Minimum Delay Spread TEQ Design in Multicarrier Systems

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***Abstract*—Recently, a time-domain equalizer (TEQ) design for multicarrier-based systems has been proposed which claims to minimize the delay spread of the overall channel impulse response. We show that this is true only in an approximate sense; depending on the channel considered, the loss in delay spread with respect to the true minimum can be significant. An iterative algorithm to find this minimum is presented, whose computational complexity is similar to that of standard TEQ designs like the MSSNR approach of Melsa *et al.* It is observed that the method iteratively and automatically seeks the time reference yielding best performance, an advantage with respect to the MSSNR design which must compute several TEQs over a range of time references in order to select the optimum.**

***Index Terms*—Channel shortening, discrete multitone, multicarrier, orthogonal frequency division multiplexing, time domain equalizers.**

I. Introduction

IN MULTICARRIER systems, intersymbol and intercarrier interference can be avoided by inserting a cyclic prefix (CP) between consecutive symbols. If the channel impulse response (CIR) spans no more than $\nu+1$ samples, where ν is the CP length, then the effect of the channel appears as a circular convolution, enabling one-tap equalization in each subcarrier. The CP reduces the data rate of the system by a factor of $N/(N+\nu)$, where N is the length of the multicarrier symbol, so ν has to be kept as small as possible. Usually the CIR is much longer than the CP, so that a time-domain equalizer (TEQ) is placed in the receiver with the purpose of appropriately shortening the overall impulse response (OIR).

Several TEQ designs have been proposed [1], [2], [5], [7], [8]. Melsa *et al.* [7] introduced the Shortening SNR (SSNR) as the ratio of the OIR energy inside a window of size $\nu+1$ to that outside the window and suggested to choose the TEQ in order to maximize the SSNR. Although this intuitive approach does not maximize the achievable bit rate, it often provides very satisfactory performance [2]. One drawback of the MSSNR method is the need to search over the range of admissible transmission delays in order to select the best one. Recently, Schur and Speidel [8] have proposed an alternative TEQ design based on minimization of the delay spread (MDS) of the OIR. This approach is independent of the CP length, and intuitively should provide an OIR which is "as squeezed as possible." This is desirable since i) the CP length could be reduced with the corresponding increase in data rate, ii) for a fixed CP length, further OIR squeezing results in additional robustness to synchronization offsets [8]. Although the MDS design,, unlike the MSSNR approach, does not require *a priori* specification of a delay index, the time reference about which the delay spread is to be defined must be selected. It was suggested in [8] to take this point as the centroid (center of mass) of the unequalized CIR.

We analyze the MDS design in order to clarify the role of the time reference. This will show that this time reference needs to be optimized as well, since setting it to the CIR centroid may yield unacceptable performance depending on the CIR. Joint optimization of delay spread and time reference is shown to result in a nonquadratic minimization problem with no closed-form solution. We present an iterative process which is computationally efficient, not requiring stepsize tuning as a gradient descent would, and typically converges within a few iterations to the desired minimum.

II. Review of the MDS TEQ Design

We define the following quantities for a given an impulse response $\{f_n, -\infty < n < \infty\}$:

$$E_f = \sum_n f_n^2 \quad \text{(energy)} \tag{1}$$

$$n_f = \frac{1}{E_f}\sum_n n f_n^2 \quad \text{(centroid)} \tag{2}$$

$$D_f = \sqrt{\frac{1}{E_f}\sum_n (n-n_f)^2 f_n^2} \quad \text{(delay spread).} \tag{3}$$

Let $\{h_n, 0 \le n < K\}$ and $\{w_n, 0 \le n < M\}$ denote the CIR and TEQ taps, respectively, and let $\{q_n, 0 \le n < K+M-2\}$ be the OIR, i.e., $q_n = h_n \star w_n$ where "$\star$" denotes linear convolution. The MDS TEQ design of [8] chooses $\mathbf{w} = [w_0\ w_1\ \cdots\ w_{M-1}]^T$ in order to minimize the following cost:

$$J_{\rm ss}(\mathbf{w},\kappa) = \frac{1}{E_q}\sum_n (n-\kappa)^2 q_n^2, \tag{4}$$

Observe that $J_{\rm ss}(\mathbf{w},\kappa)$ can be seen as the (squared) dispersion of $\{q_n\}$ about the reference κ.

Letting $\mathbf{H}$ be the convolution matrix of the unequalized channel, which is $(M+K-1)\times M$ Toeplitz with first column $[h_0\ h_1\ \cdots\ h_{K-1}\ 0\ \cdots\ 0]^T$, then the OIR vector $\mathbf{q} = [q_0\ q_1\ \cdots\ q_{K+M-2}]^T$ is given by $\mathbf{q} = \mathbf{H}\mathbf{w}$. The cost $J_{\rm ss}$ of (4) can be written as

$$J_{\rm ss}(\mathbf{w},\kappa) = \frac{\mathbf{q}^T\mathbf{\Lambda}_\kappa^2\mathbf{q}}{\mathbf{q}^T\mathbf{q}} = \frac{\mathbf{w}^T\mathbf{H}^T\mathbf{\Lambda}_\kappa^2\mathbf{H}\mathbf{w}}{\mathbf{w}^T\mathbf{H}^T\mathbf{H}\mathbf{w}} \tag{5}$$

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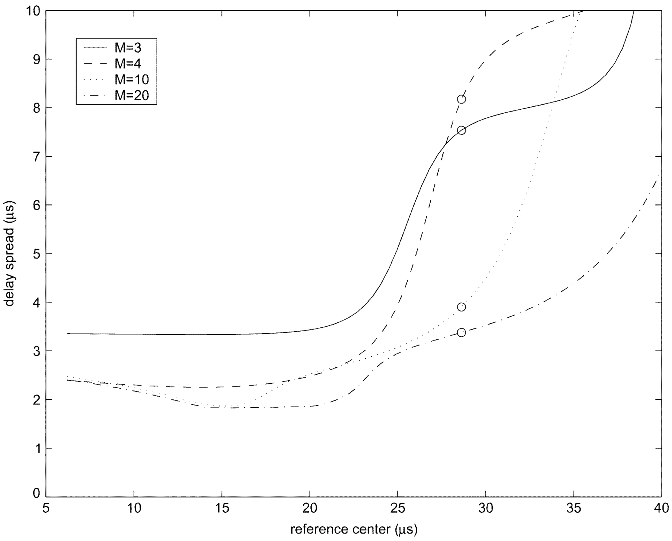

Fig. 1. Delay spread attained by minimization of $J_{\text{ss}}(\mathbf{w}, \kappa)$ over a range of time references κ.

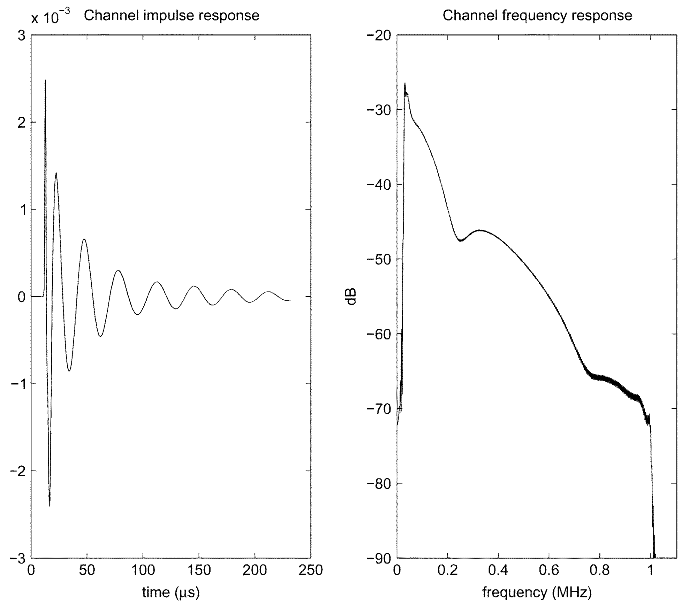

Fig. 2. Channel impulse and frequency responses.

where $\mathbf{\Lambda}_\kappa$ is a diagonal matrix given by

$$\mathbf{\Lambda}_\kappa = \text{diag}(0 \quad 1 \quad \cdots \quad K + M - 2) - \kappa \mathbf{I}. \qquad (6)$$

Minimizing (5) results in a generalized eigenvalue problem [8]. By introducing the Cholesky factorization

$$\mathbf{H}^T\mathbf{H} = \mathbf{U}^T\mathbf{U}, \quad \text{with } \mathbf{U} M \times M \text{ upper triangular} \qquad (7)$$

then the MDS TEQ $\mathbf{w}_\kappa$ is given by

$$\mathbf{w}_\kappa = \mathbf{U}^{-1}\mathbf{v}_\kappa \qquad (8)$$

where $\mathbf{v}_\kappa$ is the eigenvector of the matrix $\mathbf{U}^{-T}\mathbf{H}^T\mathbf{\Lambda}_\kappa^2\mathbf{H}\mathbf{U}^{-1}$ associated to its smallest eigenvalue.

III. Relation to the True Delay Spread

In [8], it is suggested that $\kappa = n_h$ (the CIR centroid) be taken. As the centroid of the resulting OIR need not coincide with n_h, the resulting cost (4) will not be equal in general to the actual OIR delay spread as defined by (3). In fact, from (4), some elementary algebra shows that

$$J_{\text{ss}}(\mathbf{w}, \kappa) = D_q^2 + (n_q - \kappa)^2. \qquad (9)$$

This cost penalizes not only large OIR delay spreads, but also the deviation of the OIR centroid n_q from κ, which is set *a priori*. However, we are interested in the minimization of D_q^2 alone.

Fig. 1 shows the delay spread obtained with the MDS TEQ (8) of various lengths M and for different values of κ. The channel was the standard DSL test loop CSA 1 [2] combined with a POTS splitter and a twelfth-order Chebyshev bandpass filter for the 30–1000 kHz frequency band, and truncated to 512 samples (see Fig. 2). The sampling frequency was 2.208 MHz. The CIR delay spread is 23 μs and its centroid $n_h = 28.8$ μs. Depending on κ, minimization of (4) need not yield a small delay spread. For $\kappa \gg n_h$, the term $(n_q - \kappa)^2$ dominates (9) and the MDS TEQ attempts to match the OIR centroid to κ by time shifting the CIR, without reducing the delay spread much. The circles in Fig. 1 mark the points $\kappa = n_h$, which do not result in the smallest delay spread[1]. The gap with respect to the optimum increases as M is reduced. Note that shorter TEQs are desirable since, in addition to their lower complexity, it is known that longer TEQs tend to place their zeros on the unit circle, thus killing subcarriers [6].

IV. Minimizing the True Delay Spread

Note from (9) that $D_q^2(\mathbf{w}) = J_{\text{ss}}(\mathbf{w}, \kappa) - [n_q(\mathbf{w}) - \kappa]^2$, while the difference $n_q - \kappa$ can be written as

$$n_q(\mathbf{w}) - \kappa = \frac{\mathbf{q}^T\mathbf{\Lambda}_\kappa\mathbf{q}}{\mathbf{q}^T\mathbf{q}} = \frac{\mathbf{w}^T\mathbf{H}^T\mathbf{\Lambda}_\kappa\mathbf{H}\mathbf{w}}{\mathbf{w}^T\mathbf{H}^T\mathbf{H}\mathbf{w}}. \qquad (10)$$

Therefore,

$$D_q^2(\mathbf{w}) = \frac{\mathbf{w}^T\mathbf{H}^T\mathbf{\Lambda}_\kappa^2\mathbf{H}\mathbf{w}}{\mathbf{w}^T\mathbf{H}^T\mathbf{H}\mathbf{w}} - \left[\frac{\mathbf{w}^T\mathbf{H}^T\mathbf{\Lambda}_\kappa\mathbf{H}\mathbf{w}}{\mathbf{w}^T\mathbf{H}^T\mathbf{H}\mathbf{w}}\right]^2, \qquad (11)$$

which is independent of the value of κ. In terms of $\mathbf{v} = \mathbf{U}\mathbf{w}$ with $\mathbf{U}$ the Cholesky factor defined in (7)

$$D_q^2(\mathbf{v}) = \frac{\mathbf{v}^T\mathbf{X}\mathbf{v}}{\mathbf{v}^T\mathbf{v}} - \left[\frac{\mathbf{v}^T\mathbf{Y}\mathbf{v}}{\mathbf{v}^T\mathbf{v}}\right]^2 \qquad (12)$$

where

$$\mathbf{X} = \mathbf{U}^{-T}\mathbf{H}^T\mathbf{\Lambda}_\kappa^2\mathbf{H}\mathbf{U}^{-1}, \quad \mathbf{Y} = \mathbf{U}^{-T}\mathbf{H}^T\mathbf{\Lambda}_\kappa\mathbf{H}\mathbf{U}^{-1}. \qquad (13)$$

Minimizing (12) subject to, e.g., $\mathbf{v}^T\mathbf{v} = 1$ results in a nonquadratic problem with no closed form solution. Using Lagrange multiplier theory [3], it can be shown that at any (local) minimum $\mathbf{v}_\star$ of this problem, $\mathbf{v}_\star$ is a unit-norm eigenvector of

[1]This will generally be the case, unless the CIR happens to be symmetric (or antisymmetric) about its centroid n_h.

the matrix $\mathbf{X} - 2n_{q_\star}\mathbf{Y}$ associated to its smallest eigenvalue $\lambda_\star = D_{q_\star}^2 - n_{q_\star}^2$; here $D_{q_\star}, n_{q_\star}$ denote the OIR delay spread and centroid attained with the TEQ $\mathbf{w}_\star = \mathbf{U}^{-1}\mathbf{v}_\star$. This suggests the use of the (inverse) power method to compute the minimum, but unfortunately the matrices constructed along the iterations need not be positive semidefinite, so this approach will not work. On the other hand, a simple gradient descent of $D_q^2(\mathbf{v})$ in (12) over the manifold $\mathbf{v}^T\mathbf{v} = 1$ takes the form

$$\bar{\mathbf{v}}_{i+1} = \mathbf{v}_i - \mu\left[\mathbf{X} - 2\left(\mathbf{v}_i^T\mathbf{Y}\mathbf{v}_i\right)\mathbf{Y}\right]\mathbf{v}_i$$
$$\mathbf{v}_{i+1} = \bar{\mathbf{v}}_{i+1}/\|\bar{\mathbf{v}}_{i+1}\| \quad (14)$$

where $\mathbf{X}, \mathbf{Y}$ are given by (13) using e.g., $\kappa = n_h$. This makes use of straightforward matrix-vector multiplications only (plus one initial Cholesky decomposition), but it requires careful adjustment of the stepsize μ to avoid divergence. Even when μ is optimized for fastest convergence, this may take quite a large number of iterations.

We propose an alternate approach. The main idea is to iteratively minimize the cost J_{ss} using as time reference the OIR centroid obtained at the previous iteration.

1) Set $\kappa_0 = n_h$, the unequalized CIR centroid.
2) For $i = 1, 2, \ldots$, compute $\mathbf{w}_i = \arg\min_{\mathbf{w}} J_{\text{ss}}(\mathbf{w}, \kappa_{i-1})$ and then set κ_i to the centroid of $\mathbf{H}\mathbf{w}_i$

$$\kappa_i = n_h + \frac{\mathbf{w}_i^T\mathbf{H}^T\mathbf{\Lambda}_{n_h}\mathbf{H}\mathbf{w}_i}{\mathbf{w}_i^T\mathbf{H}^T\mathbf{H}\mathbf{w}_i}.$$

With just one iteration, the design of [8] is recovered. This scheme typically converges in a few iterations to the desired minimum of the delay spread. We now show that the OIR delay spread cannot increase from iteration to iteration. Observe that by picking the time reference κ_i as the centroid of $\mathbf{H}\mathbf{w}_i$, then in view of (9) it satisfies $\kappa_i = \arg\min_\kappa J_{\text{ss}}(\mathbf{w}_i, \kappa)$ since $J_{\text{ss}}(\mathbf{w}_i, \kappa) = D_q^2(\mathbf{w}_i) + (\kappa_i - \kappa)^2$. Therefore

$$J_{\text{ss}}(\mathbf{w}_i, \kappa_i) \le J_{\text{ss}}(\mathbf{w}_i, \kappa_{i-1}) \le J_{\text{ss}}(\mathbf{w}_{i-1}, \kappa_{i-1}) \quad (15)$$

where the second inequality follows from the definition of $\mathbf{w}_i$. Since $J_{\text{ss}}(\mathbf{w}, \kappa) \ge 0$ for all $\mathbf{w}, \kappa$, (15) implies that this sequence must converge. Finally, we note that by choice of κ_{i-1} and κ_i, one has $J_{\text{ss}}(\mathbf{w}_i, \kappa_i) = D_q^2(\mathbf{w}_i)$ and $J_{\text{ss}}(\mathbf{w}_{i-1}, \kappa_{i-1}) = D_q^2(\mathbf{w}_{i-1})$. Hence from (15), $D_q^2(\mathbf{w}_i) \le D_q^2(\mathbf{w}_{i-1})$.

It must be noted that this "cyclic" minimization procedure need not converge to the optimal setting; only convergence to a (local) minimum of $J_{\text{ss}}(\mathbf{w}, \kappa)$ is guaranteed. However, for most practical channels, the initialization $\kappa_0 = n_h$ is likely to be good enough for the algorithm to find the global minimum.

The iterative method proposed requires the computation of an eigenvector at each iteration, which can be computationally demanding. If these eigenvectors are obtained via the inverse power method (InvPM) [4], then with complexity reduction in mind, it makes sense to combine the inner (inverse power) iterations with the outer ones. The resulting algorithm is as follows.

- Compute $\mathbf{C} = \mathbf{H}^T\mathbf{H}$, and set $\mathbf{w}_0 = [1/\sqrt{c_{00}}\ 0\ \cdots\ 0]^T$, with c_{00} the first element of $\mathbf{C}$.
- Set $\kappa_0 = n_h$ and initialize the matrices $\mathbf{A}_0 = \mathbf{H}^T\mathbf{\Lambda}_{\kappa_0}^2\mathbf{H}$ and $\mathbf{B}_0 = \mathbf{H}^T\mathbf{\Lambda}_{\kappa_0}\mathbf{H}$.
- For $i = 1, 2, \ldots, i_{\max}$, do
  1) Solve for $\bar{\mathbf{w}}_i$ in $\mathbf{A}_{i-1}\bar{\mathbf{w}}_i = \mathbf{C}\mathbf{w}_{i-1}$, then set $\mathbf{w}_i = \bar{\mathbf{w}}_i/\sqrt{\bar{\mathbf{w}}_i^T\mathbf{C}\bar{\mathbf{w}}_i}$.
  2) Compute $\Delta\kappa_i = \kappa_i - \kappa_{i-1} = \mathbf{w}_i^T\mathbf{B}_{i-1}\mathbf{w}_i$.
  3) Update $\mathbf{A}_i = \mathbf{A}_{i-1} + (\Delta\kappa_i)^2\mathbf{C} - 2\Delta\kappa_i\mathbf{B}_{i-1}$ and $\mathbf{B}_i = \mathbf{B}_{i-1} - \Delta\kappa_i\mathbf{C}$.

These matrix updates follow from the fact that $\mathbf{\Lambda}_{\kappa_i} = \mathbf{\Lambda}_{\kappa_{i-1}} - (\kappa_i - \kappa_{i-1})\mathbf{I}$.

Note that the computational complexity of a single eigenvector extraction via InvPM is $O(j_{\max}M^3)$, where $j_{\max}$ denotes the maximum number of iterations allowed to InvPM; while that of the proposed scheme is $O(i_{\max}M^3)$. Thus, if $i_{\max}$ and $j_{\max}$ are comparable (as they usually are), the new method has about the same computational cost as those of [7], [8].

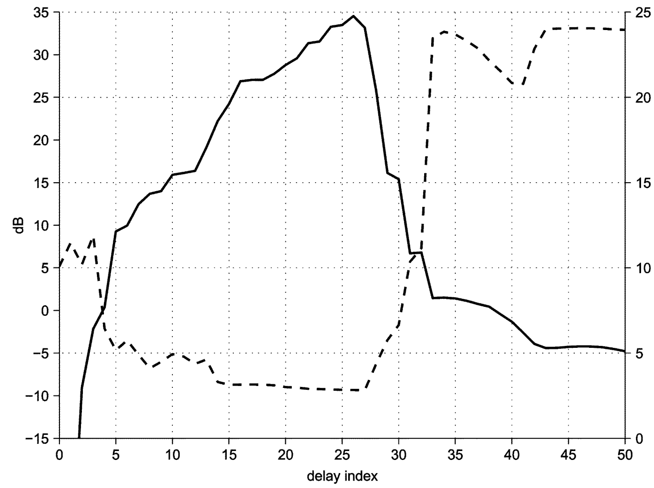

Fig. 3. SSNR (solid) and delay spread (dashed) obtained with the MSSNR design as a function of the delay index.

V. Comparison With the MSSNR Design

The MSSNR TEQ design of [7] requires the specification of the cyclic prefix size ν and the delay index. Fig. 3 shows the SSNR and the corresponding delay spread obtained with this approach as a function of delay index, for the channel in Fig. 2, TEQ length $M = 4$, and $\nu = 26$ samples. Observe that high MSSNR values correspond to low delay spread values. It turned out that the MSSNR TEQ for the optimum delay index and the TEQ obtained with the iterative MDS approach were identical, resulting in the OIR shown in Fig. 4. (The iterative MDS scheme converged in just five steps, while the gradient descent (14) with optimized stepsize took thirty iterations to reach the same TEQ). Thus, in a sense, the iterative MDS design automatically searches for the optimal window position. This has been observed for a variety of channels and TEQ lengths, and clearly constitutes an advantage with respect to the MSSNR design, for which a range of delay indices must be swept in order to pick the best one. The improvement with respect to the original MDS design of [8] also becomes clear in Fig. 4. The respective delay spreads attained were 2.4 and 8.1 μs.

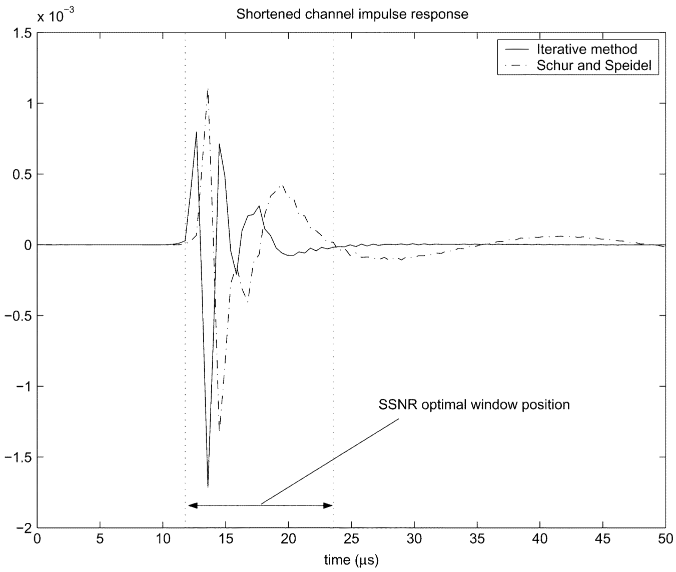

Fig. 4. Overall channel impulse response after shortening.

VI. Conclusion

We have developed an iterative approach that computes the time domain equalizer truly minimizing the delay spread of the overall channel. Convergence is much faster than that of a gradient descent, while computational complexity is similar to that of any design requiring an eigenvector computation and making use of the inverse power iteration for that goal. The solution is very close to that of the MSSNR approach with optimized delay index.

Although we have focused on channel shortening, noise effects can be incorporated in the design along the guidelines of [9]. In essence, an additional penalty term proportional to the inverse of the SNR at the equalizer output is added to the cost function to be minimized. In this way, a trade-off between channel shortening and noise enhancement can be obtained.

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