# On Simple Oversampled A/D Conversion in $L^2(I\!R)$

Zoran Cvetković, *Member, IEEE*, and Martin Vetterli, *Fellow, IEEE*

***Abstract*—Accuracy of oversampled analog-to-digital (A/D) conversion, the dependence of accuracy on the sampling interval $\tau$ and on the bit rate $R$ are characteristics fundamental to A/D conversion but not completely understood. These characteristics are studied in this paper for oversampled A/D conversion of band-limited signals in $L^2(I\!R)$. We show that the digital sequence obtained in the process of oversampled A/D conversion describes the corresponding analog signal with an error which tends to zero as $\tau^2$ in energy, provided that the quantization threshold crossings of the signal constitute a sequence of stable sampling in the respective space of band-limited functions. Further, we show that the sequence of quantized samples can be represented in a manner which requires only a logarithmic increase in the bit rate with the sampling frequency, $R = O(|\log \tau|)$, and hence that the error of oversampled A/D conversion actually exhibits an exponential decay in the bit rate as the sampling interval tends to zero.**

***Index Terms*—Accuracy, analog-to-digital (A/D), conversion, oversampling.**

## I. Introduction

Analog-to-digital (A/D) conversion involves discretization of an analog continuous-time signal in both time and amplitude. In simple A/D conversion, the signal is first discretized in time using regular sampling with an interval $\tau$, followed by uniform scalar quantization with a step $q$ (see Fig. 1). Conversion accuracy and its dependence on the bit rate are characteristics fundamental to A/D conversion but not completely understood.

If the input signal is $\pi$-band-limited, and the sampling interval $\tau$ is shorter than the Nyquist sampling interval, $\tau_N = 1$, the discretization in time is reversible, however, the amplitude discretization introduces an irreversible loss of information. To quote Gray, "Deceptively simple in its description and construction, the uniform quantizer has proved to be surprisingly difficult to analyze" [1]. The usual approach to studying the accuracy of oversampled A/D conversion is to model the quantization error as an uncorrelated, uniformly distributed additive noise, independent of the input, and to assume linear reconstruction that amounts to low-pass filtering of the sequence of quantized samples with the cutoff frequency equal to the signal bandwidth. In the 1940s, Bennett proved that the additive-noise model provides a good approximation for the quantization error under conditions such as a small quantization step size and a large number of quantization levels so that the quantizer does not overload [2]. Using the additive-noise model, one can show that the average power of the error between a band-limited signal $f$ and the signal $\tilde{f}$ which is obtained by applying the low-pass filtering to the sequence of quantized samples of $f$ is given by

$$E(|\tilde{f}(t) - f(t)|^2) = \frac{q^2}{12}\frac{\tau}{\tau_N}. \tag{1}$$

This result suggests that the conversion accuracy can be improved by improving the resolution of the discretization in either time or amplitude. Due to the costs involved in building high-resolution quantizers, high accuracy of modern techniques for A/D conversion is achieved through oversampling. Formula (1) is, however, misleading as to the actual accuracy of oversampled A/D conversion. This comes about partly because the additive-noise model has a very limited scope and is hard to justify asymptotically, when, due to fine sampling, correlations between the error samples become more pronounced. An exact analysis of quantization error by Clavier, Panter, and Grieg [3], [4], for signals which are superpositions of two sinusoids, demonstrated the nonwhite nature of the quantization error. It has also been experimentally verified that the quantization error, when measured for linear reconstruction, does not tend to zero along with $\tau$ in the manner described by (1), but rather reaches a nonzero floor level for some finite $\tau$. Moreover, the deterministic analysis of oversampled A/D conversion in [5] revealed that, in the case of signals which are superpositions of finitely many harmonic sinusoids, the digital sequence generated in the process of oversampled A/D conversion allows for reconstruction of the corresponding analog signal within an error which can be bounded in average power by a $\tau^2$ expression, provided that the signal has sufficiently many quantization threshold crossings.

The purpose of this paper is to elucidate information-theoretic properties of simple oversampled A/D conversion for a very broad realistic class of signals, such as band-limited signals in $L^2(I\!R)$. We provide an exact deterministic analysis of oversampled A/D conversion, based on results of nonharmonic Fourier analysis, and we focus on the information content of the bit stream generated in the conversion process. Our deterministic analysis demonstrates that if the quantization threshold crossings of a band-limited signal form a *sequence of stable sampling* [6] in the considered space of square-integrable band-limited functions, then the error incurred in the process of oversampled A/D conversion of this signal converges to zero as $\tau^2$ in energy.

Manuscript received June 1, 1999; revised August 28, 2000. The material in this paper was presented in part at the International Conference on Components, Speech and Signal Processing (ICASSP'96), Atlanta, GA, 1996 and the International Symposium on Circuits and Systems, Hong Kong, 1997.

Z. Cvetković was with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley CA USA. He is now with AT&T Shannon Laboratory, Florham Park, NJ 07932 USA (e-mail: zoran@research.att.com).

M. Vetterli was with is with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA. He is now with EPFL, CH-1015, Lausanne, Switzerland, and the University of California, Berkeley, CA 94720 USA (e-mail: vetterli@de.epfl.ch).

Communicated by J. A. O'Sullivan, Associate Editor for Detection and Estimation.

Publisher Item Identifier S 0018-9448(01)00593-4.

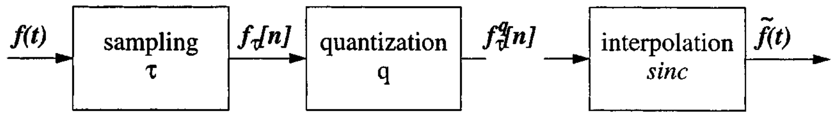


Fig. 1. Block diagram of simple oversampled A/D conversion followed by classical reconstruction. The input $\pi$-band-limited signal $f$ is first sampled with an interval $\tau$, smaller than the Nyquist sampling interval $\tau_N = 1$. The sequence of samples $f_\tau$ is then discretized in amplitude by applying uniform quantization with a step $q$. Classical reconstruction amounts to linear interpolation between samples of the quantized sequence $f_\tau^q$ using $\operatorname{sinc}(x) = \sin(\pi x)/\pi x$ interpolating functions.

Among the signals whose quantization threshold crossings do not form a sequence of stable sampling, there exist examples of signals for which the accuracy of oversampled A/D conversion does not change as $\tau$ tends to zero; but it is still not clear whether this is true for every signal of this kind. However, we demonstrate that the signals for which the $\tau^2$ conversion accuracy holds constitute a rich class of band-limited functions; moreover, we propose a dithering scheme which ensures $\tau^2$ accuracy on a very broad class of band-limited square-integrable functions.

We also reconsider rate-distortion properties of oversampled A/D conversion. It is commonly believed that even though oversampling improves conversion accuracy, it has an adverse impact on overall rate-distortion properties of the conversion due to the high rate of increase in the bit rate with oversampling. Specifically, unless some efficient entropy coding is used which exploits correlations between quantized samples, the bit rate of oversampled A/D conversion increases inversely to the sampling interval, and that results in error-rate characteristics having an inverse polynomial decay. The quantized samples of a band-limited signal, however, can be represented through information about the corresponding sequence of quantization threshold crossings. With a representation of this kind, the bit rate of oversampled A/D conversion increases only as a logarithm of the sampling interval [7], which in turn results in exponentially decaying error-rate characteristics. For completeness of our treatment of oversampled A/D conversion, we review these results in the final section of this paper.

## II. Deterministic Analysis of Oversampled A/D Conversion in $L^2(\mathbb{R})$

The signal space which we consider is the space of square-integrable $\pi$-band-limited functions. We denote this space[1] by $\mathcal{V}_\pi$

$$\mathcal{V}_\pi = \left\{ f\colon \int_{-\infty}^{+\infty} |f(t)|^2\, dt < \infty,\ |\hat{f}(\omega)| = 0 \text{ for } |\omega| > \pi \right\}.$$

Simple A/D conversion in $\mathcal{V}_\pi$ is the superposition of a sampling operator $\boldsymbol{S}_\tau$ and a uniform scalar quantizer $\boldsymbol{Q}_q$.

*Definition 1:* The uniform scalar quantizer $\boldsymbol{Q}^q$ with a quantization step $q$ is the mapping $\boldsymbol{Q}^q\colon \mathbb{R} \to q\mathbb{Z} + q/2$

$$\boldsymbol{Q}^q(x) = q\lfloor x/q \rfloor + q/2.$$

The points of discontinuities of the function $\boldsymbol{Q}^q(x)$ are said to be quantization thresholds of the quantizer $\boldsymbol{Q}^q$.

[1]We use $\hat{f}(\omega)$ to denote the Fourier transform of $f$, $\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{j\omega t}\, dt$.

*Definition 2:* The sampling operator $\boldsymbol{S}_\tau$ is the mapping $\boldsymbol{S}_\tau\colon \mathcal{V}_\pi \to \ell^2(\mathbb{Z})$

$$\boldsymbol{S}_\tau f := f_\tau = \{f_\tau[n]\colon f_\tau[n] = f(n\tau),\ n \in \mathbb{Z}\}.$$

*Definition 3:* The A/D converter $\boldsymbol{C}_\tau^q \mathcal{V}_\pi$ is the superposition of the sampling operator $\boldsymbol{S}_\tau$ and the uniform scalar quantizer $\boldsymbol{Q}^q$, $\boldsymbol{C}_\tau^q = \boldsymbol{S}_\tau \boldsymbol{Q}^q$.[2]

The digital version $f_\tau^q = \boldsymbol{C}_\tau^q f$ of a signal $f$ in $\mathcal{V}_\pi$ is traditionally viewed as a representation which gives information about the samples of $f$ at points $(n\tau)_{n\in\mathbb{Z}}$ with uncertainty $q$ in amplitude. Our analysis of oversampled A/D conversion treats the digital sequence as a representation which carries information about the instants of quantization threshold crossings of $f$ with time uncertainty equal to $\tau$.

*Definition 4:* A function $f$ in $\mathcal{V}_\pi$ is said to have a quantization threshold crossing at a point $x$ relative to a quantizer $\boldsymbol{Q}^q$ if $\boldsymbol{Q}^q(f(x^-)) \neq \boldsymbol{Q}^q(f(x^+))$.

The time instants of quantization threshold crossings of $f$ are indicated by the data changes in the sequence $f_\tau^q$. A data change of the form $f_\tau^q[m_n] \neq f_\tau^q[m_n+1]$ indicates that there exists a point $x_n$ on the interval $(m_n\tau,\ m_n\tau + \tau)$ where $f$ goes through the quantization threshold which is between $f_\tau^q[m_n]$ and $f_\tau^q[m_n+1]$.[3] The amplitude of $f$ at $x_n$ is thus known with infinite precision, $f(x_n) = (f_\tau^q[m_n] + f_\tau^q[m_n+1])/2$, whereas $x_n$ is known with uncertainty $\tau$. An alternative way to view this data change is to consider that it gives the value of $f$ at $m_n\tau$ with an error proportional to the size of the sampling interval. In particular, one obtains

$$f(m_n\tau) = l_n q + f'(\xi_n)\tau$$

where

$$l_n = (f_\tau^q[m_n] + f_\tau^q[m_n+1])/2q$$

and $\xi_n$ is a point on the interval $(m_n\tau,\ x_n)$. As the sampling interval tends to zero, the time instants of quantization threshold crossings are given with progressively better precision; and at the limit, when $\tau = 0$, the information retained in the conversion process consists of signal samples at its quantization threshold crossings. Whether this information uniquely describes the input signal $f$ and what is the accuracy with which $f$ is described by the corresponding digital sequence

[2]Note that the operators $\boldsymbol{S}_\tau$ and $\boldsymbol{Q}^q$ commute. For convenience, we define $\boldsymbol{C}_\tau^q$ as $\boldsymbol{C}_\tau^q = \boldsymbol{S}_\tau \boldsymbol{Q}^q$ rather than $\boldsymbol{C}_\tau^q = \boldsymbol{Q}^q \boldsymbol{S}_\tau$, which corresponds to real implementation.

[3]We assume that $\tau$ is small enough that $f$ cannot go through two different thresholds on an interval of length $\tau$.

for some finite $\tau$ depends on the sampling properties of the sequence of quantization threshold crossings of $f$ in $\mathcal{V}_\pi$.

In the following section we show that if the sequence of quantization threshold crossings of $f$ is uniformly discrete[4] and forms a sequence of stable sampling in $\mathcal{V}_\pi$ [6], then for a small enough sampling interval $\tau$ and any $\tilde{f}$ which in the conversion produces the same digital sequence as $f$ the energy of the difference between $f$ and $\tilde{f}$ can be bounded as

$$\int_{-\infty}^{\infty} |\tilde{f}(x) - f(x)|^2 \, dx \leq c_f \tau^2 \tag{2}$$

where $c_f$ does not depend on $\tau$. The notion of sequence of stable sampling was introduced by Landau in his work on irregular sampling of band-limited functions [6].

*Definition 5:* A sequence of real numbers $(\lambda_n)_{n\in\mathbb{Z}}$ is said to be a sequence of stable sampling in $\mathcal{V}_\pi$ if there exist two constants, $A > 0$ and $B < \infty$, such that for any $f$ in $\mathcal{V}_\pi$ the following holds:

$$A \int_{-\infty}^{\infty} |f(x)|^2 \, dx \leq \sum_n |f(\lambda_n)|^2 \leq B \int_{-\infty}^{\infty} |f(x)|^2 \, dx. \tag{3}$$

If $(\lambda_n)_{n\in\mathbb{Z}}$ is a sequence of stable sampling in $\mathcal{V}_\pi$, not only is every $f$ in $\mathcal{V}_\pi$ uniquely specified by its samples at $(\lambda_n)_{n\in\mathbb{Z}}$, but also, any two signals in $\mathcal{V}_\pi$ which are close at these points, are close everywhere; in particular

$$\int_{-\infty}^{+\infty} |\tilde{f}(x) - f(x)|^2 \, dx \leq \frac{1}{A} \sum_n |\tilde{f}(\lambda_n) - f(\lambda_n)|^2. \tag{4}$$

It is thus this stability property which makes the conversion error *converge* to zero at the $\tau^2$ rate when the quantization threshold crossings of $f$ form a uniformly discrete sequence of stable sampling. The uniform discreteness property is needed to have all the quantization threshold crossings revealed for small enough $\tau$. If the sequence of quantization threshold crossings is not uniformly discrete, then for no finite $\tau$ will all the crossings be revealed; the conversion accuracy then depends on the sampling properties of the subset of quantization threshold crossings which is detected for a given finite $\tau$.

What can we say about the conversion accuracy for signals whose quantization threshold crossings do not form a sequence of stable sampling? If quantization threshold crossings of $f$ form a sequence of uniqueness [6] but not a sequence of stable sampling, then in the limit, when $\tau = 0$, $f$ is uniquely specified by its quantization threshold crossings. In general, however, we cannot claim a specific conversion accuracy for any finite sampling interval $\tau$. This is because when $(\lambda_n)_{n\in\mathbb{Z}}$ is not a sequence of stable sampling even though it might be a sequence of uniqueness, for any positive constant $a$ there exists a signal $\tilde{f}$ in $\mathcal{V}_\pi$ such that

$$\sum_n |\tilde{f}(\lambda_n) - f(\lambda_n)|^2 \leq a$$

$$\text{while} \int_{-\infty}^{+\infty} |\tilde{f}(x) - f(x)|^2 \, dx = 1. \tag{5}$$

[4]A sequence $(\lambda_n)_{n\in\mathbb{Z}}$ is said to be uniformly discrete if $\inf_{m\neq n} |\lambda_m - \lambda_n| > 0$.

When the sequence of quantization threshold crossings of $f$, $(x_n)_{n\in\mathbb{Z}}$, fails to be even a sequence of uniqueness in $\mathcal{V}_\pi$, then there exists a space of signals $\tilde{f}$ such that $\tilde{f}(x_n) = f(x_n)$ for all $x_n$; but the difference between $\tilde{f}$ and $f$ can be arbitrarily large elsewhere. It is not clear whether for *every* $f$ whose quantization threshold crossings do not form a sequence of uniqueness we can find a signal $\tilde{f}$ different from $f$, i.e.,

$$\int_{-\infty}^{+\infty} |\tilde{f}(x) - f(x)|^2 \, dx > 0 \tag{6}$$

and such that $\boldsymbol{C}_\tau^q \tilde{f} = \boldsymbol{C}_\tau^q f$ for all $\tau$; but there do exist signals for which this holds. A trivial example is given by the positive band-limited signals with amplitudes below $q$, which all give the same sequence of quantized samples for all $\tau$; however, their amplitudes can differ up to $q$, and the total energy of their differences can be arbitrarily large. Investigating every possible scenario which results in inferior conversion accuracy can be exceedingly laborious and not relevant in practice. Instead, in Section IV, we discuss simple dithering techniques, which can be used with very general sets of signals in $\mathcal{V}_\pi$, to enforce a sufficiently dense sequence of quantization threshold crossings, thus to ensure $\tau^2$ conversion accuracy.

## III. The $\tau^2$ Conversion Accuracy

As a measure of conversion accuracy, we consider the energy of the difference between two signals which are mapped to the same digital sequence by the conversion operator $\boldsymbol{C}_\tau^q$. Two signals which have this property are said to be *consistent estimates* of each other [5].

*Definition 6:* A signal $\tilde{f}$ is said to be a consistent estimate of a signal $f$ in $\mathcal{V}_\pi$ with respect to the converter $\boldsymbol{C}_\tau^q$ if $\tilde{f}$ is in $\mathcal{V}_\pi$ and $\boldsymbol{C}_\tau^q \tilde{f} = \boldsymbol{C}_\tau^q f$.

For signals which have quantization threshold crossings which form a sequence of stable sampling in $\mathcal{V}_\pi$, the following result holds concerning conversion accuracy.

*Theorem 1:* Let the sequence of quantization threshold crossings of a signal $f$ in $\mathcal{V}_\pi$, relative to a quantizer $\boldsymbol{Q}^q$, be a uniformly discrete sequence of stable sampling in $\mathcal{V}_\pi$. There exists a $\delta > 0$ such that for any $\tau \leq \delta$ and any consistent estimate $\tilde{f}$ of $f$ with respect to $\boldsymbol{C}_\tau^q$

$$\|\tilde{f} - f\|^2 \leq c_f \|f\|^2 \tau^2 \tag{7}$$

where $c_f$ is a constant which depends on the distribution of quantization threshold crossings of $f$ but does not depend on $\tau$ or $\tilde{f}$.

*Proof:* Let $(x_n)_{n\in\mathbb{Z}}$ be the sequence of quantization threshold crossings of $f$, and $A > 0$ and $B < \infty$ the two constants, such that

$$A\|g\|^2 \leq \sum_n |g(x_n)|^2 \leq B\|g\|^2 \tag{8}$$

for all $g$ in $\mathcal{V}_\pi$. Assume that $\tau$ is small enough

$$\tau < \inf_{m\neq n} |x_n - x_m|$$

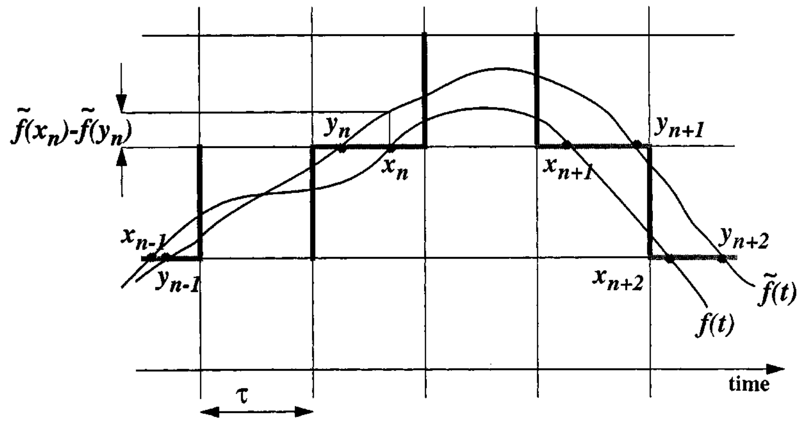


Fig. 2. Quantization threshold crossings of an analog signal $f$ and its consistent estimate $\tilde{f}$. If $f$ goes through a certain quantization threshold at a point $x_n$, then $\tilde{f}$ has to cross the same threshold at a point $y_n$ in the same sampling interval. The error amplitude at the point $x_n$ is equal to $|\tilde{f}(x_n) - f(x_n)| = |\tilde{f}(y_n) - \tilde{f}(x_n)|$.

that no two quantization threshold crossings of $f$ occur in the same sampling interval. Then, for every quantization threshold crossing $x_n$ of $f$ there exists a corresponding crossing of $\tilde{f}$ through the same quantization threshold at some point $y_n$ in the same sampling interval. For each such pair of quantization threshold crossings we, therefore, find that $f(x_n) = \tilde{f}(y_n)$ and $|x_n - y_n| < \tau$. Hence, the amplitude of the error signal $e = \tilde{f} - f$ at $x_n$ is equal to the increment of $\tilde{f}$ on the interval between $x_n$ and $y_n$ (see Fig. 2)

$$e(x_n) = \tilde{f}(x_n) - f(x_n) = \tilde{f}(y_n) - \tilde{f}(x_n). \tag{9}$$

It follows that at each quantization threshold crossing $x_n$, the amplitude of the error signal can be bounded as

$$|e(x_n)| = |\tilde{f}(y_n) - \tilde{f}(x_n)| \leq \tilde{f}'(\xi_n)\tau \tag{10}$$

where $\xi_n$ is a point in the interval between $x_n$ and $y_n$. The energy of the error signal, therefore, satisfies the following upper bound:

$$||e||^2 \leq \frac{1}{A}\sum_n |e(x_n)|^2 \leq \frac{\tau^2}{A}\sum_n |\tilde{f}'(\xi_n)|^2. \tag{11}$$

Since $(x_n)_{n\in Z}$ is a sequence of stable sampling in $\mathcal{V}_\pi$, with bounds $A$ and $B$, there exists a $\delta_{B/A} > 0$ such that any sequence $(z_n)_{n\in Z}$ which satisfies $|x_n - z_n| \leq \delta_{B/A}$ for all $n$ is itself a sequence of stable sampling in $\mathcal{V}_\pi$; moreover

$$\frac{A}{4}||g||^2 \leq \sum_n |g(z_n)|^2 \leq \frac{9B}{4}||g||^2$$

for all $g$ in $\mathcal{V}_\pi$ (see Lemma 1 in Appendix A). Since $|x_n - \xi_n| \leq \tau$ for all $n$, for $\tau \leq \delta_{B/A}$ we find that

$$\sum_n |\tilde{f}'(\xi_n)|^2 \leq z\frac{9B}{4}||\tilde{f}'||^2. \tag{12}$$

This inequality in turn gives

$$||e||^2 \leq \frac{9}{4}\frac{B}{A}\tau^2||\tilde{f}'||^2 \leq \frac{9}{4}\frac{B}{A}\pi^2\tau^2||\tilde{f}||^2. \tag{13}$$

It remains to find a bound on the energy of $\tilde{f}$. Since $|x_n - y_n| < \tau < \delta_{B/A}$, according to Lemma 1 in Appendix A, the following holds:

$$||g||^2 \leq \frac{4}{A}\sum_n |\tilde{f}(y_n)|^2 = \frac{4}{A}\sum_n |f(x_n)|^2 \leq \frac{4B}{A}||f||^2. \tag{14}$$

From this last inequality and the error bound in (13), we obtain that for any consistent estimate $\tilde{f}$ of $f$ for

$$\tau < \min(\inf_{m\neq n} |x_n - x_m|, \delta_{B/A})$$

the error energy satisfies

$$||\tilde{f} - f||^2 \leq \frac{9B^2}{A^2}\pi^2||f||^2\tau^2. \tag{15}$$

This proves the theorem. □

*Remarks:*

1) Note that to prove the $\tau^2$ upper error bound of Theorem 1, in addition to having the sequence of quantization threshold crossings constitute a sequence of stable sampling in $\mathcal{V}_\pi$, it is essential to have the error at the quantization threshold crossings of $f$ be bounded as
$$|\tilde{f}(x_n) - f(x_n)| \leq c_n\tau$$
where $(c_n)_{n\in\mathbb{Z}}$ is a square-summable sequence (see (10)).

2) Theorem 1 implies that the pointwise error power between $f$ and any consistent estimate $\tilde{f}$ of $f$ satisfies
$$|\tilde{f}(t) - f(t)|^2 \leq c_f||f||^2\tau^2 \tag{16}$$
where $c_f$ is as in the statement of the theorem. The discrepancy between this result and the traditional result
$$E(|\tilde{f}(t) - f(t)|^2) = \frac{q^2}{12}\tau$$
is due to the suboptimality of the linear reconstruction to which the traditional result pertains. That is, the linear reconstruction based on low-pass filtering does not necessarily give a consistent estimate of the original; therefore, it does not fully utilize the information contained in the digital representation. This scenario is illustrated in Fig. 3. The suboptimality of the linear reconstruction was first observed by Thao and Vetterli in the context of oversampled A/D conversion of periodic band-limited signals [5]. Algorithms for consistent reconstruction, based on alternating projections onto convex sets, were

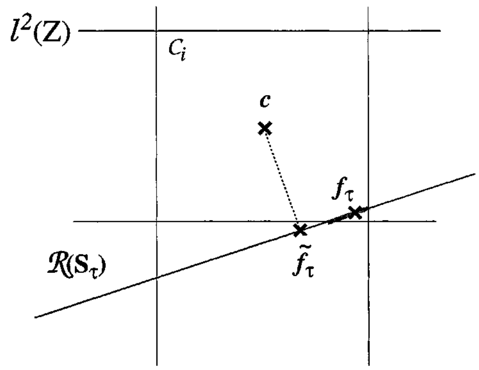


Fig. 3. Inconsistent reconstruction with linear interpolation. When the sampling interval $\tau$ is smaller than the Nyquist interval, the range of the sampling operator $\boldsymbol{S}_\tau$, $\mathcal{R}(\boldsymbol{S}_\tau)$, forms a linear subspace of $\ell^2(\mathbb{Z})$. The uniform scalar quantizer $\boldsymbol{Q}_q$ maps the sequence of samples $f_\tau$ of $f$ in $\mathcal{V}_\pi$ to a point $\boldsymbol{c}$ in the corresponding quantization cell $\mathcal{C}_i$. The linear interpolation between the quantized samples $\boldsymbol{c} = f_\tau^q$ gives a signal $\tilde{f}$, whose sampled version $\tilde{f}_\tau$ is the orthogonal projection of $\boldsymbol{c}$ onto $\mathcal{R}(\boldsymbol{S}_\tau)$. However, $\tilde{f}_\tau$ and $f_\tau$ need not be in the same quantization cell, and, therefore, $\tilde{f}$ need not be a consistent estimate of $f$, as illustrated in this figure.

studied in [5] and [9] for the case of periodic band-limited signals. Theoretically, these algorithms can be extended to band-limited signals in $L^2(\mathbb{R})$; however, these algorithms would not be practical since they require dealing with ideal low-pass filters that are noncausal and not compactly supported in time. Algorithms for local reconstruction with $\tau^2$ accuracy are currently being investigated.

## IV. Sampling Properties of Sequences of Quantization Threshold Crossings

For a given A/D converter, there exist signals which have quantization threshold crossings that constitute a sequence of stable sampling in $\mathcal{V}_\pi$ as well as signals which do not have this property. Precise description of these two classes of signals is a question which we cannot answer in a clear and concise manner. The major problem involved is giving a precise characterization of sequences of stable sampling in $\mathcal{V}_\pi$ that have uniform density equal to 1. In this section, we review some relevant results of nonharmonic Fourier analysis to provide intuition about classes of signals which exhibit the $\tau^2$ error behavior, and also to suggest modulation schemes which would ensure the $\tau^2$ conversion accuracy for a broad class of band-limited signals. We shall use results on stable sampling which are formulated in terms of frame properties of corresponding systems of complex exponentials.

*Definition 7 [8]:* A system of complex exponentials $(e^{jx_n\omega})_{n\in\mathbb{Z}}$ is said to be a *frame* in $L^2(-\pi, \pi)$ if there exist two constants $A > 0$ and $B < \infty$ such that for any $\hat{f}(\omega)$ in $L^2(-\pi, \pi)$ the following holds:

$$A \le \frac{\frac{1}{2\pi}\sum_n \left|\int_{-\pi}^{\pi} \hat{f}(\omega) e^{jx_n\omega}\, d\omega\right|^2}{\int_{-\pi}^{\pi} |\hat{f}(\omega)|^2\, d\omega} \le B. \tag{17}$$

.

Note that a sequence of real numbers $(x_n)_{n\in\mathbb{Z}}$ is a sequence of stable sampling in $\mathcal{V}_\pi$ if and only if the corresponding system of complex exponentials, $(e^{jx_n\omega})_{n\in\mathbb{Z}}$, is a frame in $L^2(-\pi, \pi)$. Analogously, $(x_n)_{n\in\mathbb{Z}}$ is a sequence of uniqueness in $\mathcal{V}_\pi$ if and only if $(e^{jx_n\omega})_{n\in\mathbb{Z}}$ is complete in $L^2(-\pi, \pi)$.

Any sequence of stable sampling is a finite union of uniformly discrete sequences with high enough densities. Recall that a sequence $(x_n)_{n\in\mathbb{Z}}$ of real numbers is said to be uniformly discrete if

$$d = \inf_{m\neq n} |x_m - x_n| > 0.$$

A quantitative characterization of the sequence density needed for stable sampling in $\mathcal{V}_\pi$ can be given based on the notion of *lower uniform density*. The lower uniform density for a uniformly discrete sequence $(x_n)_{n\in\mathbb{Z}}$ is defined as [10]

$$D^-[(x_n)_{n\in\mathbb{Z}}] = \lim_{s\to\infty} \frac{n^-(s)}{s} \tag{18}$$

where $n^-(s)$ denotes the smallest number of points of the sequence on an interval of length $s$. We say that a sequence $(x_n)_{n\in\mathbb{Z}}$ is *relatively uniformly discrete* if it can be expressed as a finite union of uniformly discrete sets. The lower uniform density of a relatively uniformly discrete sequence $(x_n)_{n\in\mathbb{Z}}$ is defined as

$$D^-[(x_n)_{n\in\mathbb{Z}}] = \sup_{(x'_n)_{n\in\mathbb{Z}}} D^-[(x'_n)_{n\in\mathbb{Z}}] \tag{19}$$

where the supremum is taken over the uniformly discrete subsequences $(x'_n)_{n\in\mathbb{Z}}$ of $(x_n)_{n\in\mathbb{Z}}$, $(x'_n)_{n\in\mathbb{Z}} \subset (x_n)_{n\in\mathbb{Z}}$. The following theorem of Jaffard [11] relates sampling properties of a sequence of real number to its density characteristics.

*Theorem 2[11]:* For a system $(e^{jx_n\omega})_{n\in\mathbb{Z}}$ to be a frame in $L^2(-\pi, \pi)$, it is necessary that $(x_n)_{n\in\mathbb{Z}}$ be relatively uniformly discrete and $D^-[(x_n)_{n\in\mathbb{Z}}] \ge 1$, and it is sufficient that $(x_n)_{n\in\mathbb{Z}}$ be relatively uniformly discrete and $D^-[(x_n)_{n\in\mathbb{Z}}] > 1$.

What are the implications of Jaffard's theorem for the accuracy of oversampled A/D conversion? We restrict ourselves to signals which have quantization threshold crossings that form relatively uniformly discrete sequences, since signals which are not in this class oscillate increasingly rapidly around zero as time tends to infinity, and are of minor practical interest. First we show that the lower uniform density of a sequence of quantization threshold crossings which corresponds to a signal in $\mathcal{V}_\pi$ cannot be larger than one.

*Theorem 3:* Let quantization threshold crossings of an $f$ in $\mathcal{V}_\pi$ with respect to a quantizer $\boldsymbol{Q}^q$ form a relatively uniformly discrete sequence $(x_n)_{n\in\mathbb{Z}}$. If the uniform density of $(x_n)_{n\in\mathbb{Z}}$ is greater than 1, $D^-[(x_n)_{n\in\mathbb{Z}}] > 1$, then $f$ is the zero function.

To prove this theorem, we note that since $f$ is square integrable, its quantization threshold crossings are eventually all zero crossings. That is, there exists a $t_f > 0$ such that $f(x_n) = 0$ for all quantization threshold crossings $x_n$ which are outside the interval $(-t_f, t_f)$. Since $(x_n)_{n\in\mathbb{Z}}$ is relatively uniformly discrete, it remains relatively uniformly discrete with the same lower uniform density after a finite number of points is removed. Considering that all except finitely many quantization threshold crossings $x_n$ are zero crossings, it follows from Theorem 2 that if $(x_n)_{n\in Z}$ forms a relatively uniformly discrete set, its lower uniform density cannot be greater than 1, except when $f$ is the zero function.

*Corollary 1:* If the sequence of quantization threshold crossings $(x_n)_{n\in\mathbb{Z}}$ of a signal $f$ in $\mathcal{V}_\pi$ forms a sequence of stable sampling, then $(x_n)_{n\in\mathbb{Z}}$ must be relatively uniformly discrete with a lower uniform density equal to 1.

However, no sufficient and necessary condition is known under which a relatively uniformly discrete sequence with a lower uniform density equal to 1 is a sequence of stable sampling in $\mathcal{V}_\pi$; this represents the main difficulty in precisely characterizing the set of signals which satisfy the postulate of Theorem 1. Below, employing a sufficient condition by Avdonin [12], we demonstrate that the signals for which Theorem 1 holds constitute a rich set of functions.

*Theorem 4 [12]:* Let $x_n = n + \delta_n$, and suppose that the sequence $(x_n)_{n\in\mathbb{Z}}$ is uniformly discrete. If there exists a positive integer $M$ and a positive number $d < 1/4$ such that

$$\left|\sum_{k=mM+1}^{(m+1)M} \delta_k\right| \leq dM$$

for all integers $m$, then the system $(e^{jx_n\omega})_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(-\pi, \pi)$.

*Corollary 2:* Assume that zero crossings of a function $f$ in $\mathcal{V}_\pi$ form a uniformly discrete sequence, and suppose that there exist positive numbers $n_0$ and $d < 1/4$ such that $f$ has a zero crossing in every interval $(n-d, n+d)$, where $n$ is an integer and $|n| > n_0$. Then, given a quantizer $\boldsymbol{Q}^q$, there exists an $\alpha_0 = \alpha_0(f, q) > 0$ such that for any $\alpha > \alpha_0$ quantization threshold crossings of the function $\alpha f$ form a uniformly discrete sequence of stable sampling in $\mathcal{V}_\pi$.

This corollary follows from the fact that, given an $f$ which satisfies the assumption of the corollary, there exists a scaling factor $\alpha_0$ such that the scaled signal $\alpha_0 f$ has at least $2n_0 + 1$ quantization threshold crossings different from zero. Hence, for any $\alpha > \alpha_0$, there exists a subsequence $(x_n)_{n\in\mathbb{Z}}$ of quantization threshold crossings of $\alpha f$ which satisfies the condition of Avdonin's theorem, and, therefore, constitutes a uniformly discrete sequence of stable sampling in $\mathcal{V}_\pi$. A class of signals which satisfy the assumption of Corollary 2 can be described using the following theorem.

*Theorem 5 [14]:* If the Fourier transform $\hat{f}(\omega)$ of a signal $f$ in $\mathcal{V}_\pi$ has bounded derivatives of order up to $k$ for every $\omega < \pi$, then $f$ has the following expansion as $t$ tends to infinity:

$$f(t) = a_1\frac{\sin(\pi t + \alpha_1)}{t} + a_2\frac{\sin(\pi t + \alpha_1)}{t^2} + \cdots + a_k\frac{\sin(\pi t + \alpha_k)}{t^k} + O\left(\frac{1}{t^{k+1}}\right)$$

where $a_l = |\hat{f}^{(l-1)}(\pi)|/\pi$ and [5]

$$\alpha_l = \arg(\hat{f}^{(l-1)}(\pi)) + (l-1)\pi/2.$$

[5] We refer to the left derivatives at $\pi$.

The implication of Theorem 5 is that if the Fourier transform $\hat{f}(\omega)$ of $f$ has bounded derivatives of order up to $k$ for every $\omega < \pi$, and $|\hat{f}^{(l)}(\pi)| \neq 0$ for some $l < k$ then $f$ behaves asymptotically as

$$f(t) \sim A\frac{\sin(\pi t + \alpha)}{t^{L+1}} + O\left(\frac{1}{t^{L+2}}\right)$$

where $L$ is the smallest $l$, $l < k$, such that $|\hat{f}^{(l)}(\pi)| \neq 0$. Zero crossings of $f$, therefore, satisfy the assumption of Corollary 2 and, for a large enough scaling factor $\alpha$, zero crossings of $\alpha f$ form a uniformly discrete sequence of stable sampling in $\mathcal{V}_\pi$. This argument proves the following theorem.

*Theorem 6:* Let $f$ be a function in $\mathcal{V}_\pi$. Assume that the Fourier transform $\hat{f}(\omega)$ of $f$ has bounded derivatives of order up to $k$, $k \geq 1$, for all $\omega \in (-\pi, \pi)$, and that $|\hat{f}^{(l)}(\pi)| > 0$ for an $l < k$. Then, given a quantizer $\boldsymbol{Q}^q$, there exists a positive number $\alpha_0 = \alpha_0(f, q)$ such that for any $\alpha > \alpha_0$ the quantization threshold crossings of $\alpha f$ form a uniformly discrete sequence of stable sampling in $\mathcal{V}_\pi$.

Theorem 6 describes only a subset of band-limited signals which exhibit the $\tau^2$ error behavior in oversampled A/D conversion, namely, only those satisfying the sufficient condition given by Avdonin's theorem. Another way to characterize signals for which Theorem 1 holds is to consider signals $f$ given by

$$f(x_n) = l_n q, \qquad l_n \in \mathbb{N}, \ \sum_{n\in\mathbb{Z}} l_n^2 < \infty$$

for all sequences $(x_n)_{n\in\mathbb{Z}}$ which satisfy Avdonin's theorem, but these considerations exceed the scope of this paper. The purpose of this discussion is merely to demonstrate that signals for which Theorem 1 holds do not belong to some very restricted set but rather constitute a rich class of band-limited square-integrable functions. We now focus on techniques which can be used to ensure the $\tau^2$ conversion accuracy on a broader class of signals.

There are two modulation techniques, commonly used in the practice of A/D conversion, that can be used to enforce the desired distribution of quantization threshold crossings on a very broad class of signals in $\mathcal{V}_\pi$. One technique is the so-called *ordered dither*, and the other technique relies on signal adaptive amplification [15]. The latter technique consists of estimating signal energy on given time intervals and then amplifying the portions of the signal which have low energy content. In effect, this approach is equivalent to employing a uniform quantizer with a quantization step which is not a constant but rather a piecewise-constant function that decays following the decay of the input signal. In Appendix B, we give an alternative, intuitive proof of Theorem 1 for the case of A/D conversion with time-adaptive uniform quantization.

The ordered dither approach consists in introducing a perturbation, and we consider one possible deterministic dither in the form of a linear combination of $\mathrm{sinc}$ functions.

*Theorem 7:* If the Fourier transform of a signal $f$ in $\mathcal{V}_\pi$ has bounded variation, then given a quantizer $\boldsymbol{Q}^q$ there exist a positive number $\alpha_0$ and a positive integer $N_0$ such that the quantization threshold crossings of the dithered signal $s = f + d$, where

$$d(t) = \alpha q \sum_{n=-N}^{N} (-1)^n \frac{\sin(\pi(t-n))}{\pi(t-n)} \tag{20}$$

form a uniformly discrete sequence of stable sampling in $\mathcal{V}_\pi$ for all $\alpha > \alpha_0$ and $N > N_0$.

To show that the dither of Theorem 7 would enforce a pattern of quantization threshold crossings that form a sequence of stable sampling, we note that if the Fourier transform of a function $f$ in $\mathcal{V}_\pi$ is of bounded variation, then $f$ decays as $1/t$ as $t$ tends to infinity. That is, there exist a $t_0 > 0$ and a $c > 0$ such that $|f(t)| < c/|t|$ for all $t$, $|t| > t_0$ [14]. It follows then that there exists a dither function of the form (20) and a positive number $n_0$ such that for all $n$, $|n| > n_0$, the following holds:

i)

$$|f(n \pm 1/8)| < |d(n \pm 1/8)|$$
$$\mathrm{sgn}[d(n+1/8)] = -\mathrm{sgn}[d(n-1/8)].$$

ii)

$$|d'(t)| > |f'(t)|, \qquad \text{for all } t \in (n - 1/8, n + 1/8).$$

It follows further that the dithered function $f + d$ has one zero crossing in intervals $(n-1/8, n+1/8)$ for all integer $n$, $|n| > n_0$. For $\alpha$ large enough, the dithered signal also has sufficiently many, i.e., $2n_0+1$, quantization threshold crossings in the interval $(-n_0 - 1/4, n_0 + 1/4)$. Hence, quantization threshold crossings of the dithered signal form a uniformly discrete sequence, a subset of which satisfies Avdonin's theorem. The oversampled A/D conversion applied to the dithered signal produces, therefore, a digital sequence which describes $f$ with the $\tau^2$ accuracy.

*Remarks:*

1) The dither function in (20) can be defined with either $\alpha = 1$ or $N = 1$, but having both parameters be variable makes it possible to tailor the dither signal better according to different classes of signals, so that desirable patterns of quantization threshold crossings can be attained using dither with smaller amplitude.
2) Note that for all $f$ in $\mathcal{V}_\pi$ with amplitude less than some $a > 0$, the zero crossings of the dithered signal $s(t) = f(t) + a \sin \pi t$ form a uniformly discrete sequence of stable sampling in $\mathcal{V}_\pi$. This kind of dither was previously used by Logan [13] in his work on reconstruction of band-limited signals from their zero crossings. However, to establish the $\tau^2$ bound on the energy of the conversion error, it is essential that all signals involved have bounded energy; and for that reason in Theorem 7 we propose a square-integrable dither in the form of a linear combination of $\mathrm{sinc}$ functions.

## V. Efficient Lossless Coding

The idea of efficient lossless encoding of digital sequences generated in the process of oversampled A/D conversion originates in the observation that it is the information about positions of quantization threshold crossings of the input signal that is successively refined as the sampling interval tends toward zero; and that representing that information directly provides an efficient way to encode the sequence of quantized samples [7]. To demonstrate this, consider quantization threshold crossings of an input signal $f$ grouped on consecutive time intervals of a given length $T$. If for a given sampling interval $\tau_0$ at most $B_m$ bits are needed to represent the information about the locations and values of the quantization threshold crossings on an interval of length $T$, then as the sampling interval tends toward zero, the required bit rate can be bounded as

$$R \le \frac{B_m}{T}\left(2 + \log_2 \frac{\tau_0}{\tau}\right). \tag{21}$$

The bit rate thus increases only as a logarithm of the sampling interval. Note that if the quantized sample values are themselves encoded, the bit rate increases inversely to the sampling interval, $R = O(1/\tau)$. The implication of the bit-rate bound in (21) together with the error bound in (7) is that in oversampled A/D conversion with the quantization threshold crossings-based coding, the error is an exponentially decaying function of the bit rate

$$\|e\|^2 \le K 2^{-2\beta R}. \tag{22}$$

To assess parameters of this error-rate characteristic, let us consider details of a specific encoding scheme. Specifying the position of a quantization threshold crossing with resolution $\tau$ on an interval of length $T$ requires at most $1 + \log_2(T/\tau)$ bits. The level of the quantization threshold can be represented with respect to the level of the preceding threshold crossing. For this information, one additional bit is needed to denote the direction of the crossing. We assume that $\tau$ is small enough so that $f$ cannot go through more than one quantization threshold on a time interval of length $\tau$. Hence, specifying the information about quantization threshold crossings of $f$ on an interval where $Q$ of them occur requires at most $Q(2 + \log_2(T/\tau)) + C_0$ bits. The constant $C_0$ represents the number of bits used to specify the level of the first crossing on the interval. The bit rate can thus be bounded as

$$R \le \frac{Q_m}{T}\left(2 + \log_2\left(\frac{T}{\tau}\right)\right) + \frac{C_0}{T} \tag{23}$$

where $Q_m$ denotes the maximal number of the crossings of $f$ on an interval of length $T$. Based on this it can be shown that the constants $\beta$ and $K$ in (22) are given as $\beta = T/Q_m$ and $K = 16 c_f T^2 2^{2C_0/Q_m}$, where $c_f$ is the signal-dependent constant which appears in (7).

In order to estimate the exponent $\beta$, we consider the two types of quantization threshold crossings, denoting them as $d$-crossings and $s$-crossings. A quantization threshold crossing is said to be a $d$-crossing if it is preceded by a crossing of a different quantization threshold, and an $s$-crossing if it is preceded by a crossing of the same threshold. The total number of quantization threshold crossings of a $\pi$-band-limited signal $f$ on an interval $T$ is the sum of these two types of crossings. The count of $d$-crossings, $Q_d$, depends on the slope of $f$ as well as on the quantization step size $q$. The slope of $f$ can be bounded as $|f'(x)| \leq \pi\|f\|$, which gives

$$\frac{Q_d}{T} \leq \frac{\pi}{q}\|f\|. \tag{24}$$

As for the count of $s$-crossings, $Q_s$, from the considerations in the previous section we conclude that for signals in $\mathcal{V}_\pi$ for which the $\tau^2$ error bound holds

$$\lim_{T\to\infty} \frac{Q_s}{T} \to 1. \tag{25}$$

Hence, the exponent $\beta$ in (22) has the form $\beta = 1/(\alpha_1 + \alpha_2)$ where $\alpha_1 = Q_d/T \leq \pi\|f\|/q$ and the order of magnitude of $\alpha_2 = Q_s/T$ is 1. However, considering that there is only a finite number of $d$-crossings, for large $T$ we obtain that $\alpha_1 \approx 0$, and, therefore, $\beta \approx 1$.

Recall that the average error power of A/D conversion with a fixed sampling interval and quantization refinement behaves as $E(|e(x)|^2) = O(2^{-2R})$, and observe that we approach this error-rate characteristic also in the case of oversampled A/D conversion and quantization threshold crossings-based coding.

## VI. Conclusion

Let us summarize results about the dependence of accuracy of oversampled A/D conversion on the sampling interval size and on the bit rate. Recall that the accuracy of the linear reconstruction is characterized by average error power and that in a certain range of conversion parameters [2] it is given by $E(|e(t)|^2) = O(\tau)$. On the other hand, results on accuracy, assuming consistent estimation, are obtained using deterministic analysis, which asserts that the error energy behaves as $\|e\|^2 = c\tau^2$. Results concerning the dependence of the bit rate on the sampling interval size are established without reference to a particular kind of reconstruction. Thus, without any entropy coding, the bit-rate oversampled A/D conversion is $R = O(1/\tau)$, and with the quantization threshold crossings-based coding the bit rate becomes $R = O(-\log\tau)$. The overall error-rate characteristic of oversampled A/D conversion with the quantization threshold crossings-based coding thus exhibits an exponential decay $\|e\|^2 \sim ce^{-2\beta R}$, where $\beta$ is close to 1.

## Appendix A

Estimates of bounds on the quantization error are derived using the following two lemmas, which are adapted from [8].

*Lemma 1 [8]:* Let $(e^{j\lambda_n\omega})_{n\in\mathbb{Z}}$ be a frame in $L^2(-\pi, \pi)$, with bounds $A > 0$ and $B < \infty$, and $\delta$ a positive number. If a sequence $(\mu_n)_{n\in\mathbb{Z}}$ satisfies $|\lambda_n - \mu_n| \leq \delta$ for all $n$, then for every $f$ in $\mathcal{V}_\pi$

$$A\left(1-\sqrt{C}\right)^2\|f\|^2 \leq \sum_n |f(\mu_n)|^2 \leq B\left(1+\sqrt{C}\right)^2\|f\|^2 \tag{26}$$

where

$$C = \frac{B}{A}(e^{\pi\delta} - 1)^2. \tag{27}$$

*Remark:* If $\delta$ in the statement of Lemma 1 is chosen small enough, so that $C < 1$, then $(e^{j\mu_n\omega})_{n\in\mathbb{Z}}$ is also a frame in $L^2(-\pi, \pi)$. Moreover, there exists a $\delta_{1/4} > 0$, such that whenever $\delta < \delta_{1/4}$, $(e^{j\mu_n\omega})_{n\in\mathbb{Z}}$ is a frame with frame bounds $A/4$ and $9B/4$.

*Lemma 2 [8]:* Let $(e^{j\lambda_n\omega})_{n\in\mathbb{Z}}$ be a frame in $L^2(-\pi, \pi)$. If $M$ is a positive number and $(\mu_n)_{n\in\mathbb{Z}}$ is a sequence satisfying $|\mu_n - \lambda_n| \leq M$ for all $n$, then there exists a number $C$ which depends on $M$ and $(\lambda_n)_{n\in\mathbb{Z}}$ such that

$$\frac{\sum_n |f(\mu_n)|^2}{\sum_n |f(\lambda_n)|^2} \leq C \tag{28}$$

for every function $f$ in $\mathcal{V}_\pi$.

## Appendix B

Assume that the quantization step $q$ changes following the decay of $f$. Let $A$ and $B$ be the lower and the upper bound, respectively, of the frame of complex exponentials $(e^{jx_n\omega})_{n\in\mathbb{Z}}$, where $(x_n)_{n\in\mathbb{Z}}$ is the sequence of quantization threshold crossings of $f$, and let $\tilde{f}$ be a consistent estimate of $f$. Following the discussion in the proof of Theorem 1 (see (11)), the error norm can be bounded as

$$\|e\|^2 < \frac{\tau^2}{A}\sum_n |\tilde{f}'(\xi_n)|^2. \tag{29}$$

Since $\tau$ is smaller then the Nyquist sampling interval, $|x_n - \xi_n| < 1$ for all $n$. According to Lemma 2, there exists a number $C$ such that for all sampling intervals $\tau < 1$

$$\sum_n |\tilde{f}'(\xi_n)|^2 \leq C\sum_n |\tilde{f}'(x_n)|^2 \tag{30}$$

which implies the following error bound:

$$\|e\|^2 \leq \pi^2\frac{BC}{A}\|\tilde{f}\|^2\tau^2. \tag{31}$$

Being a consistent estimate of $f$, $\tilde{f}$ cannot differ from $f$ by more than $q(n\tau)$ at time instants $n\tau$. By considering samples of $\tilde{f}$ at these points, the energy of $\tilde{f}$ can be bounded as follows:

$$\begin{aligned}\|\tilde{f}\|^2 &= \tau\sum_n |\tilde{f}(n\tau)|^2\\ &\leq \tau\sum_n \left(|f(n\tau)|^2 + 2|q(n\tau)f(n\tau)| + |q(n\tau)|^2\right)\\ &= \|f\|^2 + E_s(\tau)\end{aligned}$$

where

$$E_s(\tau) = \tau\sum_n \left(2|q(n\tau)f(n\tau)| + |q(n\tau)|^2\right).$$

When $\tau$ tends toward zero, the sum $E_s(\tau)$ converges to the integral

$$\lim_{\tau\to 0} E_s(\tau) = \int_{-\infty}^{+\infty} \left(2|q(t)f(t)| + |q(t)|^2\right)\, dt \tag{32}$$

which has to be finite since $q(x) = O(f(x))$ and $f$ is square-integrable. Therefore, there exists a $\tau_0$ and an $E > 0$ such that $E_s(\tau) < E$ for all $\tau < \tau_0$, which finally gives the error bound

$$||e||^2 \leq \pi^2 \frac{BC}{A} (||f||^2 + E)\tau^2. \quad (33)$$

## ACKNOWLEDGMENT

The authors wish to thank J. Lagarias, H. Landau, and N. Thao for their fruitful discussions on the present topic.

## REFERENCES

[1] R. M. Gray, *Source Coding Theory*. Boston, MA: Kluwer, 1990.
[2] W. R. Bennett, "Spectra of quantized signals," *Bell Syst. Tech. J.*, vol. 27, pp. 446–472, July 1948.
[3] A. G. Clavier, P. F. Panter, and D. D. Grieg, "Distortion in a pulse count modulation system," *AIEE Trans.*, vol. 66, pp. 989–1005, 1947.
[4] ——, "PCM distortion analysis," *Elec. Eng.*, pp. 1110–1122, Nov. 1947.
[5] N. T. Thao and M. Vetterli, "Reduction of the MSE in R-times oversampled A/D conversion from $O(1/R)$ to $O(1/R^2)$," *IEEE Trans. Signal Processing*, vol. 42, pp. 200–203, Jan. 1994.
[6] H. J. Landau, "Sampling, data transmission, and the Nyquist rate," *Proc. IEEE*, vol. 55, pp. 1710–1706, Oct. 1967.
[7] Z. Cvetković and M. Vetterli, "Error-rate characteristics of oversampled analog-to-digital conversion," *IEEE Trans. Inform. Theory*, vol. 44, pp. 1961–1964, Sept. 1998.
[8] R. J. Duffin and A. C. Schaeffer, "A class of nonharmonic Fourier series," *Trans. Amer. Math. Soc.*, vol. 72, pp. 341–366, Mar. 1952.
[9] S. Hein and A. Zakhor, "Reconstruction of oversampled band-limited signals from $\Sigma\Delta$ encoded binary sequences," *IEEE Trans. Signal Processing*, vol. 42, pp. 799–811, Apr. 1994.
[10] A. Beurling, "The collected works of Arne Beurling," in *Harmonic Analysis*, L. Carleson, P. Malliavin, J. Neuberger, and J. Wermer, Eds. Boston, MA: Birkhäuser, 1989, pp. 341–365.
[11] S. Jaffard, "A density criterion for frames of complex exponentials," *Michigan Math. J.*, vol. 38, pp. 339–348, 1991.
[12] S. A. Avdonin, "On the question of Riesz bases of exponential functions in $L^2$," *Vestnik Leningrad. Univ. Ser. Math.*, vol. 13, 1974. English translation in *Vest. Leningrad Univ. Math.*, vol. 7, pp. 5–12, 1979.
[13] B. F. Logan, "Signals designed for recovery after clipping—I. Localization of infinite products," *AT&T Bell Lab. Tech. J.*, vol. 63, no. 2, pp. 261–285, Feb. 1984.
[14] A. Papoulis, *Signal Analysis*. New York: McGraw-Hill, 1972.
[15] N. S. Jayant and P. Noll, *Digital Coding of Waveforms*. Englewood Cliffs, NJ: Prentice-Hall, 1984.