Stochastic Calculus for Fractional Brownian Motion.
I: Theory

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Abstract

This paper describes some of the results in [5] for a stochastic calculus for a fractional Brownian motion with the Hurst parameter in the interval (1/2, 1). Two stochastic integrals are defined with explicit expressions for their first two moments. Multiple and iterated integrals of a fractional Brownian motion are defined and various properties of these integrals are given. A square integrable functional on a probability space of a fractional Brownian motion is expressed as an infinite series of multiple integrals.

1 Introduction

Fractional Brownian motion is a family of Gaussian processes that are indexed by the Hurst parameter $H$ in the interval $(0, 1)$. These processes were introduced by Kolmogorov [10]. The first application of these processes was made by Hurst [7], [8] who used them to model the long term storage capacity of reservoirs along the Nile River. Mandelbrot [12] used these processes to model some economic time series and most recently these processes have been used to model telecommunication traffic (e.g., [11]). Two important properties of these Gaussian processes for modeling are self similarity and, for $H \in (1/2, 1)$, a long range dependence. The self similarity means that if $a > 0$ then $(B^H(at), t \geq 0)$ and $(a^H B^H(t), t \geq 0)$ have the same probability law where $(B^H(t), t \geq 0)$ is a (standard) fractional Brownian motion. The long range dependence means that if $r(n) = E[B^H(1)(B^H(n + 1) - B^H(n))]$ then $\sum_{n=1}^{\infty} r(n) = \infty$.

Now a fractional Brownian motion is defined. For each $H \in (0, 1)$, a real-valued Gaussian process $(B^H(t), t \geq 0)$ is defined such that $E[B^H(t^2)] = 0$ and

$$E[|B^H(t)B^H(s)|] = \frac{1}{2}(|t|^2H + |s|^{2H} - |t - s|^{2H})$$

for all $t, s \in \mathbb{R}_+$. If $H = 1/2$ then the fractional Brownian motion is a standard Brownian motion (Wiener process). These processes have a version with continuous sample paths. In this paper $H$ is restricted to the interval $(1/2, 1)$. The $p$th variation of such a process is nonzero and finite for $p = 1/H$, that is, if $(P_n, n \in \mathbb{N})$ is sequence of partitions of $[0, 1]$ that are refinements of the previous and become dense in $[0, 1]$ then

$$\lim_{n \to \infty} \sum_{i=1}^{n} |B^H(t_i^{(n)}) - B^H(t_{i-1}^{(n)})|^p = c(p) \quad \text{a.s.}$$

where $P_n = \{t_0^{(n)}, \ldots, t_n^{(n)}\}$ and $c(p) = E|B^H(1)|^p$ (e.g., [13]). For $H > 1/2$, $(B^H(t), t \geq 0)$ is not a semimartingale and not Markov. These facts require that a different stochastic calculus be used.

In this paper some results of a stochastic calculus from [5] are described. This description complements [4]. Some other approaches to stochastic calculus have been given in [1], [2], [3]. In Section 2, a directional derivative in the path space is given and two stochastic integrals with respect to a fractional Brownian motion are defined. The Wick product and the Hermite polynomials are introduced. In Section 3, multiple and iterated integrals with respect to a fractional Brownian motion are shown to satisfy many properties that are satisfied for the analogous integrals with respect to a Brownian motion. A square integrable functional on a probability space of a fractional Brownian motion is expressed as an infinite series of multiple integrals, which generalizes the well known result for Brownian motion.

2 Some Methods for Stochastic Calculus

Let $\Omega = C_0(\mathbb{R}_+, \mathbb{R})$ be the Fréchet space of real-valued continuous functions on $\mathbb{R}_+$ with initial value zero and the topology of local uniform convergence. There is a probability measure, $P^H$, on $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the Borel $\sigma$-algebra such that on the probability space $(\Omega, \mathcal{F}, P^H)$, the coordinate process is a fractional Brow-
nian motion, that is,

\[ B^H(t, \omega) = \omega(t) \]

for each \( t \in \mathbb{R}_+ \) and (almost all) \( \omega \in \Omega \).

Let \( \phi : \mathbb{R} \rightarrow \mathbb{R}_+ \) be given by

\[ \phi(t) = H(2H - 1)|t|^{2H - 2}. \] \( (1) \)

It follows directly that

\[ \mathbb{E}[B^H(t)B^H(s)] = \int_0^t \int_0^s \phi(u - v)du dv. \] \( (2) \)

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be Borel measurable. The function \( f \in L^2_{\phi} \) if

\[ |f|^2 = \int_0^\infty \int_0^\infty f(s)f(t)\phi(s - t)dsdt < \infty \] \( (3) \)

The Hilbert space \( L^2_{\phi} \) is naturally associated with the Gaussian process \( (B^H(t), t \geq 0) \). The inner product on \( L^2_{\phi} \) is denoted by \( \langle \cdot, \cdot \rangle_\phi \).

A notion of directional derivative in \( \Omega \) in directions associated with \( L^2_{\phi} \) is important in some computations with stochastic integrals.

**Definition 2.1** The \( \phi \)-derivative of a random variable \( F \in L^p \) in the direction \( \Phi g \) for \( g \in L^2_{\phi} \) is defined as

\[ D_{\Phi g}F(\omega) = \lim_{\delta \to 0} \frac{1}{\delta} \left[ F(\omega + \delta \int_0^t (\Phi g)(s)ds) - F(\omega) \right] \]

if the limit exists in \( L^p \) where

\[ (\Phi g)(t) = \int_0^\infty \phi(t - u)g(u)du \]

and \( t \geq 0 \). Furthermore, if there is a process \( (D^s_{\Phi}F, s \geq 0) \) such that

\[ D_{\Phi g}F = \int_0^\infty D^s_{\Phi}Fg(s)ds \]

for each \( g \in L^2_{\phi} \), then the random variable \( F \) is said to be \( \phi \)-differentiable.

The notion of \( \phi \)-differentiability is also defined for a process.

**Definition 2.2** The process \( (F(t), t \geq 0) \) is said to be \( \phi \)-differentiable if for each \( t \in \mathbb{R}_+ \), \( F(t) \) is \( \phi \)-differentiable and \( D^s_{\Phi}F : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R} \) is jointly measurable.

The Wick product of two random variables is denoted \( \diamond \). This product is important in the construction of the stochastic integrals (of Itô type).

**Definition 2.3** Let \( \mathcal{L}(0, T) \) be the family of processes on \( [0, T] \) such that \( F \in \mathcal{L}(0, T) \) if \( \mathbb{E}[F^2] < \infty \), \( F \) is \( \phi \)-differentiable, the trace of \( (D^s_{\phi}F, s, t \in [0, T]) \) exists and \( \mathbb{E}\int_0^T (D^s_{\phi}F_s)^2ds < \infty \) and for each sequence of partitions \( (\pi_n, n \in \mathbb{N}) \) of \( [0, T] \) such that \( |\pi_n| \to 0 \) as \( n \to \infty \)

\[ \sum_{i=0}^{n-1} \mathbb{E} \left[ \int_{\pi_{i+1}}^{\pi_i} |D^s_{\phi}F_{\pi_i} - D^s_{\phi}F_{\pi_{i+1}}|^2ds \right] \]

and

\[ \mathbb{E}[F^\pi - F]_{\phi}^2 \]

tend to 0 as \( n \to \infty \) where \( \pi_n = \{t_0^{(n)}, \ldots, t_n^{(n)}\} \) and \( F^\pi \) is the simple process induced by \( \pi_n \).

The stochastic integral of \( F \in \mathcal{L}(0, T) \) is constructed from Riemann sums using the Wick product as

\[ \sum_{i=0}^{n-1} F_{\pi_i} \diamond (B^H(t_{i+1}) - B^H(t_i)). \] \( (4) \)

**Theorem 2.1** Let \( F \) be a process in \( \mathcal{L}(0, T) \). The limit in \( L^2(P) \) of Riemann sums of the form \( (4) \) exists for each sequence of partitions \( (\pi_n, n \in \mathbb{N}) \) such that \( |\pi_n| \to 0 \) as \( n \to \infty \) and the limit does not depend on the sequence of partitions. This limit is denoted as \( \int_0^T FdB^H \). Furthermore, \( \mathbb{E}[\int_0^T FdB^H]^2 = 0 \) and

\[ \mathbb{E} \left[ \int_0^T FdB^H \right]^2 = \mathbb{E} \left[ \left( \int_0^T D^s_{\phi}F_s ds \right)^2 + |F|_{\phi}^2 \right]. \] \( (5) \)

A stochastic integral of Stratonovich type is now defined.

**Definition 2.4** Let \( (\pi_n, n \in \mathbb{N}) \) be a sequence of partitions of \( [0, T] \) such that \( |\pi_n| \to 0 \) as \( n \to \infty \) and is dense. If the sequence of random variables

\[ \left( \sum_{i=0}^{n-1} F(t_{i+1}^{(n)}) (B^H(t_{i+1}^{(n)}) - B^H(t_i^{(n)})) \right) \]

converges in \( L^2(P) \) to the same limit for each sequence of partitions, then this limit is called the stochastic integral of Stratonovich type and the limit is denoted \( \int_0^T FdB^H \).

The two stochastic integrals are related in the following result.

**Theorem 2.2** If \( F \in \mathcal{L}(0, T) \), then the stochastic integral of Stratonovich type exists and the following equality is satisfied

\[ \int_0^T FdB^H = \int_0^T FdB^H + \int_0^T D^s_{\phi}F_s ds \text{ a.s.} \] \( (6) \)
The sequence of Hermite polynomials \((H_n, n \in \mathbb{N})\) where \(\deg H_n = n\) can be defined by a generating function as

\[
e^{tx-(1/2)x^2} = \sum_{n=0}^{\infty} \frac{t^n H_n(x)}{n!}.
\]

Define

\[
\tilde{f}(t) = |f[0,t]|_\phi^{-1} \int_0^t f dB^H
\]

and

\[
H_n^\phi f(t) = |f[0,t]|_\phi^n H_n(\tilde{f}(t)).
\]

As an application of an Itô formula for fractional Brownian motion (Theorem 4.3, [5]) there is the following result.

**Proposition 2.1** If \(f[0,T] \in L^2_\phi\), then the following equality is satisfied

\[
dH_n^\phi f(t) = nH_{n-1}^\phi f(t) dB^H(t)
\]

### 3 Multiple Integrals

Let \(f \in L^2_\phi\) be such that \(|f|_\phi = 1\). The Wick exponential, \(\exp^\phi\), and the Wick logarithm, \(\log^\phi\), are defined as

\[
\exp^\phi \left( \int_0^\infty f dB^H \right) := \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_0^\infty f dB^H \right)^n
\]

and

\[
\log^\phi \left( 1 + \int_0^\infty f dB^H \right) := \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} \left( \int_0^\infty f dB^H \right)^n
\]

where \(\left( \int_0^\infty f dB^H \right)^n\) is the \(n\)th Wick power of \(\int_0^\infty f dB^H\). This \(n\)th Wick power can be expressed in terms of the Hermite polynomial \(H_n\).

**Lemma 3.1** If \(f \in L^2_\phi\) with \(|f|_\phi = 1\)

\[
\left( \int_0^\infty f dB^H \right)^n = H_n \left( \int_0^\infty f dB^H \right)^n
\]

for each \(n \in \mathbb{N}\) where \(H_n\) is the Hermite polynomial of degree \(n\).

More generally, if \(f \in L^2_\phi\) then

\[
\left( \int_0^\infty f dB^H \right)^n = |f|_\phi^n \left( \int_0^\infty f dB^H \right)^n
\]

\[
= |f|_\phi^n H_n \left( \int_0^\infty f dB^H \right)^n.
\]

The Wick exponential can be expressed in terms of the usual exponential as follows.

**Proposition 3.1** If \(f \in L^2_\phi\), then

\[
\exp^\phi \left( \int_0^\infty f dB^H \right) = \exp \left( \int_0^\infty f dB^H - \frac{1}{2} |f|_\phi^2 \right).
\]

This exponential (7) is the Radon-Nikodym derivative of the following translate of a fractional Brownian motion

\[
X(t) = B^H(t) + \int_0^t \Phi f(s) ds
\]

and

\[
(\Phi f)(t) = \int_0^\infty \phi(t,u) f(u) du.
\]

The following expectation is useful in computations with multiple integrals of a fractional Brownian motion.

**Lemma 3.2** If \(f_1, \ldots, f_n, g_1, \ldots, g_m \in L^2_\phi\), then the following equality is satisfied

\[
\mathbb{E} \left[ \left( \int_0^\infty f_1 dB^H \diamond \cdots \diamond \int_0^\infty f_n dB^H \right)^m \right]
\]

\[
\times \left( \int_0^\infty g_1 dB^H \diamond \cdots \diamond \int_0^\infty g_m dB^H \right)^m
\]

\[
= \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{n! \sum_{\sigma} \langle f_1, g_{\sigma(1)} \rangle_\phi \cdots \langle f_n, g_{\sigma(n)} \rangle_\phi} & \text{if } m = n
\end{cases}
\]

where \(\sum_{\sigma}\) denotes the sum over all permutations \(\sigma\) of \(\{1, \ldots, n\}\).

The Hilbert space \(L^2_\phi\) is extended to its \(n\)th symmetric tensor product, that is,

\[
L^2_{\phi,n} := L^2_\phi \otimes \cdots \otimes L^2_\phi.
\]

If \(f \in L^2_{\phi,n}\), that is, \(f : \mathbb{R}_+^n \to \mathbb{R}\) and is symmetric in its arguments then

\[
\langle f, f \rangle_\phi := \int_{\mathbb{R}_+^n} \phi(u_1 - v_1) \cdots \phi(u_n - v_n) f(u_1, \ldots, u_n)
\]

\[
\times f(v_1, \ldots, v_n) du_1 \cdots du_n dv_1 \cdots dv_n.
\]

If \(f \in L^2_{\phi,n}\) is of the form

\[
f(s_1, \ldots, s_n) = \sum a_{k_1 \cdots k_n} e_{k_1}(s_1) \cdots e_{k_n}(s_n)
\]

and \((e_n, n \in \mathbb{N})\) is a complete orthonormal basis of \(L^2_\phi\), then the multiple integral of \(f, I_n(f)\) is defined as

\[
I_n(f) = \sum a_{k_1 \cdots k_n} \int_0^\infty e_{k_1} dB^H \cdots \int_0^\infty e_{k_n} dB^H.
\]

This definition of multiple integral is easily extended to an arbitrary \(f \in L^2_{\phi,n}\).

The following result gives the expectation of a product of two multiple integrals.
Lemma 3.3 If \( f \in L^2_{\phi,n} \) and \( g \in L^2_{\phi,m} \), then
\[
\mathbb{E}[I_n(f)I_m(g)] = \begin{cases} 
\langle f, g \rangle_\phi & \text{if } m = n \\
0 & \text{if } m \neq n
\end{cases}
\]
The iterated integral can be defined by the natural recursion
\[
\int_{0 \leq s_1 < \cdots < s_n \leq t} f(s_1, \ldots, s_n) dB^H(s_1) \cdots dB^H(s_n)
= \int_0^t \left( \int_{0 \leq s_1 < \cdots < s_n} f(s_1, \ldots, s_n) \right)
\times dB^H(s_1) \cdots dB^H(s_{n-1}) dB^H(s_n)
\]
(9)
The following result relates this iterated integral (9) and the multiple integral (8).

Theorem 3.1 If \( f \in L^2_{\phi,n} \), then the iterated integral (9) exists and
\[
I_n(f) = n! \int_{0 \leq s_1 < \cdots < s_n \leq t} f(s_1, \ldots, s_n) dB^H(s_1) \cdots
\times dB^H(s_n).
\]
If \( f \in L^2_{\phi,n} \) is a simple function of the form
\[
f(t_1, \ldots, t_n) = \sum a_{i_1, \ldots, i_n} f_{i_1}(t_1) \cdots f_{i_n}(t_n),
\]
then the \( \phi \)-trace \( \text{Tr}_\phi \) and its powers \( \text{Tr}_\phi^k \) for \( k \in \{1, 2, \ldots, \lfloor n/2 \rfloor \} \) are defined as
\[
\text{Tr}_\phi^k f(t_1, \ldots, t_{n-2k})
= \int_0^\infty \cdots \int_0^\infty f(s_1, \ldots, s_{2k}, t_1, \ldots, t_{n-2k})
\times \phi(s_1 - s_2) \cdots \phi(s_{2k-1} - s_{2k})
\times ds_1 \cdots ds_{2k}.
\]
To define the trace in general let \( \gamma_\varepsilon \) be an approximation to the Dirac function, that is,
\[
\lim_{\varepsilon \to 0} \int \gamma_\varepsilon(s,t) f(s) ds = f(t)
\]
in some sense and
\[
\int_0^\infty \int_0^\infty \gamma_\varepsilon(s,t) ds dt < \infty.
\]
If \( f \in L^2_{\phi,n} \), then \( f^\varepsilon \in L^2_{\phi,n} \) where
\[
f^\varepsilon(t_1, \ldots, t_n)
= \int_{\mathbb{R}_+^n} f(s_1, \ldots, s_n) \gamma_\varepsilon(s_1, t_1) \cdots \gamma_\varepsilon(s_n, t_n) ds_1 \cdots ds_n.
\]
Let
\[
\rho_\varepsilon(s,t) = \int_0^\infty \gamma_\varepsilon(s,u) \gamma_\varepsilon(t,u) du.
\]
The \( k \)-th \( \phi \)-trace of \( f^\varepsilon \) is
\[
\text{Tr}_\phi^k f^\varepsilon(t_1, \ldots, t_{n-2k})
= \int_{\mathbb{R}_+^k} f(s_1, \ldots, s_n) \rho_\varepsilon(s_1, s_2) \cdots \rho_\varepsilon(s_{2k-1}, s_{2k})
\times \gamma_\varepsilon(s_{2k-1}, t_1) \cdots \gamma_\varepsilon(s_{2n}, t_{n-2k})
\times ds_1 \cdots ds_{2n}.
\]
The \( k \)-th trace of \( f \) is said to exist if
\[
\text{Tr}_\phi^k f(t_1, \ldots, t_{n-2k}) = \lim_{\varepsilon \to 0} \text{Tr}_\phi^k f^\varepsilon(t_1, \ldots, t_{n-2k}).
\]
Now multiple Stratonovich integrals of a fractional Brownian motion are defined. Let
\[
(B^H(t))^\varepsilon = \int_0^\infty \gamma_\varepsilon(t,s) dB^H(s)
\]
and \( f \in L^2_{\phi,n} \). Define
\[
S_n^\varepsilon(f) = \int_{\mathbb{R}_+^n} f(s_1, \ldots, s_n)(B^H(s_1))^\varepsilon \cdots (B^H(s_n))^\varepsilon
\times ds_1 \cdots ds_n.
\]
(10)
If \( S_n^\varepsilon(f) \) converges in \( L^2(P) \) as \( \varepsilon \to 0 \), then the multiple Stratonovich integral is said to exist and is denoted
\[
S_n(f) = \int_{\mathbb{R}_+^n} f(s_1, \ldots, s_n) dB^H(s_1) \cdots dB^H(s_n).
\]
(11)
It follows that
\[
\left( \int_0^\infty f dB^H \right)^n = \sum_{k \leq \lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!} I_{n-2k} \left( \text{Tr}_\phi^k f^\otimes n \right)
\]
where \( f^\otimes n \) is the symmetric tensor product of \( f \). More generally, if \( f_1, \ldots, f_n \in L^2_{\phi} \) and \( f \in L^2_{\phi,n} \) is the symmetrization of \( f_1, \ldots, f_n \), then
\[
\int_0^\infty f_1 dB^H \cdots \int_0^\infty f_n dB^H
= \sum_{k \leq \lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!} I_{n-2k} \left( \text{Tr}_\phi^k(f) \right)
\]
and \( S_n^\varepsilon(f) \) can be defined as in (10). If for \( k \in \{1, \ldots, \lfloor n/2 \rfloor \} \)
\[
\int_{\mathbb{R}_+^n} f(s_1, \ldots, s_n) \gamma_\varepsilon(s_1, s_2) \cdots \gamma_\varepsilon(s_{2k-1}, s_{2k})
\times \gamma_\varepsilon(s_{2k+1}, t_1) \cdots \gamma_\varepsilon(s_{2n}, t_{n-2k}) ds_1 \cdots ds_{2n}.
\]
converges to a function $T_{\phi,n}^k f$ in $L^2_{\phi,n-2k}$ as $\varepsilon \to 0$ then $(S_n^\varepsilon(f), n \in \mathbb{N})$ converges in $L^2(P)$ and the limit, which is called the extended Hu-Meyer formula [6], is

$$S_n(f) = \sum_{k \leq [n/2]} \frac{n!}{2^k k!(n-2k)!} I_{n-2k} \left( T_{\phi,n}^k f \right).$$

For Brownian motion, there is a well known expansion of any square integrable functional on Wiener space in terms of multiple Wiener integrals [9] or Hermite polynomials. The following result is the analogue for a fractional Brownian motion with $H \in (1/2, 1)$.

**Theorem 3.2** If $F \in L^2(P)$, then there is a sequence $f_n \in L^2_{\phi,n}, n \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} |f_n|_\phi^2 < \infty$$

and

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} \int_{\mathbb{R}_+^n} f_n(s_1, \ldots, s_n) \times dB_H^H(s_1) \cdots dB_H^H(s_n) \quad \text{a.s.}$$

(12)

The multiple integrals on the right hand side of (12) can be expressed as iterated integrals so that $F$ can be expressed as a sum of a constant and a stochastic integral. This result has many applications in stochastic analysis.

**References**


