.1. Partial conservation laws

Suppose to each job class and feasible high-priority set $j \in S \in \mathcal{F}$ is associated a coefficient $w_j^S > 0$, so that $\sum_{j \in S} w_j^S \, x_j^u$ represents a measure of the system's *workload* corresponding to S-jobs, or S-*workload*, under policy u. We shall refer to w_j^S as the *marginal* S-*workload of class* j. Denote the *minimal* S-*workload* by

$$b^S = \inf \left\{ \sum_{j \in S} w_j^S \, x_j^u \, : u \in \mathcal{U} \right\}, \quad S \in \mathcal{F}.$$

Definition 5.2 (Partial conservation laws). We say that performance vector $\mathbf{x}^u$ satisfies *partial conservation laws (PCLs)* relative to $\mathcal{F}$-policies if the following holds:
(i) for $S \in \mathcal{F} \setminus \{J\}$,

$$\sum_{j \in S} w_j^S \, x_j^{\pi} = b^S, \quad \text{under any } \pi \in \Pi(\mathcal{F}) \text{ giving priority to } S\text{-jobs.}$$

(ii) $\sum_{j \in J} w_j^J \, x_j^{\pi} = b^J, \quad$ under any $\pi \in \Pi(\mathcal{F})$.

Remark 5.2.

1. Satisfaction of the above PCLs means that, for each $S \in \mathcal{F}$, the S-workload is minimized by *any* $\mathcal{F}$-policy which gives priority to S-jobs.
2. The *generalized conservation laws (GCLs)* in [2] are recovered in the case $\mathcal{F} = 2^N$. The *strong conservation laws* in [30] are further recovered when $w_j^S \equiv 1$.

Assume in what follows that $\mathbf{x}^u$ satisfies PCLs as above. This gives a *partial* characterization of achievable performance region $\mathcal{X}$, based on polytope $P(\mathcal{F})$ in (4.2).

Theorem 5.1 (Achievable performance). *$P(\mathcal{F})$ is an $\mathcal{F}$-extended polymatroid, satisfying $\mathcal{X} \subseteq P(\mathcal{F})$. The performance vectors $\mathbf{x}^{\pi}$ of $\mathcal{F}$-policies π are vertices of $P(\mathcal{F})$.*

Proof. PCLs imply $\mathcal{X} \subseteq P(\mathcal{F})$. Let $\pi \in \Pi(\mathcal{F})$. By PCL, performance vector $\mathbf{x}^{\pi}$ is the solution of (4.3). Since $\mathbf{x}^{\pi} \in \mathcal{X} \subseteq P(\mathcal{F})$, Definition 4.2 implies that $P(\mathcal{F})$ is an $\mathcal{F}$-extended polymatroid. By Lemma 4.1, $\mathbf{x}^{\pi}$ is a vertex of $P(\mathcal{F})$.

□

Remark 5.3.

1. In the GCL case ($\mathcal{F} = 2^J$), it holds that $\mathcal{X} = P(2^J)$. See Theorem 4 in [2].
2. By Theorem 5.1, (4.4) is an *LP relaxation* of (5.1), hence $v^{\mathrm{LP}} \leq v$. It further implies optimality of $\mathcal{F}$-policies for (5.1) under *some* cost vectors $\mathbf{c}$, so that $v^{\mathrm{LP}} = v$; and, in particular, under *$\mathcal{F}$-admissible cost vectors* $\mathbf{c} \in \mathcal{C}(\mathcal{F})$ of LP (4.4).

We show next that, under PCLs, the scheduling problem is solved by an index policy with the postulated structure, under appropriate linear objectives. Let $\mathbf{c} \in \mathcal{C}(\mathcal{F})$, and let $\boldsymbol{\pi} \in \Pi(\mathcal{F})$ and $\boldsymbol{\nu} = (\nu_j)_{j \in J}$ be produced by any index algorithm in Section 4 on input $\mathbf{c}$. Let S_k be given by (4.1), and let v^{LP} be the optimal LP value given by (4.4).

Theorem 5.2 (Optimality of index $\mathcal{F}$-policies). *The $\boldsymbol{\pi}$-priority policy, giving higher priority to classes with larger indices ν_j, is optimal. Its value is $v = v^{\mathrm{LP}}$.*

Proof. The result follows directly by combining Theorem 4.2 and Theorem 5.1. □

Remark 5.4. Note that, by Theorem 4.2, under any policy $u \in \mathcal{U}$ it holds that

$$\sum_{j \in J} c_j \, x_j^u = \nu_{\pi_1} \sum_{j \in S_1} w_j^{S_1} \, x_j^u + \sum_{k=2}^{n} (\nu_{\pi_k} - \nu_{\pi_{k-1}}) \sum_{j \in S_k} w_j^{S_k} \, x_j^u .$$

5.2. Multi-project scheduling and index decomposition

This section considers the case where problem (5.1) represents a *multi-project scheduling* model, which represents the natural setting for application of the decomposition property in Section 4.4. We shall apply a special case of the result below in Section 6. Decomposition results have been previously established in [2] (under GCLs), and in [26] (under PCLs). The following is a refined version of the latter.

Consider a finite collection of $m \geq 2$ *projects*, with project $k \in K = \{1, \ldots, m\}$ evolving through finite *state space* N_k. Effort is dynamically allocated to projects through a *scheduling policy* $u \in \mathcal{U}$, where $\mathcal{U}$ is the space of *admissible policies*, prescribing which of two actions to take at each project: *engage* it ($a_k = 1$) or *rest* it ($a_k = 0$). Project k's states are partitioned as $N_k = N_k^{\{0,1\}} \cup N_k^{\{1\}}$. When the state i_k lies in the *controllable state space* $N_k^{\{0,1\}}$, the project can be either engaged or rested, whereas it must be engaged when it lies in the *uncontrollable state space* $N_k^{\{1\}}$. We assume that project state spaces are disjoint. Write $J_k = N_k^{\{0,1\}}$.

The performance of policy u over state $j_k \in J_k$ of project k is given by performance measure $x_{j_k}^{k,u} \geq 0$. Write $\mathbf{x}^{k,u} = (x_{j_k}^{k,u})_{j_k \in J_k}$.

The *multi-project scheduling problem* of concern is

$$v = \min \left\{ \sum_{k=1}^{m} \sum_{j_k \in J_k} c_{j_k}^k \, x_{j_k}^{k,u} : u \in \mathcal{U} \right\},$$

namely, find a policy that minimizes the stated linear performance objective. This problem fits formulation (5.1), by letting job classes correspond to project states.

The PCL framework requires a notion of priority among classes. In the current setting, this follows from the natural notion of priority among projects. We thus interpret each full string $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$, where $n = |J|$, as a corresponding $\boldsymbol{\pi}$-priority policy.

We must further specify a set system $(J, \mathcal{F})$, defining the family of $\mathcal{F}$-policies. Assume we are given a family of policies for operating each project k *in isolation*, i.e., prescribing in which controllable states it should be engaged, given as an appropriate set system $(J_k, \mathcal{F}_k)$. Project k's $\mathcal{F}_k$*-policies* are obtained by associating to each set $S_k \in \mathcal{F}_k$ a corresponding S_k*-active policy*, which engages the project when its state lies in $S_k \cup N^{\{1\}}$, and rests it otherwise. Construct now $(J, \mathcal{F})$ as in (4.11).

Assume further that (i) performance vector $\mathbf{x}^u = (x_j^u)_{j \in J}$ satisfies PCLs relative to $\mathcal{F}$-policies; and that (ii) marginal workloads w_j^S satisfy Assumption 4.3.

Suppose every project k's cost vector $\mathbf{c}^k = (c_{j_k}^k)_{j_k \in J_k}$ is $\mathcal{F}_k$-admissible. Then, Theorem 4.4(a) gives that $\mathbf{c} = (c_j)_{j \in J}$, where $c_{j_k} = c_{j_k}^k$, is $\mathcal{F}$-admissible. Let $\boldsymbol{\nu}^k = (\nu_{j_k}^k)_{j_k \in J_k}$ be project k's index vector. The following result follows from Theorem 4.4.

Theorem 5.3 (Index decomposition for multi-project scheduling). *Any $\mathcal{F}$-policy $\boldsymbol{\pi}$ giving higher priority to projects k whose states j_k have larger indices $\nu_{j_k}^k$ is optimal.*

6. PCL-indexable RBs

In this section we return to the RB model discussed in Section 3. We shall resolve the issues raised in Section 3.3 by deploying the PCL framework.

6.1. Standard LP formulation and pure passive-cost normalization

We review next the standard LP formulation of *ν-charge problem* (3.2), arising as the dual of the LP formulation of its DP equations (3.3). We shall use it to reduce the problem to a *pure passive-cost* normalized version, on which we shall focus our analyses.

The standard LP formulation of ν-charge problem (3.2) is

$$v_i(\nu) = \min \mathbf{x}^0 \mathbf{h}^0 + \mathbf{x}^1 (\mathbf{h}^1 + \nu \boldsymbol{\theta}^1) \tag{6.1}$$

subject to

$$\mathbf{x}^0 (\mathbf{I} - \beta \mathbf{P}^0) + \mathbf{x}^1 (\mathbf{I} - \beta \mathbf{P}^1) = \mathbf{e}_i \tag{6.2}$$

$$x_j^0 = 0, \quad j \in N^{\{1\}}, \tag{6.3}$$

$$\mathbf{x}^0, \mathbf{x}^1 \geq \mathbf{0}$$

where $\mathbf{x}^a = (x_j^a)_{j \in N}$, $\boldsymbol{\theta}^1 = (\theta_j^1)_{j \in N}$, and $\mathbf{e}_i$ is the ith unit coordinate vector in $\mathbb{R}^N$. Vectors are in row or column form as required. In such LP, variable x_j^a corresponds to the standard *state-action occupation measure*

$$x_{ij}^{a,u} = E_i^u \left[\sum_{t=0}^{\infty} 1\{X(t) = j, a(t) = a\} \beta^t \right],$$

giving the expected total discounted number of times action a is taken in state j under policy u, starting at i. Thus, (6.3) says the project must be active at uncontrollable states.

The LP constraints (6.2) imply that

$$\mathbf{x}^1 = \mathbf{e}_i\,(\mathbf{I} - \beta\,\mathbf{P}^1)^{-1} - \mathbf{x}^0\,(\mathbf{I} - \beta\,\mathbf{P}^0)\,(\mathbf{I} - \beta\,\mathbf{P}^1)^{-1},$$

and hence its objective (6.1) can be reformulated as

$$\mathbf{e}_i\,(\mathbf{I} - \beta\,\mathbf{P}^1)^{-1}\,\mathbf{h}^1 + \mathbf{x}^0\,\hat{\mathbf{h}}^0 + \nu\,\mathbf{x}^1\,\boldsymbol{\theta}^1 = v_i^{N^{\{0,1\}}} + \mathbf{x}^0\,\hat{\mathbf{h}}^0 + \nu\,\mathbf{x}^1\,\boldsymbol{\theta}^1, \tag{6.4}$$

where $\hat{\mathbf{h}}^0 = \left(\hat{h}_j^0\right)_{j \in N}$ is the *normalized passive-cost vector* given by

$$\hat{\mathbf{h}}^0 = \mathbf{h}^0 - (\mathbf{I} - \beta\,\mathbf{P}^0)\,(\mathbf{I} - \beta\,\mathbf{P}^1)^{-1}\,\mathbf{h}^1. \tag{6.5}$$

Note further that identity (6.5) and the definition of uncontrollable states gives

$$\hat{h}_j^0 = 0, \quad j \in N^{\{1\}}.$$

We shall focus henceforth on the following *normalized ν-charge problem*

$$\hat{v}_i(\nu) = \min\left\{\sum_{j \in N^{\{0,1\}}} \hat{h}_j^0\,x_{ij}^{0,u} + \nu\,b_i^u : u \in \mathcal{U}\right\}, \tag{6.6}$$

whose optimal value is related to $v_i(\nu)$ by

$$v_i(\nu) = v_i^{N^{\{0,1\}}} + \hat{v}_i(\nu).$$

6.2. PCLs for normalized ν-charge problem

We shall next cast problem (6.6) into the multi-project scheduling case of the PCLs in Section 5.2. Reinterpret (6.6) as a *two-project* scheduling model, by adding to the original project a *calibrating project* with a *single state* $*$. One project must be engaged at each time, where the calibrating project is engaged when the original project is rested.

As in Section 5.2, let the controllable state space of the two-project model be

$$J^* = N^{\{0,1\}} \cup \{*\}.$$

We shall seek to establish PCLs for performance vector $\mathbf{x}_i^u = (x_{ij}^u)_{j \in J^*}$, where

$$x_{ij}^u = \begin{cases} x_{ij}^{0,u} & \text{if } j \in N^{\{0,1\}} \\ b_i^u & \text{if } j = *. \end{cases} \tag{6.7}$$

Note that the normalized ν-charge problem can then be formulated as

$$\hat{v}_i(\nu) = \min\left\{\sum_{j \in N^{\{0,1\}}} \hat{h}_j^0\,x_{ij}^u + \nu\,x_{i*}^u : u \in \mathcal{U}\right\}.$$

Regarding *priorities*'s interpretation, note that, e.g., giving higher priority to calibrating project's state $*$ over original project's state j means that the latter is rested in state j.

Recall from Section 3.3 that we are given an appropriate set system $(N^{\{0,1\}}, \mathcal{F})$ defining the family of *$\mathcal{F}$-policies* (cf. Definition 3.1). Proceeding as in Section 5.2, construct a set system $(J^*, \mathcal{F}^*)$ for the two-project model by letting

$$\mathcal{F}^* = \left\{S^* = S_1 \cup S_2 : S_1 \in \mathcal{F}, S_2 \in \{\emptyset, \{*\}\}\right\}.$$

We shall seek to establish that performance vector $\mathbf{x}_i^u$ satisfies PCLs relative to $\mathcal{F}^*$, for which suitable coefficients $w_j^{S^*}$ and b^{S^*} must be defined.

We start by defining *marginal workloads* w_j^S, for $j \in N$, $S \subseteq N^{\{0,1\}}$, in terms of *activity measures* b_i^S (cf. Section 3.1). The latter are characterized by

$$b_i^S = \begin{cases} \theta_i^1 + \beta \sum_{j \in N} p_{ij}^1 \, b_j^S & \text{if } i \in S \cup N^{\{1\}} \\ \beta \sum_{j \in N} p_{ij}^0 \, b_j^S, & \text{if } i \in N^{\{0,1\}} \setminus S; \end{cases} \tag{6.8}$$

We shall use below the following notation: given $\mathbf{d} = (d_j)_{j \in N}$, $\mathbf{A} = (a_{i,j})_{i,j \in N}$, and $S, T \subseteq N$, we shall write $\mathbf{d}_S = (d_j)_{j \in S}$ and $\mathbf{A}_{ST} = (a_{ij})_{i \in S, j \in T}$. We can thus reformulate the above equations as

$$\begin{aligned} \mathbf{b}_{S \cup N^{\{0,1\}}}^S &= \boldsymbol{\theta}_{S \cup N^{\{0,1\}}}^1 + \beta \, \mathbf{P}_{S \cup N^{\{0,1\}}, N}^1 \, \mathbf{b}^S \\ \mathbf{b}_{N^{\{0,1\}} \setminus S}^S &= \beta \, \mathbf{P}_{N^{\{0,1\}} \setminus S, N}^0 \, \mathbf{b}^S. \end{aligned}$$

Let now

$$w_i^S = \theta_i^1 \, 1\{i \in N^{\{0,1\}}\} + \beta \sum_{j \in N} (p_{ij}^1 - p_{ij}^0) \, t_j^S, \quad i \in N; \tag{6.9}$$

i.e.,

$$\begin{aligned} \mathbf{w}_{N^{\{0,1\}}}^S &= \boldsymbol{\theta}_{N^{\{0,1\}}}^1 + \beta \left(\mathbf{P}_{N^{\{0,1\}}, N}^1 - \mathbf{P}_{N^{\{0,1\}}, N}^0 \right) \mathbf{b}^S \\ \mathbf{w}_{N^{\{1\}}}^S &= \beta \left(\mathbf{P}_{N^{\{1\}}, N}^1 - \mathbf{P}_{N^{\{1\}}, N}^0 \right) \mathbf{b}^S = \mathbf{0}, \end{aligned} \tag{6.10}$$

where the last identity follows from the assumption $p_{ij}^1 = p_{ij}^0$ for $i \in N^{\{1\}}$. Coefficient w_i^S thus represents the *marginal increment in activity measure b^S resulting from a passive-to-active action interchange in initial state i*.

We proceed with a preliminary result, giving further relations between $\mathbf{b}^S$ and $\mathbf{w}^S$. The proof is omitted, as it follows by straightforward algebra from the above.

Lemma 6.1. *The following identities hold:*

$$\begin{aligned} (\mathbf{I} - \beta \, \mathbf{P}^0) \, \mathbf{b}^S &= \begin{bmatrix} \mathbf{w}_S^S \\ \mathbf{0}_{N^{\{0,1\}} \setminus S} \\ \boldsymbol{\theta}_{N^{\{1\}}}^1 \end{bmatrix} \\ \boldsymbol{\theta}^1 - (\mathbf{I} - \beta \, \mathbf{P}^1) \, \mathbf{b}^S &= \begin{bmatrix} \mathbf{0}_{S \cup N^{\{1\}}} \\ \mathbf{w}_{N^{\{0,1\}} \setminus S}^S \end{bmatrix}. \end{aligned} \tag{6.11}$$

Motivated by Assumption 4.3, we complete the marginal workload definitions by letting, for $j \in J^*$ and $S^* = S \cup \{*\}$, with $S \subseteq N^{\{0,1\}}$,

$$w_j^{S^*} = \begin{cases} w_j^S & \text{if } j \in N^{\{0,1\}} \\ 1 & \text{if } j = *. \end{cases}$$

It remains to define the function $b_i^{S^*}$ arising in the right-hand side of the PCLs (which now depends on initial state i). Let, for $S^* \subseteq J^*$,

$$b_i^{S^*} = \begin{cases} b_i^S & \text{if } S^* = S \cup \{*\}, \emptyset \neq S \subseteq N^{\{0,1\}} \\ 0 & \text{otherwise.} \end{cases}$$

The next result gives a set of *workload decomposition laws*, i.e., linear equations relating workload terms corresponding to the active and the passive action.

Proposition 6.1 (Workload decomposition laws). *For $u \in \mathcal{U}$ and $S \subseteq N^{\{0,1\}}$,*

$$b_i^u + \sum_{j \in S} w_j^S x_{ij}^{0,u} = b_i^S + \sum_{j \in N^{\{0,1\}} \setminus S} w_j^S x_{ij}^{1,u}.$$

Proof. Using in turn equations (6.2) and (6.11), we have

$$\begin{aligned} 0 &= \left[\mathbf{x}_i^{0,u} (\mathbf{I} - \beta \mathbf{P}^0) + \mathbf{x}_i^{1,u} (\mathbf{I} - \beta \mathbf{P}^1) - \mathbf{e}_i\right] \mathbf{b}^S \\ &= \mathbf{x}_i^{0,u} (\mathbf{I} - \beta \mathbf{P}^0) \mathbf{b}^S + \mathbf{x}_i^{1,u} \left[(\mathbf{I} - \beta \mathbf{P}^1) \mathbf{b}^S - \boldsymbol{\theta}^1\right] - \mathbf{e}_i \mathbf{b}^S + \mathbf{x}_i^{1,u} \boldsymbol{\theta}^1 \\ &= \mathbf{x}_{i,S}^{0,u} \mathbf{w}_S^S - \mathbf{x}_{i,N \setminus S}^{1,u} \mathbf{w}_{N \setminus S}^S - b_i^S + b_i^u, \end{aligned}$$

which gives the required result, after simplification using Lemma 6.1. □

The relation between coefficients b_j^S's and w_j^S's is further clarified next.

Corollary 6.1. *For $i \in N$ and $S \subseteq N^{\{0,1\}}$,*

$$b_i^{S \cup \{j\}} = b_i^S + w_j^S x_{ij}^{1, S \cup \{j\}}, \quad j \in N^{\{0,1\}} \setminus S$$
$$b_i^S = b_i^{S \setminus \{j\}} + w_j^S x_{ij}^{0, S \setminus \{j\}}, \quad j \in S.$$

Proof. It follows by letting $u = S \cup \{j\}$ and $u = S \setminus \{j\}$ in Proposition 6.1, respectively. □

Proposition 6.1 suggests the following conditions for satisfaction of PCLs.

Assumption 6.1. *Marginal workloads w_j^S satisfy the following: for $S \in \mathcal{F}$,*

$$w_j^S > 0, \quad j \in N^{\{0,1\}}.$$

Assumption 6.1 represents a monotonicity property of b^u, as shown next.

Proposition 6.2. *Assumption 6.1 is equivalent to the following: for $S \in \mathcal{F}$,*

$$\begin{aligned} b_j^S &< b_j^{S \cup \{j\}}, \quad j \in N^{\{0,1\}} \setminus S \\ b_j^S &> b_j^{S \setminus \{j\}}, \quad j \in S. \end{aligned} \tag{6.12}$$

Proof. The result follows from Corollary 6.1, by noting that $x_{jj}^{1, S \cup \{j\}} > 0$, for $j \in N^{\{0,1\}} \setminus S$, and $x_{jj}^{0, S \setminus \{j\}} > 0$, for $j \in S$.

□

We are now ready to established the required PCLs.

Theorem 6.2 (PCLs). *Under Assumption 6.1, performance vector $\mathbf{x}_i^u$ satisfies PCLs relative to $\mathcal{F}^*$-policies.*

Proof. The result follows by combining Proposition 6.1 with Assumption 6.1. Consider, e.g., the case $S^* = S \cup \{*\}$, where $\emptyset \neq S \in \mathcal{F}$. Under any policy $u \in \mathcal{U}$,

$$\begin{aligned} \sum_{j \in \mathcal{F}^*} w_j^{S^*} x_{ij}^u &= b_i^u + \sum_{j \in S} w_j^S x_{ij}^{0,u} \\ &= b_i^S + \sum_{j \in N^{\{0,1\}} \setminus S} w_j^S x_{ij}^{1,u} \\ &\geq b_i^S = b_i^{S^*}, \end{aligned}$$

with equality attained in the last inequality if *priority* is given to S^**-jobs*, i.e., if the passive action is taken at states $j \in N^{\{0,1\}} \setminus S$. Other cases follow similarly.

□

We next define a class of RBs that will be shown to be indexable. Let $n = |N^{\{0,1\}}|$.

Definition 6.1 (PCL-indexable RBs). We say the RB is *PCL-indexable* relative to activity measure b^u and $\mathcal{F}$-policies if the following conditions holds:
(i) *Positive marginal workloads:* Assumption 6.1 holds.
(ii) *Index monotonicity:* Let $(ADMISSIBLE, \boldsymbol{\pi}, \boldsymbol{\nu})$ be the output of any index algorithm in Section 4 on input $\hat{\mathbf{h}}_{N^{\{0,1\}}}^0$. Then, the indices satisfy

$$\nu_{\pi_1} \leq \cdots \leq \nu_{\pi_n}, \tag{6.13}$$

i.e., $ADMISSIBLE = TRUE$, or $\hat{\mathbf{h}}_{N^{\{0,1\}}}^0 \in \mathcal{C}(\mathcal{F})$.

Remark 6.1. The definition of PCL-indexability in [26] is recovered in the case $\theta_j^1 \equiv 1$.

Assume below that the RB is PCL-indexable. Feed any index algorithm with input $\hat{\mathbf{h}}_{N^{\{0,1\}}}^0$ to get $\mathcal{F}$-string $\boldsymbol{\pi}$ and index vector $\boldsymbol{\nu}$. Let $S_k = \{\pi_k, \ldots, \pi_n\}$, for $1 \leq k \leq n$. The next result shows that PCL-indexability implies indexability (cf. Definition 3.2).

Theorem 6.3 (PCL-indexability $\Longrightarrow$ indexability). *The RB is indexable, and the dynamic allocation index of state j is ν_j, for $j \in N^{\{0,1\}}$.*

Proof. Theorem 5.3 applies to the two-project formulation of the normalized ν-charge problem. It follows that (i) the priority index of the calibrating project's state is $\nu_* = \nu$; and (ii) the dynamic allocation index for the original project's controllable state j is ν_j. The result now follows by interpreting Theorem 5.3 in terms of Definition 3.2. □

Several consequences follow from the above, starting with a reformulation of the ν-charge problem as an LP over an $\mathcal{F}^*$-extended polymatroid. Consider the polyhedron $P_i(\mathcal{F}^*) \subset \mathbb{R}^{J^*}$ defined as in (4.2), relative to parameters $w_j^{S^*}$ and $b_i^{S^*}$ as above.

Corollary 6.2. *$P_i(\mathcal{F}^*)$ is an $\mathcal{F}^*$-extended polymatroid. The ν-charge problem can be reformulated as the LP*

$$v_i(\nu) = v_i^{N^{\{0,1\}}} + \min \left\{ \sum_{j \in N^{\{0,1\}}} \hat{h}_j^0 x_j + \nu x_* : \mathbf{x} \in P_i(\mathcal{F}^*) \right\}.$$

The next result, illustrated in Figure 6.1, characterizes $v_i(\nu)$. Let $S_{n+1} = \emptyset$.

Corollary 6.3. *Function $v_i(\nu)$ is continuous, concave and piecewise linear on ν, and*

$$\begin{aligned} v_i(\nu) &= \min \left\{ v_i^{S_k}(\nu) : 0 \le k \le n \right\} \\ &= \begin{cases} v_i^{S_1}(\nu) = v_i^{S_1} + \nu\, b_i^{S_1}, & \text{if } \nu \in (-\infty, \nu_{\pi_1}] \\ v_i^{S_k}(\nu) = v_i^{S_k} + \nu\, b_i^{S_k}, & \text{if } \nu \in [\nu_{\pi_{k-1}}, \nu_{\pi_k}],\ 2 \le k \le n \\ v_i^{S_{n+1}}(\nu) = v_i^{S_{n+1}} + \nu\, b_i^{S_{n+1}}, & \text{if } \nu \in [\nu_{\pi_n}, +\infty). \end{cases} \end{aligned}$$

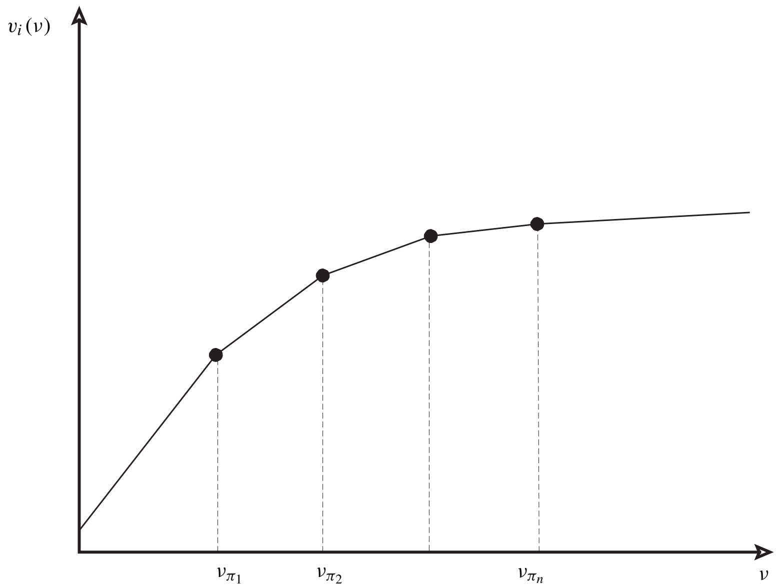

Fig. 6.1. Dependence on activity charge ν of optimal value function $v_i(\nu)$

Proof. The identities follow from Theorem 6.3 and Definition 3.2. They imply $v_i(\nu)$ is continuous concave piecewise linear on ν, being the minimum of linear functions of ν. □

6.3. Marginal costs

Recall that in Section 4.5 we introduced *marginal costs* c_j^S's to simplify index calculations. This section discusses further properties of such coefficients in the RB setting.

We start by defining coefficients c_j^S, for $j \in N$, $S \subseteq N^{\{0,1\}}$, in terms of value measure v_i^S. For every S, the v_i^S's are characterized by the linear equations

$$v_i^S = \begin{cases} h_i^1 + \beta \sum_{j \in N} p_{ij}^1 \, v_j^S & \text{if } i \in S \\ h_i^0 + \beta \sum_{j \in N} p_{ij}^0 \, v_j^S & \text{if } i \in N \setminus S; \end{cases}$$

or, in vector notation,

$$\begin{aligned} \mathbf{v}_S^S &= \mathbf{h}_S^1 + \beta \, \mathbf{P}_{SN}^1 \, \mathbf{v}^S \\ \mathbf{v}_{N \setminus S}^S &= \mathbf{h}_{N \setminus S}^0 + \beta \, \mathbf{P}_{N \setminus S, N}^0 \, \mathbf{v}^S. \end{aligned} \tag{6.14}$$

Define now

$$c_i^S = h_i^0 - h_i^1 + \beta \sum_{j \in N} (p_{ij}^0 - p_{ij}^1) \, v_j^S, \quad i \in N, \tag{6.15}$$

i.e.,

$$\mathbf{c}^S = \mathbf{h}^0 - \mathbf{h}^1 + \beta \left(\mathbf{P}^0 - \mathbf{P}^1 \right) \mathbf{v}^S, \tag{6.16}$$

Coefficient c_i^S thus represents the *marginal increment in cost measure v^S resulting from a passive-to-active action interchange in initial state i.*

It immediately follows that

$$c_j^S = 0, \quad j \in N^{\{1\}}. \tag{6.17}$$

Furthermore, (6.14)–(6.16) readily yields the following counterpart of Lemma 6.1.

Lemma 6.2. *The following identities hold:*

$$\begin{aligned} \mathbf{h}^0 - (\mathbf{I} - \beta \, \mathbf{P}^0) \, \mathbf{v}^S &= \begin{bmatrix} \mathbf{c}_S^S \\ \mathbf{0}_{N \setminus S} \end{bmatrix} \\ (\mathbf{I} - \beta \, \mathbf{P}^1) \, \mathbf{v}^S - \mathbf{h}^1 &= \begin{bmatrix} \mathbf{0}_S \\ \mathbf{c}_{N \setminus S}^S \end{bmatrix}. \end{aligned} \tag{6.18}$$

The next result is a cost analog of Proposition 6.1.

Proposition 6.3 (Cost decomposition laws). *For $u \in \mathcal{U}$ and $S \subseteq N^{\{0,1\}}$,*

$$v_i^S + \sum_{j \in S} c_j^S x_{ij}^{0,u} = v_i^u + \sum_{j \in N^{\{0,1\}} \setminus S} c_j^S x_{ij}^{1,u}. \tag{6.19}$$

Proof. Using in turn equations (6.2) and (6.18), we have

$$\begin{aligned} 0 &= \left[\mathbf{x}_i^{0,u}\,(\mathbf{I} - \beta\,\mathbf{P}^0) + \mathbf{x}_i^{1,u}\,(\mathbf{I} - \beta\,\mathbf{P}^1) - \mathbf{e}_i\right] \mathbf{v}^S \\ &= \mathbf{x}_i^{0,u}\left[(\mathbf{I} - \beta\,\mathbf{P}^0)\,\mathbf{v}^S - \mathbf{h}^0\right] + \mathbf{x}_i^{1,u}\left[(\mathbf{I} - \beta\,\mathbf{P}^1)\,\mathbf{v}^S - \mathbf{h}^1\right] \\ &\quad - \mathbf{e}_i\,\mathbf{v}^S + \mathbf{x}_i^{1,u}\,\mathbf{h}^1 + \mathbf{x}^{0,u}(i)\,\mathbf{h}^0 \\ &= -\mathbf{x}_{i,S}^{0,u}\,\mathbf{c}_S^S + \mathbf{x}_{i,N\setminus S}^{1,u}\,\mathbf{c}_{N\setminus S}^S - v_i^S + v_i^u, \end{aligned}$$

which yields the result, using (6.17). □

The relation between coefficients v_j^S's and c_j^S's is clarified next (cf. Corollary 6.1).

Corollary 6.4. *The following identities hold: for $i \in N$ and $S \subseteq N^{\{0,1\}}$,*

$$\begin{aligned} v_i^S &= v_i^{S\cup\{j\}} + c_j^S x_{ij}^{1,S\cup\{j\}}, \quad j \in N^{\{0,1\}} \setminus S \\ v_i^{S\setminus\{j\}} &= v_i^S + c_j^S x_{ij}^{0,S\setminus\{j\}}, \quad j \in S. \end{aligned}$$

Proof. It follows by letting $u = S \cup \{j\}$ and $u = S \setminus \{j\}$ in Proposition 6.3, respectively. □

The next result sheds further light on the relation between time and value measures, and between marginal workloads and marginal costs.

Proposition 6.4. *Under Assumption 6.1, the following holds: for $j \in S \in \mathcal{F}$,*

(a) $\mathbf{v}^{S\setminus\{j\}} - \mathbf{v}^S = \dfrac{c_j^S}{w_j^S}\left(\mathbf{b}^S - \mathbf{b}^{S\setminus\{j\}}\right) = \dfrac{c_j^{S\setminus\{j\}}}{w_j^{S\setminus\{j\}}}\left(\mathbf{b}^S - \mathbf{b}^{S\setminus\{j\}}\right).$

(b) $\dfrac{c_j^S}{w_j^S} = \dfrac{c_j^{S\setminus\{j\}}}{w_j^{S\setminus\{j\}}}.$

(c) $\mathbf{c}^S - \mathbf{c}^{S\setminus\{j\}} = \dfrac{c_j^S}{w_j^S}\left(\mathbf{w}^S - \mathbf{w}^{S\setminus\{j\}}\right).$

Proof. (a) This part follows from Proposition 6.1 and Proposition 6.3.

(b) The result follows from (a) and Proposition 6.2.

(c) From (6.10), we readily obtain

$$\mathbf{w}^S - \mathbf{w}^{S\setminus\{j\}} = \beta\left(\mathbf{P}^1 - \mathbf{P}^0\right)\left(\mathbf{b}^S - \mathbf{b}^{S\setminus\{j\}}\right). \tag{6.20}$$

Similarly, by (6.16), we have

$$\mathbf{c}^S - \mathbf{c}^{S\setminus\{j\}} = \beta\left(\mathbf{P}^0 - \mathbf{P}^1\right)\left(\mathbf{v}^S - \mathbf{v}^{S\setminus\{j\}}\right). \tag{6.21}$$

The result now follows by combining part (a) with (6.20)–(6.21). □

Remark 6.2.

1. Proposition 6.4 shows that the c_j^S's defined by (6.15) extend those defined by (4.19).
2. It follows by construction that the c_j^S's are *symmetric* (cf. Definition 4.5). Therefore, marginal workloads satisfy the recursion in Proposition 4.3.
3. Note that, by combining identities (6.5), (6.14) and (6.18), it follows that

$$\hat{\mathbf{h}}^0 = \mathbf{c}^J. \tag{6.22}$$

6.4. PCL-indexability as a law of diminishing marginal returns

This section discusses the intuitive interpretation of PCL-indexability (cf. Definition 6.1) as a form of the classic economic law of *diminishing marginal returns*. Suppose the project is PCL-indexable as above, and let $\boldsymbol{\pi}$, $\boldsymbol{\nu}$ and S_k be as in Section 6.2. Assume the initial state is drawn from a probability distribution assigning a *positive mass* $p_i > 0$ *to each state* $i \in N$. Write $\mathbf{p} = (p_i)_{i \in N}$, $b^S = \sum_{i \in N} p_i\, b_i^S$, and $v^S = \sum_{i \in N} p_i\, v_i^S$.

Theorem 6.4 (Index characterization and diminishing marginal returns).

(a)

$$b^{S_{n+1}} < b^{S_n} < \cdots < b^{S_1}.$$

(b) For $1 \le k \le n$, *dynamic allocation index* ν_{π_k} *is given by*

$$\begin{aligned} \nu_{\pi_k} &= \frac{v^{S_{k+1}} - v^{S_k}}{b^{S_k} - b^{S_{k+1}}} \\ &= \min\left\{ \frac{v^{S_k \setminus \{j\}} - v^{S_k}}{b^{S_k} - b^{S_k \setminus \{j\}}} : j \in S_k \right\} \\ &= \max\left\{ \frac{v^{S_k} - v^{S_k \cup \{j\}}}{b^{S_k \cup \{j\}} - b^{S_k}} : j \in N^{\{0,1\}} \setminus S_k \right\}. \end{aligned}$$

(c) Diminishing marginal returns:

$$\frac{v^{S_2} - v^{S_1}}{b^{S_1} - b^{S_2}} \le \frac{v^{S_3} - v^{S_2}}{b^{S_2} - b^{S_3}} \le \cdots \le \frac{v^{S_{n+1}} - v^{S_n}}{b^{S_n} - b^{S_{n+1}}}.$$

Proof. (a) This part follows from Proposition 6.2 and $\mathbf{p} > \mathbf{0}$.

(b) The first identity follows from Proposition 6.4, identity (4.16), and part (a). The second identity then follows from (4.18) in Proposition 4.2. The third identity further follows from Corollary 6.3.

(c) The result follows from parts (a), (b) and the inequalities in (4.7).

□

Remark 6.3.

1. Part (a) shows that the *busy* or *active* time (as measured by b^u) is strictly increasing along the set/policy sequence $\emptyset = S_{n+1} \subset S_n \subset \cdots \subset S_1 = N^{\{0,1\}}$.

2. Part (b) characterizes index ν_{π_k} as a *locally optimal marginal cost rate*: it is the *minimal rate of marginal cost increase from* v^{S_k} *per unit marginal activity decrease from* b^{S_k} *resulting from an active-to-passive action interchange on some state* $j \in S_k$. Furthermore, ν_{π_k} is the *maximal rate of marginal cost decrease from* v^{S_k} *per unit marginal activity increase from* b^{S_k} *resulting from a passive-to-active action interchange on some state* $j \in N^{\{0,1\}} \setminus S_k$.
3. Part (c) shows that the optimal rate of marginal cost decrease per unit marginal active time increase diminishes on the base active time. It thus represents a form of the law of diminishing marginal returns.

Figure 6.2 illustrates the result by an *activity-cost plot*, where the shaded area represents the *region of achievable activity-cost pairs* (b^u, v^u). We further have the following index characterization under Assumption 4.6 on *nondecreasing marginal workloads*.

Theorem 6.5. *Under Assumption 4.6,*

$$\nu_j = \max \left\{ \frac{v_j^{S\setminus\{j\}} - v_j^S}{b_j^S - b_j^{S\setminus\{j\}}} : j \in S \in \{S_1, \dots, S_n\} \right\}, \quad j \in N^{\{0,1\}}. \tag{6.23}$$

Proof. The result follows directly from Theorem 4.7 and Proposition 6.4. □

Remark 6.4.

1. Theorem 6.5 represents an RB counterpart of the *Gittins index* characterization for classic bandits ($\mathbf{P}^0 = \mathbf{I}$ and $\mathbf{h}^0 = \mathbf{0}$) as an *optimal average cost (reward) rate per*

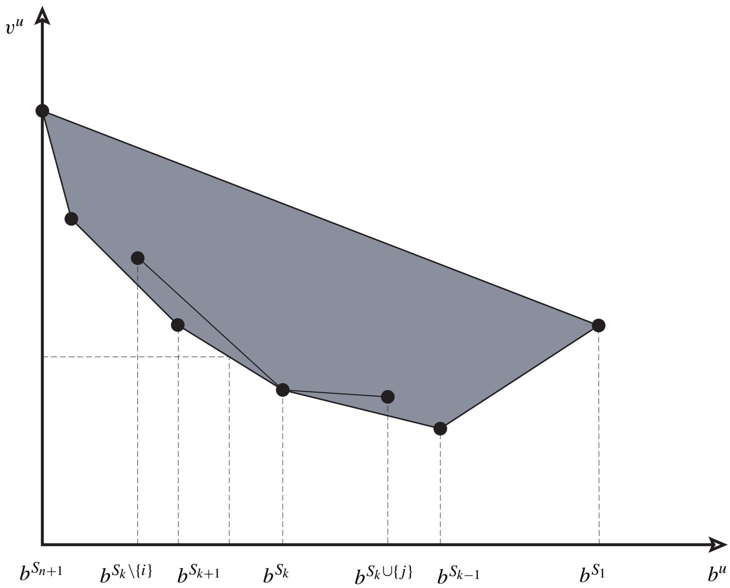

Fig. 6.2. Activity-cost plot: PCL-indexability and diminishing marginal returns

unit time, first given in [15]. In the classic case Theorem 6.5 gives

$$\nu_j = \max\left\{\frac{-v_j^S}{b_j^S} : j \in S \in \{S_1, \ldots, S_n\}\right\}, \quad j \in N^{\{0,1\}},$$

since $b_j^{S\setminus\{j\}} = v_j^{S\setminus\{j\}} = 0$. Actually, the Gittins index characterization in [15] is

$$\nu_j = \max\left\{\frac{-v_j^S}{b_j^S} : j \in S \in 2^N\right\}, \quad j \in N,$$

2. We pose the *open problem*: Find conditions under which (6.23) extends to

$$\nu_j = \max\left\{\frac{v_j^{S\setminus\{j\}} - v_j^S}{b_j^S - b_j^{S\setminus\{j\}}} : j \in S \in \mathcal{F}\right\}, \quad j \in N^{\{0,1\}}.$$

6.5. Extension to the long-run average criterion

The results above for the time-discounted criterion readily extend to the long-run average criterion under suitable *ergodicity* conditions, by standard limiting (Tauberian) arguments. Assume that the model is *communicating*, i.e., every state can be reached from every other state under some stationary policy. Assume further that, for every $S \in \mathcal{F}$, the S-active policy is *unichain*, i.e., it induces a single recurrent class plus a (possibly empty) set of transient states. Then, it is well known that measures $b_i^u(\beta)$, $v_i^u(\beta)$, $x_{ij}^{a,u}(\beta)$ (where we have made explicit the dependence on β), when scaled by factor $1 - \beta$, converge to limiting values independent of the initial state i, given by

$$\bar{b}^u = \lim_{T\to\infty} \frac{1}{T} E_i^u\left[\sum_{t=0}^{T} \theta_{X(t)}^1\, a(t)\right] = \lim_{\beta\nearrow 1} (1-\beta)\, b_i^u(\beta),$$

$$\bar{v}^u = \lim_{T\to\infty} \frac{1}{T} E_i^u\left[\sum_{t=0}^{T} h_{X(t)}^{a(t)}\right] = \lim_{\beta\nearrow 1} (1-\beta)\, v_i^u(\beta),$$

$$\bar{x}_j^{a,u} = \lim_{T\to\infty} \frac{1}{T} E_i^u\left[\sum_{t=0}^{T} 1\{X(t) = j, a(t) = a\}\right] = \lim_{\beta\nearrow 1} (1-\beta)\, x_{ij}^{a,u}(\beta).$$

Hence, $\bar{b}^u$, $\bar{v}^u$ and $\bar{x}_j^{a,u}$ are the corresponding *long-run average*, or *steady-state*, measures.

We next argue that the *unscaled* quantities $w_i^S(\beta)$ and $c_i^S(\beta)$ converge to finite limits $\bar{w}_i^S$ and $\bar{c}_i^S$ as $\beta \nearrow 1$. Start with marginal workload $w_i^S(\beta)$. It is well known that we can write, for $i \in N$ and $S \in \mathcal{F}$,

$$b_i^S(\beta) = \frac{\bar{b}^S}{1-\beta} + a_i^S + O(1-\beta), \quad \text{as } \beta \nearrow 1, \tag{6.24}$$

where the values a_i^S are determined, up to an additive constant, by the equations

$$\bar{b}^S + a_i^S = \begin{cases} \theta_i^1 + \sum_{j \in N} p_{ij}^1 \, a_j^S & \text{if } i \in S \cup N^{\{1\}} \\ \sum_{j \in N} p_{ij}^0 \, a_j^S & \text{if } i \in N^{\{0,1\}} \setminus S. \end{cases}$$

Now, substituting for $b_i^S(\beta)$ as given by (6.24) in (6.9), and letting $\beta \nearrow 1$, gives

$$\bar{w}_i^S = \lim_{\beta \nearrow 1} w_i^S(\beta) = \theta_i^1 \, 1\{i \in N^{\{0,1\}}\} + \sum_{j \in N} (p_{ij}^1 - p_{ij}^0) \, a_j^S, \quad i \in N.$$

We proceed analogously with marginal costs $c_i^S(\beta)$. Write

$$v_i^S(\beta) = \frac{\bar{v}^S}{1-\beta} + f_i^S + O(1-\beta), \quad \text{as } \beta \nearrow 1, \tag{6.25}$$

where the values f_i^S are determined, up to an additive constant, by the equations

$$\bar{v}^S + f_i^S = \begin{cases} h_i^1 + \sum_{j \in N} p_{ij}^1 \, f_j^S & \text{if } i \in S \\ h_i^0 + \sum_{j \in N} p_{ij}^0 \, f_j^S & \text{if } i \in N \setminus S. \end{cases}$$

Now, substituting for $v_i^S(\beta)$ as given by (6.25) in (6.15), and letting $\beta \nearrow 1$, gives

$$\bar{c}_i^S = \lim_{\beta \nearrow 1} c_i^S(\beta) = h_i^0 - h_i^1 + \sum_{j \in N} (p_{ij}^0 - p_{ij}^1) \, f_j^S, \quad i \in N.$$

Thus, previous results carry over to the long-run average case.

6.6. *Optimal control subject to an activity constraint*

In applications, it is often of interest to impose a constraint on the mean rate of activity. See, e.g., [19] and the references therein. This is particularly relevant under the *long-run average* criterion discussed above, on which we focus next.

The *constrained control problem* of concern is to find a stationary policy minimizing cost measure $\bar{v}^u$, among those whose long-run average activity rate is $\bar{b}^u = t$:

$$\bar{v}_t = \min \left\{ \bar{v}^u : \bar{b}^u = t, u \in \mathcal{U} \right\}. \tag{6.26}$$

Assume the project is PCL-indexable as in Section 6.4, and let $\boldsymbol{\pi}$, $\boldsymbol{\nu}$ and S_k be its optimal $\mathcal{F}$-string, index vector and active sets. Suppose that, for some $1 \le k \le n$,

$$\bar{b}^{S_{k+1}} < t < \bar{b}^{S_k},$$

and let

$$p = \frac{t - \bar{b}^{S_{k+1}}}{\bar{b}^{S_k} - \bar{b}^{S_{k+1}}}, \qquad q = 1 - p.$$

Denote by (S_{k+1}, π_k, p) the stationary policy that is: active on states $j \in S_{k+1} \cup N^{\{1\}}$; active on state π_k with probability p; and passive otherwise. The next result follows immediately from Section 6.4, and hence its proof is omitted. See also Figure 6.2.

Proposition 6.5. *The following holds:*

(a) Policy (S_{k+1}, π_k, p) is optimal for problem (6.26); its optimal value is

$$\bar{v}_t = (1 - p)\, \bar{v}^{S_{k+1}} + p\, \bar{v}^{S_k}.$$

(b) Function $\bar{v}_t$ is piecewise linear concave on t, with

$$\frac{d}{dt}\bar{v}_t = \nu_{\pi_k}, \quad \bar{b}^{S_{k+1}} < t < \bar{b}^{S_k}.$$

Remark 6.5. Proposition 6.5(b) characterizes the index ν_{π_k} as a derivative of the optimal constrained value function v_t with respect to the required activity level t.

7. Admission control problem: PCL-indexability analysis

This section returns to the admission control model introduced in Section 2. We shall resolve the issues raised in Section 2.3 by deploying a PCL-indexability analysis. See the Appendix for important yet ancillary material relevant to this section.

In what follows, we shall write $\Delta x_i = x_i - x_{i-1}$,

$$d_i = \mu_i - \lambda_i,$$

and

$$\rho_i = \frac{\lambda_i}{\mu_{i+1}}.$$

We next state the regularity conditions we shall require of model parameters.

Assumption 7.1. *The following conditions hold:*
(i) *Concave nondecreasing d_i:* $0 \le \Delta d_{i+1} \le \Delta d_i, \quad 1 \le i \le n - 1$, *and* $\Delta d_1 > 0$.
(ii) *Convex nondecreasing h_i:* $\Delta h_{i+1} \ge \Delta h_i \ge 0, \quad 1 \le i \le n - 1$.

Remark 7.1. Assumption 7.1 is significantly less restrictive than Chen and Yao's conditions in [6]. Besides requiring d_i to be nondecreasing in their condition (5.5a), they require μ_i to be concave nondecreasing in their condition (5.5b). They further impose additional conditions, including *linearity* of holding costs.

Calculation of $w_i^{S_1}$:

$$\frac{w_0^{S_1}}{\lambda_0} = \frac{\alpha + \Delta d_1}{\alpha + \mu_1}; \quad \frac{w_i^{S_1}}{\lambda_i} = \frac{\alpha + \Delta d_{i+1} + \frac{w_{i-1}^{S_1}}{\rho_{i-1}}}{\alpha + \mu_{i+1}}, \quad 1 \le i \le n-1$$

Calculation of $w_i^{S_2}$:

$$\frac{w_0^{S_2}}{\lambda_0} = \frac{\alpha + \Delta d_1}{\alpha + \lambda_0 + \mu_1}; \quad \frac{w_i^{S_2}}{\lambda_i} = \frac{\alpha + \Delta d_{i+1} + \frac{w_{i-1}^{S_2}}{\rho_{i-1}}}{\alpha + \mu_{i+1}}, \quad 1 \le i \le n-1$$

Calculation of $w_i^{S_k c 1}$'s, for $2 \le k \le n$:

$$\frac{w_{k-1}^{S_{k+1}}}{\lambda_{k-1}} = \frac{1}{a_k} \frac{\alpha + \Delta d_k + \frac{w_{k-2}^{S_k}}{\rho_{k-2}}}{\alpha + \lambda_{k-1} + \mu_k}; \quad \frac{w_{k-2}^{S_{k+1}}}{\rho_{k-2}} = -(\alpha + \Delta d_k) + \frac{\alpha + \lambda_{k-1} + \mu_k}{\lambda_{k-1}} w_{k-1}^{S_{k+1}}$$

$$\frac{w_i^{S_{k+1}}}{\lambda_i} = \frac{\alpha + \Delta d_{i+1} + \frac{w_{i-1}^{S_{k+1}}}{\rho_{i-1}}}{\alpha + \mu_{i+1}}, \quad k \le i \le n-1$$

$$\frac{w_i^{S_{k+1}}}{\rho_i} = -(\alpha + \Delta d_{i+2}) + \frac{\alpha + \lambda_{i+1} + \mu_{i+2}}{\lambda_{i+1}} w_{i+1}^{S_{k+1}} - w_{i+2}^{S_{k+1}}, \quad 0 \le i \le k-3$$

Fig. 7.1. Recursive calculation of marginal workloads $w_i^{S_k}$

7.1. PCL-indexability analysis under the discounted criterion

We shall establish PCL-indexability of the model relative to the family of *threshold policies*, given by set system $(N^{\{0,1\}}, \mathcal{F})$, where $\mathcal{F} = \{S_1, \ldots, S_{n+1}\}$ is given by (2.5)–(2.6). *Activity measure* b_i^u is given by (2.2), which corresponds to letting $\theta_j^1 = \lambda_j/(\alpha + \Lambda)$ in (3.1), for $j \in N$, where Λ is the *uniformization rate* (cf. Appendix A).

We must first calculate marginal workload coefficients $w_i^{S_k}$, for which a complete recursion is given in Figure 7.1. It involves coefficients a_i, given by (B.3).

Proposition 7.1. *Marginal workloads w_i^S, for $i \in N^{\{0,1\}}$ and $S \in \mathcal{F}$, are calculated by the recursion shown in Figure 7.1.*

Proof. The result follows by reformulating in terms of the w_i^S's the equations on terms $\Delta b^S(i)$'s given in Lemma B.2 and Lemma B.5 in Appendix B.1, using identity (B.2). □

Remark 7.2. The recursion in Figure 7.1 further yields coefficients w_i^S when $\alpha = 0$. These are the *long-run average marginal workloads* discussed in Section 7.2.

The next result establishes the required properties of marginal workloads.

Proposition 7.2 (Positive nondecreasing w_i^S's). *Under Assumption 7.1(i):*
(a) $w_i^S > 0$, for $i \in N^{\{0,1\}}$, $S \in \mathcal{F}$, and hence Assumption 6.1 holds.
(b) w_i^S is nondecreasing on $S \in \mathcal{F}$, for $i \in S$ fixed, and hence Assumption 4.6 holds.

$$
\begin{array}{ccccccccc}
\boxed{w_0^{S_1}} & > & \boxed{w_0^{S_2}} & < & \cdots & \cdots & \cdots & < & w_0^{S_{n+1}} \\
\downarrow & & \downarrow & \searrow & & & & & \uparrow \\
w_1^{S_1} & > & w_1^{S_2} & > & \ddots & \cdots & \cdots & < & w_1^{S_{n+1}} \\
\downarrow & & \downarrow & & & \ddots & & & \uparrow \\
\vdots & & \vdots & & & & \ddots & & \vdots \\
\downarrow & & \downarrow & & & & & \searrow & \uparrow \\
w_{n-1}^{S_1} & > & w_{n-1}^{S_2} & > & \cdots & \cdots & \cdots & > & \boxed{w_{n-1}^{S_{n+1}}}
\end{array}
$$

Fig. 7.2. Relations between marginal workloads $w_i^{S_k}$

Proof. Both parts follow directly from Lemma B.6 in Appendix B.1. □

Figure 7.2 illustrates the recursions and inequalities established in Appendix B.1 on marginal workloads (arrows indicate the direction of calculations). *Pivot* terms, forming the backbone of the recursion, are enclosed in boxes.

Marginal cost analyses are given in Appendix B.2, yielding the following recursion.

Proposition 7.3. *Marginal costs $c_k^{S_{k+2}}$, for $0 \le k \le n-1$, are calculated by*

$$c_0^{S_2} = \frac{\lambda_0}{\alpha + \lambda_0 + \mu_1}\, \Delta h_1$$

$$c_k^{S_{k+2}} = \frac{\lambda_k}{a_{k+1}} \frac{\Delta h_{k+1} + \dfrac{c_{k-1}^{S_{k+1}}}{\rho_{k-1}}}{\alpha + \lambda_k + \mu_{k+1}}, \quad 1 \le k \le n-1.$$

We are now ready to establish the model's PCL-indexability, and to calculate its indices. Construct $\nu_0, \ldots, \nu_{n-1}$ recursively by

$$\nu_0 = \frac{\Delta h_1}{\alpha + \Delta d_1}$$
$$\nu_j = \nu_{j-1} + \frac{\Delta h_{j+1} - \nu_{j-1}\,(\alpha + \Delta d_{j+1})}{\alpha + \Delta d_{j+1} + \dfrac{w_{j-1}^{S_{j+1}}}{\rho_{j-1}}}, \quad 1 \le j \le n-1. \tag{7.1}$$

We shall need the following preliminary result.

Lemma 7.1. *Under Assumption 7.1, the following holds:*

(a) $\dfrac{\Delta h_j}{\alpha + \Delta d_j} \le \dfrac{\Delta h_{j+1}}{\alpha + \Delta d_{j+1}}, \quad 1 \le j \le n-1.$

(b) $\nu_j \le \dfrac{\Delta h_{j+1}}{\alpha + \Delta d_{j+1}}, \quad 0 \le j \le n-1.$

(c) $\nu_0 \le \nu_1 \le \cdots \le \nu_{n-1}.$

Proof. (a) The result follows directly from Assumption 7.1.

(b) Proceed by induction on j. The case $j = 0$ holds by (7.1). Suppose now

$$\nu_{j-1} \leq \frac{\Delta h_j}{\alpha + \Delta d_j}.$$

It then follows, by part (a), that

$$\nu_{j-1} \leq \frac{\Delta h_{j+1}}{\alpha + \Delta d_{j+1}}.$$

Notice now that the last identity in (7.1) can be reformulated as

$$\nu_j = \frac{\Delta h_{j+1}}{\alpha + \Delta d_{j+1}} + \frac{\frac{w_{j-1}^{S_{j+1}}}{\rho_{j-1}}}{\alpha + \Delta d_{j+1} + \frac{w_{j-1}^{S_{j+1}}}{\rho_{j-1}}} \left[\nu_{j-1} - \frac{\Delta h_{j+1}}{\alpha + \Delta d_{j+1}} \right].$$

Since $\alpha + \Delta d_{j+1} > 0$ and $w_{j-1}^{S_{j+1}} > 0$, it follows from the last identity that

$$\nu_{j-1} \leq \frac{\Delta h_{j+1}}{\alpha + \Delta d_{j+1}} \Longleftrightarrow \nu_j \leq \frac{\Delta h_{j+1}}{\alpha + \Delta d_{j+1}},$$

which completes the induction.

(c) This follows from parts (a) and (b), together with (7.1).

□

We are now ready to establish the main result of this section.

Theorem 7.2 (PCL-indexability: discounted criterion). *Under Assumption 7.1, the admission control model is PCL-indexable relative to threshold policies and rejection measure b^u. Its dynamic allocation indices are the ν_j's given by (7.1), and satisfy (6.23).*

Proof. Using (4.16) and Proposition 6.4(b), we must show that

$$\nu_j = \frac{c_j^{S_{j+2}}}{w_j^{S_{j+2}}}, \quad 0 \leq j \leq n-1.$$

This readily follows by induction on j, drawing on Proposition 7.1 and Proposition 7.3. Furthermore, Proposition 7.2 and Theorem 6.23 imply that the index satisfies (6.23).

□

7.2. PCL-indexability under the time-average criterion

As in Section 6.5, the PCL-indexability analysis above extends to the long-run average version of the admission control model. The relevant rejection and cost measures are

$$\bar{b}^u = \lim_{T\to\infty} \frac{1}{T} E^u \left[\int_0^T \lambda_{L(t)}\, a(t)\, dt \right],$$

$$\bar{v}^u = \lim_{T\to\infty} \frac{1}{T} E^u \left[\int_0^T h_{L(t)}\, dt \right].$$

The limiting values of $w_j^{S_k}$, $c_j^{S_k}$ and ν_j as $\alpha \searrow 0$ (equivalent to letting $\beta \nearrow 1$ in Section 6.5) are obtained by setting $\alpha = 0$ in the given recursions. The next result follows.

Corollary 7.1 (PCL-indexability: time-average criterion). *Under Assumption 7.1, the admission control model, under the long-run average criterion, is PCL-indexable relative to threshold policies and rejection measure $\bar{b}^u$. Its indices satisfy*

$$\bar{\nu}_j = \max \left\{ \frac{\bar{v}^{S\setminus\{j\}} - \bar{v}^S}{\bar{b}^S - \bar{b}^{S\setminus\{j\}}} : j \in S \in \{S_1, \ldots, S_n\} \right\}, \quad j \in N^{\{0,1\}}.$$

7.3. The case $\lambda_j = \lambda$, $\mu_j = \mu$, $\alpha = 0$

This section derives the long-run average indices when $\lambda_j = \lambda$, $\mu_j = \mu$. Note that $\rho_j = \rho = \lambda/\mu$. As we shall see in Section 8, the case $\rho > 1$ is often of interest in applications.

The following results follow easily by induction, and hence we omit their proof. Note first that the coefficients a_j, defined by (B.3), are given, for $1 \le j \le n-1$, by

$$a_j = \frac{1}{1+\rho} \frac{1+\cdots+\rho^j}{1+\cdots+\rho^{j-1}} = \begin{cases} \dfrac{1}{1+\rho} \dfrac{\rho^{j+1}-1}{\rho^j - 1} & \text{if } \rho \neq 1 \\[2ex] \dfrac{1}{2} \dfrac{j+1}{j} & \text{if } \rho = 1. \end{cases}$$

Regarding marginal workloads, we have

$$\begin{aligned} w_j^{S_1} &= \lambda, \quad 1 \le j \le n-1 \\ w_0^{S_2} &= \frac{1}{1+\rho} \lambda \\ w_{j-1}^{S_{j+1}} &= \frac{1}{1+\rho} \frac{w_{j-2}^{S_j}}{a_j}, \quad 2 \le j \le n. \end{aligned}$$

Such recursion gives

$$w_{j-1}^{S_{j+1}} = \frac{\lambda}{(1+\rho)^j \prod_{i=1}^{j} a_i} = \frac{\lambda}{1+\cdots+\rho^j}, \quad 1 \le j \le n.$$

Hence, index recursion (7.1) reduces to

$$\nu_0 = \frac{\Delta h_1}{\mu}$$
$$\nu_j = \nu_{j-1} + \Delta h_{j+1} \frac{1+\cdots+\rho^j}{\mu}, \quad 1 \le j \le n-1,$$

which yields

$$\nu_j = \frac{1}{\mu}\sum_{i=1}^{j+1} \Delta h_i \left(1+\cdots+\rho^{i-1}\right) = \begin{cases} \frac{1}{\mu}\sum_{i=1}^{j+1} \Delta h_i \frac{\rho^i - 1}{\rho - 1} & \text{if } \rho \neq 1 \\ \frac{1}{\mu}\sum_{i=1}^{j+1} i \, \Delta h_i & \text{if } \rho = 1. \end{cases} \tag{7.2}$$

Remark 7.3. A consequence of (7.2) is that, in this setting, *the index is monotonic, and hence the model is PCL-indexable, under the relaxed assumption that cost rates h_i be only nondecreasing*: they need *not* be convex as in Assumption 7.1(ii).

In the *linear cost* case $h_j = h\, j$, we obtain

$$\nu_j = \frac{h}{\mu}\sum_{i=1}^{j+1}\left(1+\cdots+\rho^{i-1}\right) = \begin{cases} \frac{h}{\mu}\left[\frac{\rho^{j+2}-1}{(\rho-1)^2} - \frac{j+2}{\rho-1}\right] & \text{if } \rho \neq 1 \\ \frac{h}{\mu}\frac{(j+1)\,(j+2)}{2} & \text{if } \rho = 1. \end{cases} \tag{7.3}$$

In the *quadratic cost* case $h_j = h\, j^2$, we obtain, when $\rho \neq 1$,

$$\nu_j = \frac{h}{\mu}\left[\left(\frac{2j+1}{(\rho-1)^2} - \frac{2}{(\rho-1)^3}\right)\rho^{j+2} - \frac{j\,(j+2)}{\rho-1} + \frac{3}{(\rho-1)^2} + \frac{2}{(\rho-1)^3}\right] \tag{7.4}$$

and, when $\rho = 1$,

$$\nu_j = \frac{h}{\mu}\frac{(j+1)\;(j+2)\;(4j+3)}{6}.$$

8. Applications to routing and make-to-stock scheduling in queueing systems

In this section we apply the admission control index obtained in Section 7 to develop new *heuristic* index policies for two hard queueing control problems.

8.1. An index policy for admission control and routing to parallel queues

Consider a system at which customers arrive as a Poisson stream with rate λ. Upon arrival, a customer may be either rejected, or routed to one of m queues for service. Queue k has a finite buffer holding at most n_k customers. Its service times are exponential, with rate $\mu_k(j_k)$ when it holds $L_k(t) = j_k$ customers at time $t \geq 0$, for $j_k \in N_k = \{0, \ldots, n_k\}$. When all buffers are full, an arriving customer is lost.

Customers in queue k incur holding costs at rate $h_k(j_k)$ while $L_k(t) = j_k$, discounted in time at rate $\alpha > 0$. Furthermore, a *rejection charge* ν is incurred per lost customer. The problem of concern is to find a stationary *admission control and routing policy* prescribing whether to admit each arriving customer and, if so, to which nonfull queue to route it, in order to minimize the expected total discounted sum of holding costs and rejection charges incurred over an infinite horizon.

We assume model parameters satisfy the following conditions.

Assumption 8.1. *For* $1 \leq k \leq m$, *the following holds:*
(i) *Concave nondecreasing* $\mu_k(j_k)$: $0 \leq \Delta\mu_k(j_k + 1) \leq \Delta\mu_k(j_k), \quad 1 \leq j_k < n_k$.
(ii) *Convex nondecreasing* $h_k(j_k)$: $0 \leq \Delta h_k(j_k) \leq \Delta h_k(j_k + 1), \quad 1 \leq j_k < n_k$.

We aim to design a well-grounded and tractable *heuristic* policy, for which we shall use the *admission control index* developed in Section 7. The idea is to note that this model is an RBP made up of m single-queue *admission control RBs* as studied before where, *at each time, at most one of the m entry gates must be open.*

Let $\nu_k(j_k)$ be queue k's admission control index, representing the *fair rejection charge* for a customer finding queue k in state $j_k < n_k$. Such interpretation leads to the following *admission control and routing index policy*:

1. Route an arriving customer to a nonfull queue k whose current state $j_k < n_k$ has the *smallest index* $\nu_k(j_k)$ satisfying $\nu_k(j_k) < \nu$, if any is available.
2. Otherwise, reject the customer.

In the case where queues are symmetric (ignoring possibly different buffer lengths), and the admission control capability is removed (by letting $\nu = \infty$), such policy reduces to the celebrated *shortest queue routing* policy. The latter is known to be optimal under appropriate assumptions. See [38, 18, 20].

In the case of constant service rates $\mu_k(j_k) = \mu_k$ and linear holding costs $h_k(j_k) = h_k\, j_k$, under the long-run average criterion ($\alpha = 0$), identity (7.3) yields the *routing index*

$$\nu_k(j_k) = \frac{h_k}{\mu_k}\left[\frac{\rho_k^{j_k+2} - 1}{(\rho_k - 1)^2} - \frac{j_k + 2}{\rho_k - 1}\right], \tag{8.1}$$

where $\rho_k = \lambda/\mu_k$. The *heavy traffic* case $\rho_k > 1$, where each queue lacks the capacity to process all the traffic, is of considerable interest in applications; in such case, when there are 2 queues, the *switching curve* in state space (j_1, j_2) determined by such policy is *asymptotically linear with limiting slope* $\ln \rho_1 / \ln \rho_2$ as $j_1, j_2 \to \infty$. The index policy above readily extends to models with *infinite buffers*.

Note that the standard heuristic in the linear cost case routes customers to the queue with smallest index $\hat{\nu}_k(j_k) = h_k\,(j_k + 1)/\mu_k$.

8.2. An index policy for scheduling a multiclass make-to-stock queue with lost sales

We next consider a model for scheduling a multiclass make-to-stock queue (cf. [5, Ch. 4]) in the *lost sales* case, which extends a simpler model studied by Veatch and Wein in [35] (having constant production and demand rates, and linear holding costs).

A flexible production facility makes m products, labeled by $k = 1, \ldots, m$, in a make-to-stock mode. The facility can work on at most an item at a time. Finished product k items are stored in a dedicated stock, holding up to n_k items. When this contains $L_k(t) = j_k$ units, the facility can work at rate $\mu_k(j_k)$ on such products, and corresponding *customer orders* arrive at rate $\lambda_k(j_k)$. We assume mutually independent, exponential production and interarrival times. A product k's order is immediately filled from stock if $j_k \geq 1$, and is otherwise lost. At each time, the facility can either stay idle, or engage in production of an item, by following a stationary policy.

Product k incurs state-dependent *stock holding costs*, at rate $c_k(j_k)$ per unit time; *stockout costs*, at rate s_k per lost order; and is sold for a state-dependent *price* $r_k(j_k)$. The resulting product k's *net cost rate* per unit time in state j_k is thus

$$h_k(j_k) = c_k(j_k) + s_k\,\lambda_k(0)\,1\{j_k = 0\} - r_k(j_k)\,\lambda_k(j_k)\,1\{j_k > 0\}.$$

We further assume that *production is subsidized at rate ν per completed item*. Costs and rewards are discounted in time at rate $\alpha > 0$.

We shall assume that model parameters satisfy the following conditions (cf. Assumption 7.1). Let $d_k(j_k) = \lambda_k(j_k) - \mu_k(j_k)$ for $j_k \geq 1$. For consistency with previous analyses, write $\Delta d_k(1) = \lambda_k(1) - \Delta\mu_k(1)$.

Assumption 8.2. *For $1 \leq k \leq m$, the following holds:*
(i) *Concave nondecreasing $d_k(j_k)$:* $0 \leq \Delta d_k(j_k + 1) \leq \Delta d_k(j_k)$, $\quad 1 \leq j_k < n_k$, *and* $\Delta d_k(1) > 0$.
(iii) *Convex nondecreasing $h_k(j_k)$:* $0 \leq \Delta h_k(j_k) \leq \Delta h_k(j_k + 1)$, $\quad 0 \leq j_k < n_k$.

The goal is to design a state-dependent *production scheduling policy*, which dynamically prescribes whether to engage in production and, if so, of which product, so as to minimize the expected total discounted value of costs accrued over an infinite horizon.

The admission control index derived before readily yields a heuristic index policy for such problem. The idea is to note that the present model is an RBP made up of m single-queue admission control projects as studied before, *with the roles of parameters λ's and μ's interchanged*. Thus, opening queue k's *entry gate* corresponds to making product k. One must then, *at each time, open at most one entry gate*.

Let $\nu_k(j_k)$ be queue k's admission control index, representing the *critical production subsidy* under which one should be indifferent between idling and making product k in state j_k. Such interpretation leads to the following *production control index policy*:

1. Make a product k with a nonfull stock level $j_k < n_k$ having the *smallest index* $\nu_k(j_k)$ satisfying $\nu_k(j_k) < \nu$, if any is available.
2. Otherwise, idle the facility.

Note that one may equivalently regard $-\nu$ as a *production cost rate per completed item*. Hence, the indices $-\nu_k(j_k)$ represent *critical production costs* for product k. Note further that, in the case of identical products, such policy prescribes to make the product k having the *least stock* j_k available, as long as $\nu_k(j_k) < \nu$.

We next draw on the results in Section 7.3 to give explicit formulae for the index in some special cases, corresponding to constant arrival and service rates $\lambda_k(j_k) = \lambda_k$, $\mu_k(j_k) = \mu_k$, under the *long-run average* criterion $\alpha = 0$. Let $\rho_k = \lambda_k/\mu_k \neq 1$.

Consider first the case of *linear stock holding costs* and *constant selling prices*,

$$h_k(j_k) = c_k\, j_k + s_k\, \lambda_k\, 1\{j_k = 0\} - r_k\, \lambda_k\, 1\{j_k > 0\}.$$

The results in Section 7.3 then yield the *production index*

$$\nu_k(j_k) = \frac{c_k}{\mu_k}\left[\frac{\rho_k^{-j_k-1} - 1}{(1-\rho_k)^2} - \frac{j_k+1}{1-\rho_k}\right] - r_k - s_k. \tag{8.2}$$

Remark 8.1.

1. The index (8.2) equals Whittle's in [35] *scaled by factor* $1/\mu_k$. Yet, although both indices give the same (optimal) policy for a single-product problem, such factor causes them to give *distinct* policies for the multi-product problem if the μ_k's differ.
2. The index policy idles the facility when the number of units in stock for each product lies at or above a corresponding critical *base-stock* level. The *idling policy* is thus characterized by the *hedging-point* (cf. [35]) consisting of such base-stocks.
3. The index in (8.2) also gives a policy for a model with *unlimited storage capacity* ($n_k = \infty$). In such setting, if $\rho_k > 1$ for *some* product k, then $\nu_k(j_k) < 0$. Hence, in the case $\nu = 0$, the facility will never idle.

Consider next the case where stock holding costs are *quadratic*, so that

$$h_k(j_k) = c_k\, j_k^2 + s_k\, \lambda_k\, 1\{j_k = 0\} - r_k\, \lambda_k\, 1\{j_k > 0\}.$$

One then obtains, via (7.4), the production index

$$\begin{aligned}\nu_k(j_k) = \frac{c_k}{\mu_k}&\left[\left(\frac{2j_k+3}{(1-\rho_k)^2} - \frac{2}{(1-\rho_k)^3}\right)\rho_k^{-j_k-1}\right.\\ &\left.-\frac{(j_k+1)^2}{1-\rho_k} - \frac{1}{(1-\rho_k)^2} + \frac{2}{(1-\rho_k)^3}\right] - r_k - s_k.\end{aligned} \tag{8.3}$$

9. Concluding remarks

We have developed a polyhedral approach to the development of dynamic allocation indices in a variety of stochastic scheduling problems. In our view, such results offer a glimpse of the untapped potential which polyhedral methods have to offer in the field of stochastic optimization. We highlight two avenues for further research, which are the subject of ongoing work: test empirically the proposed heuristic index policies, as in [3]; and provide approximate and asymptotic analyses of their performance, as in [16].

A. Discrete-time reformulation

We reformulate the model of concern into discrete time by deploying the standard *uniformization* technique (cf. [22]), which proceeds in two steps: (i) the original process $L(t)$ is reformulated into an equivalent *uniformized process* $\tilde{L}(t)$, having *uniform transition rate* Λ; process $\tilde{L}(t)$ is obtained by sampling $L(t)$ at time epochs corresponding to a Poisson process with rate Λ; these includes *real* as well as *virtual* transitions, in which no state change occurs; and (ii) process $\tilde{L}(t)$ is reformulated into a *discrete-time process* $X(t)$, by viewing inter-transition intervals as discrete time periods.

Note that $\Lambda > 0$ is a valid *uniform transition rate* iff it satisfies

$$\lambda_i + \mu_i \leq \Lambda, \quad i \in N.$$

The resulting discrete-time process $X(t)$, for $t = 0, 1, \ldots$, is an RB (cf. Section 3) characterized by the following elements:

-State space: $N = \{0, 1, \ldots, n\}$; $N^{\{0,1\}} = \{0, \ldots, n-1\}$; $N^{\{1\}} = \{n\}$.
-Actions: $a = 0$ (passive; open entry gate) and $a = 1$ (active; shut entry gate).
-Transition probability matrices: Under action $a = 1$,

$$\mathbf{P}^1 = \frac{1}{\Lambda} \begin{bmatrix} \Lambda & & & \\ \mu_1 & \Lambda - \mu_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \ddots \\ & & \mu_n & \Lambda - \mu_n \end{bmatrix};$$

and, under action $a = 0$,

$$\mathbf{P}^0 = \frac{1}{\Lambda} \begin{bmatrix} \Lambda - \lambda_0 & \lambda_0 & & & \\ \mu_1 & \Lambda - \lambda_1 - \mu_1 & \lambda_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & \mu_n & \Lambda - \mu_n \end{bmatrix}.$$

-One-period holding costs: $\mathbf{c}^0 = \mathbf{c}^1 = \dfrac{1}{\alpha + \Lambda} \mathbf{h}$.

-Discount factor: $\beta = \dfrac{\Lambda}{\alpha + \Lambda}$.

B. Marginal workload and cost analysis

B.1. Marginal workloads: calculation and properties

We next address the tasks of calculating marginal workloads $w_i^{S_k}$ for the admission control model, and of establishing their required properties.

Calculation of scaled $w_i^{S_k}$'s

To avoid dependence on uniformization rate Λ, the coefficients w_i^S we shall calculate correspond to those defined by (6.9) after *scaling* by factor $\alpha + \Lambda$. Since

$$\mathbf{P}^1 - \mathbf{P}^0 = \frac{1}{\Lambda} \begin{bmatrix} \lambda_0 & -\lambda_0 & & & & \\ & \lambda_1 & -\lambda_1 & & & \\ & & \ddots & \ddots & & \\ & & & \lambda_{n-1} & -\lambda_{n-1} \\ & & & & 0 & 0 \end{bmatrix}, \qquad \text{(B.1)}$$

we have

$$w_i^S = \begin{cases} \lambda_i \left[1 - \Delta b_{i+1}^S\right] & \text{if } 0 \le i \le n-1 \\ 0 & \text{if } i = n. \end{cases} \qquad \text{(B.2)}$$

Calculation of the w_i^S's thus reduces to that of the Δb_i^S's. To study the latter, we start by characterizing the coefficients $b_i^{S_k}$, through their defining equations in (6.8). We shall denote by λ_i^S the *birth rate* in state i under the S-active policy, i.e.,

$$\lambda_i^S = \lambda_i \, 1\{i \in N^{\{0,1\}} \setminus S\}, \quad i \in N.$$

Note that $\lambda_i^{S_k} = \lambda_i \, 1\{0 \le i < k-1\}$, for $1 \le k \le n+1$.

Lemma B.1. *For $1 \le k \le n+1$, coefficients $b_i^{S_k}$ are characterized by the equations*

$$(\alpha + \Lambda)\, b_0^{S_k} = \lambda_0 - \lambda_0^{S_k} + (\Lambda - \lambda_0^{S_k})\, b_0^{S_k} + \lambda_0^{S_k}\, b_1^{S_k}$$
$$(\alpha + \Lambda)\, b_i^{S_k} = \lambda_i - \lambda_i^{S_k} + \mu_i\, b_{i-1}^{S_k} + (\Lambda - \lambda_i^{S_k} - \mu_i)\, b_i^{S_k} + \lambda_i^{S_k}\, b_{i+1}^{S_k}, \ 1 \le i \le n-1$$
$$(\alpha + \Lambda)\, b_n^{S_k} = \lambda_n + \mu_n\, b_{n-1}^{S_k} + (\Lambda - \mu_n)\, b_n^{S_k}.$$

The next result, characterizing coefficients $\Delta b_i^{S_k}$, follows immediately.

Lemma B.2. *For $1 \le k \le n+1$, coefficients $\Delta b_i^{S_k}$ are characterized by the equations*

$$(\alpha + \lambda_0^{S_k} + \mu_1)\, \Delta b_1^{S_k} = \Delta\lambda_1 - \Delta\lambda_1^{S_k} + \lambda_1^{S_k}\, \Delta b_2^{S_k}$$
$$(\alpha + \lambda_{i-1}^{S_k} + \mu_i)\, \Delta b_i^{S_k} = \Delta\lambda_i - \Delta\lambda_i^{S_k} + \mu_{i-1}\, \Delta b_{i-1}^{S_k} + \lambda_i^{S_k}\, \Delta b_{i+1}^{S_k}, \ 2 \le i \le n-1$$
$$(\alpha + \lambda_{n-1}^{S_k} + \mu_n)\, \Delta b_n^{S_k} = \Delta\lambda_n + \lambda_{n-1}^{S_k} + \mu_{n-1}\, \Delta b_{n-1}^{S_k}.$$

We next develop a recursive procedure to solve the equations in Lemma B.2, based on the following observations: (i) the equations give

$$\Delta b_1^{S_1} = \frac{\Delta \lambda_1}{\alpha + \mu_1},$$

from which remaining $\Delta b_i^{S_1}$'s are calculated; (ii) for $1 \leq k \leq n$, once *pivot coefficient* $\Delta b_k^{S_{k+1}}$ is available, they give the remaining $\Delta b_i^{S_{k+1}}$'s; and (iii) the first pivot is

$$\Delta b_1^{S_2} = \frac{\lambda_1}{\alpha + \lambda_0 + \mu_1}.$$

Hence, if we can express pivot $\Delta b_{k+1}^{S_{k+2}}$ in terms of $\Delta b_k^{S_{k+1}}$, for $1 \leq k \leq n-1$, this would complete a recursion to calculate all coefficients $\Delta b_i^{S_k}$.

We next seek to relate successive pivots, drawing on [6]. Consider, for $1 \leq k \leq n-1$, the vectors (where $\mathbf{x}^T$ denotes the transpose of vector $\mathbf{x}$)

$$\begin{aligned}
\Delta \mathbf{b}^k &= \left(\Delta b_1^{S_{k+1}}, \ldots, \Delta b_k^{S_{k+1}}\right)^T \\
\Delta \hat{\mathbf{b}}^k &= \left(\Delta b_1^{S_{k+2}}, \ldots, \Delta b_k^{S_{k+2}}\right)^T \\
\mathbf{b}^k &= \frac{\lambda_k}{\alpha + \lambda_{k-1} + \mu_k}\, \mathbf{e}_k \\
\hat{\mathbf{b}}^k &= \frac{\lambda_k\, \Delta b_{k+1}^{S_{k+2}}}{\alpha + \lambda_{k-1} + \mu_k}\, \mathbf{e}_k,
\end{aligned}$$

where $\mathbf{e}_k$ is the kth unit coordinate vector in $\mathbb{R}^k$. Let further $\mathbf{B}^k$ be the $k \times k$ matrix

$$\mathbf{B}^k = \begin{bmatrix}
0 & \frac{\lambda_1}{\alpha+\lambda_0+\mu_1} & & & \\
\frac{\mu_1}{\alpha+\lambda_1+\mu_2} & 0 & \frac{\lambda_2}{\alpha+\lambda_1+\mu_2} & & \\
 & \ddots & \ddots & \ddots & \\
 & & \ddots & \ddots & \ddots \\
 & & & \frac{\mu_{k-1}}{\alpha+\lambda_{k-1}+\mu_k} & 0
\end{bmatrix},$$

with $\mathbf{B}^1 = 0$. The next result reformulates some equations in Lemma B.2.

Lemma B.3. *For* $1 \leq k \leq n-1$*:*

(a) $\Delta \mathbf{b}^k = \mathbf{b}^k + \mathbf{B}^k\, \Delta \mathbf{b}^k$.
(b) $\Delta \hat{\mathbf{b}}^k = \hat{\mathbf{b}}^k + \mathbf{B}^k\, \Delta \hat{\mathbf{b}}^k$.

To proceed, introduce coefficients

$$a_k = \begin{cases} 1 & \text{if } k = 1 \\[2ex] \dfrac{\det\left(\mathbf{I} - \mathbf{B}^k\right)}{\det\left(\mathbf{I} - \mathbf{B}^{k-1}\right)} & \text{if } 2 \leq k \leq n. \end{cases} \tag{B.3}$$

Lemma B.4. *Under Assumption 7.1(i), the following holds:*

(a) $a_k > 0$, *for* $1 \le k \le n$.

(b) The a_k*'s can be computed recursively by letting* $a_1 = 1$ *and*

$$a_k = 1 - \frac{\lambda_{k-1}\,\mu_{k-1}}{(\alpha+\lambda_{k-2}+\mu_{k-1})\,(\alpha+\lambda_{k-1}+\mu_k)}\,\frac{1}{a_{k-1}}, \quad 2 \le k \le n.$$

(c) $\dfrac{\alpha+\mu_k}{\alpha+\lambda_{k-1}+\mu_k} < a_k < 1, \quad 2 \le k \le n.$

Proof. (a) Under Assumption 7.1(i) the row sums of $\mathbf{B}^k$ are less than unity, and hence so is its spectral radius. It follows that $\det\left(\mathbf{I}-\mathbf{B}^k\right) > 0$, which proves the result.

(b) The recursion follows from the definition of a_k and the identity

$$\det(\mathbf{I}-\mathbf{B}^k) = \det(\mathbf{I}-\mathbf{B}^{k-1}) - \frac{\lambda_{k-1}\,\mu_{k-1}}{(\alpha+\lambda_{k-2}+\mu_{k-1})\,(\alpha+\lambda_{k-1}+\mu_k)}\,\det(\mathbf{I}-\mathbf{B}^{k-2})$$

(c) Let $2 \le k \le n$. It follows from (a) and (b) that $a_k < 1$. We next show that

$$a_k > \frac{\alpha+\mu_k}{\alpha+\lambda_{k-1}+\mu_k}, \quad 1 \le k \le n,$$

by induction on k. The case $k = 1$ is trivial. Assume the result holds for $k-1$, i.e.,

$$a_{k-1} > \frac{\alpha+\mu_{k-1}}{\alpha+\lambda_{k-2}+\mu_{k-1}}.$$

Then, part (b) and the induction hypothesis yield

$$\begin{aligned} a_k &= 1 - \frac{\lambda_{k-1}}{\alpha+\lambda_{k-1}+\mu_k}\,\frac{\dfrac{\mu_{k-1}}{\alpha+\lambda_{k-2}+\mu_{k-1}}}{a_{k-1}} \\ &> 1 - \frac{\lambda_{k-1}}{\alpha+\lambda_{k-1}+\mu_k} = \frac{\alpha+\mu_k}{\alpha+\lambda_{k-1}+\mu_k}, \end{aligned}$$

which completes the proof.

□

We are now ready to relate successive pivots.

Lemma B.5. *For* $1 \le k \le n-1$,

$$a_{k+1}\left[1-\Delta b_{k+1}^{S_{k+2}}\right] = \frac{\alpha+\Delta d_{k+1}+\mu_k\left[1-\Delta b_k^{S_{k+1}}\right]}{\alpha+\lambda_k+\mu_{k+1}};$$

or, equivalently,

$$w_k^{S_{k+2}} = \frac{\lambda_k}{a_{k+1}}\,\frac{\alpha+\Delta d_{k+1}+\dfrac{w_{k-1}^{S_{k+1}}}{\rho_{k-1}}}{\alpha+\lambda_k+\mu_{k+1}}.$$

Proof. Fix $1 \le k \le n-1$. By Lemma B.3 and the definitions of $\mathbf{h}^k$, $\hat{\mathbf{h}}^k$, we have

$$\begin{aligned}\Delta \mathbf{b}^k - \Delta \hat{\mathbf{b}}^k &= (\mathbf{I} - \mathbf{B}^k)^{-1} (\mathbf{b}^k - \hat{\mathbf{b}}^k) \\ &= \frac{\lambda_k \left[1 - \Delta b_{k+1}^{S_{k+2}}\right]}{\alpha + \lambda_{k-1} + \mu_k} (\mathbf{I} - \mathbf{B}^k)^{-1} \mathbf{e}^k.\end{aligned} \tag{B.4}$$

Now, noting that the element in position (k, k) of matrix $\left(\mathbf{I} - \mathbf{B}^k\right)^{-1}$ is

$$\frac{\det\left(\mathbf{I} - \mathbf{B}^{k-1}\right)}{\det\left(\mathbf{I} - \mathbf{B}^k\right)},$$

which by definition equals $1/a_k$, it follows from the last identity above that

$$\Delta b_k^{S_{k+1}} - \Delta b_k^{S_{k+2}} = \frac{1}{a_k} \frac{\lambda_k \left(1 - \Delta b_{k+1}^{S_{k+2}}\right)}{\alpha + \lambda_{k-1} + \mu_k}. \tag{B.5}$$

Combining the previous identity with

$$\Delta b_{k+1}^{S_{k+2}} = \frac{\lambda_{k+1}}{\alpha + \lambda_k + \mu_{k+1}} + \frac{\mu_k}{\alpha + \lambda_k + \mu_{k+1}} \Delta b_k^{S_{k+2}},$$

(cf. Lemma B.2), and substituting for a_k in terms of a_{k+1} (cf. Lemma B.4), yields the required identities (after straightforward algebra).

□

Properties of marginal workloads

We next set out to establish properties of marginal workloads which are invoked in Section 7.

Lemma B.6. *Under Assumption 7.1(i), the following holds, for $\alpha \ge 0$:*

(a) $w_{k-1}^{S_{k+1}} > 0, \quad 1 \le k \le n.$

(b) $w_{i-1}^{S_{k+2}} > w_{i-1}^{S_{k+1}}, \quad 1 \le i \le k \le n-1.$

(c) $w_{i-1}^{S_{k+1}} > 0 \Longrightarrow w_i^{S_{k+1}} > 0, \quad 1 \le k \le i \le n-1.$

(d) $w_i^{S_{k+1}} > w_i^{S_{k+2}}, \quad 1 \le k \le i \le n-1.$

Proof. (a) Proceed by induction on k. The case $k = 1$ holds by the expression for $w_0^{S_2}$ in Figure 7.1 and Assumption 7.1(i). Suppose now $w_{k-2}^{S_k} > 0$. We have

$$\frac{w_{k-1}^{S_{k+1}}}{\lambda_{k-1}} = \frac{1}{a_k} \frac{\alpha + \Delta d_k + \dfrac{w_{k-2}^{S_k}}{\rho_{k-2}}}{\alpha + \lambda_{k-1} + \mu_k} > 0,$$

where the identity is taken from Figure 7.1, and the inequality follows from the induction hypothesis, along with Assumption 7.1(i) and $a_k > 0$ (Lemma B.4).

(b) Using (B.2), we can rewrite identity (B.4) as

$$\Delta \mathbf{b}^k - \Delta \hat{\mathbf{b}}^k = \frac{w_k^{S_{k+2}}}{\alpha + \lambda_{k-1} + \mu_k} \, (\mathbf{I} - \mathbf{B}^k)^{-1} \, \mathbf{e}_k.$$

Now, since the spectral radius of $\mathbf{B}^k$ is less than unity (cf. Lemma B.4's proof), matrix $\left(\mathbf{I} - \mathbf{B}^k\right)^{-1}$ is positive componentwise, and hence $\left(\mathbf{I} - \mathbf{B}^k\right)^{-1} \mathbf{e}_k > \mathbf{0}$. Combining this with part (b) and the last identity above yields $\Delta \mathbf{b}^k - \Delta \hat{\mathbf{b}}^k > \mathbf{0}$, i.e.,

$$\Delta b_i^{S_{k+1}} > \Delta b_i^{S_{k+2}}, \quad 1 \le i \le k.$$

By (B.2), these inequalities give the required result.

(c) The result follows from

$$\frac{w_i^{S_{k+1}}}{\lambda_i} = \frac{\alpha + \Delta d_{i+1} + \dfrac{w_{i-1}^{S_{k+1}}}{\rho_{i-1}}}{\alpha + \mu_{i+1}}, \quad k \le i \le n-1$$

(cf. Figure 7.1), and Assumption 7.1(i).

(d) By (B.2), the result is equivalent to

$$\Delta b_{i+1}^{S_{k+1}} < \Delta b_{i+1}^{S_{k+2}}, \quad 1 \le k \le i \le n-1. \tag{B.6}$$

Now, it follows from Lemma B.2 that, for $1 \le k \le n-1$,

$$(\alpha + \mu_i) \, \Delta b_i^{S_{k+1}} = \mu_{i-1} \, \Delta b_{i-1}^{S_{k+1}}, \quad k+1 \le i \le n-1$$
$$(\alpha + \mu_i) \, \Delta b_i^{S_{k+2}} = \mu_{i-1} \, \Delta b_{i-1}^{S_{k+2}}, \quad k+2 \le i \le n-1,$$

hence

$$\Delta b_i^{S_{k+2}} - \Delta b_i^{S_{k+1}} = \frac{\mu_{i-1}}{\alpha + \mu_i} \, (\Delta b_{i-1}^{S_{k+2}} - \Delta b_{i-1}^{S_{k+}}), \quad k+2 \le i < n.$$

In light of the last identity, to prove (B.6) it is enough to show that

$$\Delta b_{k+1}^{S_{k+2}} - \Delta b_{k+1}^{S_{k+1}} > 0,$$

which we establish next. Consider the case $k = 0$. By Lemma B.2, we have

$$\begin{aligned} \Delta b_1^{S_2} - \Delta b_1^{S_1} &= \frac{\lambda_1}{\alpha + \lambda_0 + \mu_1} - \frac{\Delta \lambda_1}{\alpha + \mu_1} \\ &= \frac{\lambda_0}{\alpha + \mu_1} \, \frac{\alpha + \Delta d_1}{\alpha + \lambda_0 + \mu_1} > 0, \end{aligned}$$

where the inequality follows by Assumption 7.1(i). Consider now the case $k \ge 1$. Drawing again on Lemma B.2, we have

$$(\alpha + \lambda_k + \mu_{k+1}) \, \Delta b_{k+1}^{S_{k+2}} = \lambda_{k+1} + \mu_k \, \Delta b_k^{S_{k+2}}$$
$$(\alpha + \mu_{k+1}) \, \Delta b_{k+1}^{S_{k+1}} = \Delta \lambda_{k+1} + \mu_k \, \Delta b_k^{S_{k+1}}.$$

Using in turn the last two identities, (B.5) and (B.2), part (a) and Lemma B.4(c), yields

$$
\begin{aligned}
(\alpha + \mu_{k+1})\,(\Delta b_{k+1}^{S_{k+2}} - \Delta b_{k+1}^{S_{k+1}}) &= \lambda_k \left[1 - \Delta b_{k+1}^{S_{k+2}}\right] + \mu_k\,(\Delta b_k^{S_{k+2}} - \Delta b_k^{S_{k+2}}) \\
&= \left[1 - \frac{\mu_k / a_k}{\alpha + \lambda_{k-1} + \mu_k}\right] w_k^{S_{k+2}} > 0,
\end{aligned}
$$

as required. This completes the proof.

□

B.2. Marginal cost calculation

We set out in this section to calculate marginal costs $c_i^{S_k}$, proceeding similarly as before for marginal workloads. Again, to eliminate the dependence on uniformization rate Λ, the terms c_i^S below correspond to those defined by (6.15) after *scaling* by factor $\alpha + \Lambda$.

We start by relating coefficients v_i^S's and c_i^S's. From (6.15) and (B.1), we obtain

$$
c_i^S = \lambda_i\,\Delta v_{i+1}^S, \quad 0 \leq i \leq n-1.
$$

We must thus calculate the $\Delta v_i^{S_k}$'s. Start by calculating the $v_i^{S_k}$'s through (6.14).

Lemma B.7. *For* $1 \leq k \leq n+1$, *coefficients* $v_i^{S_k}$ *are characterized by the equations*

$$
\begin{aligned}
(\alpha + \Lambda)\,v_0^{S_k} &= h_0 + (\Lambda - \lambda_0^{S_k})\,v_0^{S_k} + \lambda_0^{S_k}\,v_1^{S_k} \\
(\alpha + \Lambda)\,v_i^{S_k} &= h_i + \mu_i\,v_{i-1}^{S_k} + (\Lambda - \lambda_i^{S_k} - \mu_i)\,v_i^{S_k} + \lambda_i^{S_k}\,v_{i+1}^{S_k}, \quad 1 \leq i \leq n-1 \\
(\alpha + \Lambda)\,v_n^{S_k} &= h_n + \mu_n\,v_{n-1}^{S_k} + (\Lambda - \mu_n)\,v_n^{S_k}.
\end{aligned}
$$

It follows that coefficients $\Delta v_i^{S_k}$ are characterized as shown next.

Lemma B.8. *For* $1 \leq k \leq n+1$, *coefficients* $\Delta v_i^{S_k}$ *are characterized by the equations*

$$
\begin{aligned}
(\alpha + \lambda_0^{S_k} + \mu_1)\,\Delta v_1^{S_k} &= \Delta h_1 + \lambda_1^{S_k}\,\Delta v_2^{S_k} \\
(\alpha + \lambda_{i-1}^{S_k} + \mu_i)\,\Delta v_i^{S_k} &= \Delta h_i + \mu_{i-1}\,\Delta v_{i-1}^{S_k} + \lambda_i^{S_k}\,\Delta v_{i+1}^{S_k}, \quad 2 \leq i \leq n-1 \\
(\alpha + \lambda_{n-1}^{S_k} + \mu_n)\,\Delta v_n^{S_k} &= \Delta h_n + \mu_{n-1}\,\Delta v_{n-1}^{S_k}.
\end{aligned}
$$

We next develop a recursion to calculate *pivot* terms $\Delta v_k^{S_{k+1}}$, along the lines followed in Appendix B.1 to calculate the $\Delta b_k^{S_{k+1}}$'s. Note that Lemma B.8 yields

$$
\Delta v_1^{S_1} = \frac{\Delta h_1}{\alpha + \mu_1},
$$

and hence

$$
c_0^{S_1} = \frac{\lambda_0}{\alpha + \mu_1}\,\Delta h_1.
$$

It further yields the first such pivot as

$$\Delta v_1^{S_2} = \frac{\Delta h_1}{\alpha + \lambda_0 + \mu_1},$$

so that

$$c_0^{S_2} = \frac{\lambda_0}{\alpha + \lambda_0 + \mu_1}\, \Delta h_1.$$

We next set out to relate successive pivots. Associate with $1 \leq k \leq n-1$ the vectors

$$\begin{aligned}
\Delta \mathbf{v}^k &= \left(\Delta v_1^{S_{k+1}}, \ldots, \Delta v_k^{S_{k+1}}\right)^T \\
\Delta \hat{\mathbf{v}}^k &= \left(\Delta v_1^{S_{k+2}}, \ldots, \Delta v_k^{S_{k+2}}\right)^T \\
\mathbf{h}^k &= \left(\frac{\Delta h_1}{\alpha + \lambda_0 + \mu_1}, \ldots, \frac{\Delta h_k}{\alpha + \lambda_{k-1} + \mu_k}\right) \\
\hat{\mathbf{h}}^k &= \mathbf{h}^k + \frac{\lambda_k\, \Delta v_{k+1}^{S_{k+2}}}{\alpha + \lambda_{k-1} + \mu_k}\, \mathbf{e}_k.
\end{aligned}$$

The next result is a counterpart to Lemma B.3.

Lemma B.9. *For* $1 \leq k \leq n-1$,

(a) $\Delta \mathbf{v}^k = \mathbf{h}^k + \mathbf{B}^k\, \Delta \mathbf{v}^k$;

(b) $\Delta \hat{\mathbf{v}}^k = \hat{\mathbf{h}}^k + \mathbf{B}^k\, \Delta \hat{\mathbf{v}}^k$.

The relation between pivots $\Delta v_k^{S_{k+1}}$ and $\Delta v_{k+1}^{S_{k+2}}$, and its marginal cost reformulation is given next. The proof is similar to that of Lemma B.5, and is hence omitted.

Lemma B.10. *For* $1 \leq k \leq n-1$,

$$a_{k+1}\, \Delta v_{k+1}^{S_{k+2}} = \frac{\Delta h_{k+1}}{\alpha + \lambda_k + \mu_{k+1}} + \frac{\mu_k}{\alpha + \lambda_k + \mu_{k+1}}\, \Delta v_k^{S_{k+1}};$$

or, equivalently,

$$c_k^{S_{k+2}} = \frac{\lambda_k}{a_{k+1}}\, \frac{\Delta h_{k+1} + \dfrac{c_{k-1}^{S_{k+1}}}{\rho_{k-1}}}{\alpha + \lambda_k + \mu_{k+1}}.$$

C. Possible inconsistency of the Whittle index relative to threshold policies

The reader may wonder whether the extra flexibility provided by parameters θ_j^1 in the new index introduced in Definition 3.3 significantly expands the scope of the original Whittle index. We argue next that such is the case by showing, in the setting of the admission control model, that the Whittle index does *not* rank the states in a manner consistent with threshold policies, under the parameter range given by Assumption 7.1.

Recall that the Whittle index arises from the appropriate ν-charge problem obtained by charging costs at rate ν *while the entry gate is shut*. Namely, the corresponding activity measure b^u obtains by letting $\theta_j^1 = 1$, for $j \in N = \{0, \ldots, n\}$

We shall consider that an index policy for the admission control model is *consistent with threshold policies* if index ν_j is *nondecreasing* on $j \in \{0, \ldots, n-1\}$.

Consider the case where the buffer size is $n = 2$, service rates are $\mu_j = \mu$, and cost rates are $h_j = h\, j$. Suppose arrival rates λ_j are strictly decreasing on j, namely

$$\Delta\lambda_2 < 0, \Delta\lambda_1 < 0. \tag{C.1}$$

It then follows that Assumption 7.1 holds.

Take, in particular, $\lambda_0 = 1$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$, $\mu = \frac{3}{2}$, $\alpha = \frac{1}{33}$, $h = 1$. Pick the uniformization rate $\Lambda = 3$, so that $\beta = \Lambda/(\alpha + \Lambda) = \frac{99}{100}$.

The corresponding RB is indexable, in Whittle's sense, and has Whittle indices

$$\nu_2 = 0 < \nu_1 = \frac{3300}{6767} < \nu_0 = \frac{11\,022}{19\,111}.$$

They thus give a state ranking which is *inconsistent* with threshold policies.

Such inconsistency only arises, however, under state-dependent arrival rates; under a constant arrival rate λ, the extended index equals Whittle's scaled by factor $1/\lambda$.

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