

The BFKL high energy asymptotics in the next-to-leading approximation

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Abstract:

We discuss the high energy asymptotics in the next-to-leading (NLO) BFKL equation. We find a general solution for Green functions and consider two properties of the NLO BFKL kernel: running QCD coupling and large NLO corrections to the conformal part of the kernel. Both of these effects lead to Regge-BFKL asymptotics only in the limited range of energy ($y = \ln(s/qq_0) \leq (\alpha_S)^{-\frac{5}{3}}$) and change the energy behaviour of the amplitude for higher values of energy. We confirm the oscillation in the total cross section found in Ref. [7] in the NLO BFKL asymptotics, which shows that the NLO BFKL has a serious pathology.

1 Introduction

The next-to-leading order corrections to the kernel of the BFKL equation [1] (NLO BFKL) have recently been presented in two papers [2][3], and have been formulated in a compact and elegant form by Fadin and Lipatov [3]. It turns out that the next-to-leading order corrections are large and they strongly modify the leading order result. The large value of the next-to-leading order corrections suggests that even high order corrections may be essential, leaving us without a reliable calculation of the parameters of the BFKL asymptotics except at very small values of the QCD coupling constant, which unfortunately are not presently attainable.

Before arriving at a pessimistic conclusion, it is necessary to understand the qualitative alteration due to the next-to-leading order corrections that occurs in the BFKL asymptotics, i.e. we wish to know how the NLO BFKL Pomeron differs from the LO BFKL one. Other questions which we would like to answer are: Does the NLO BFKL Pomeron still manifest itself as a Reggeon - like exchange at high energy as the LO BFKL Pomeron does? Can the NLO BFKL Pomeron be described as the diffusion cascading process in $\log k_{\perp}$ where k_{\perp} is the partonic transverse momentum? Can we calculate the NLO BFKL Pomeron in the framework of perturbative QCD or the nonperturbative correction would destroy our approach at so small value of energy that we cannot use the BFKL asymptotic? In short, we would like to understand what the NLO BFKL Pomeron is ?

This paper is an attempt to find a general solution to the NLO BFKL equation, and to discuss the main properties of the high energy asymptotics that follows from this solution. The NLO BFKL kernel contains two parts: a conformally invariant part and a running coupling part. In studying the running coupling effect we reproduce non-Regge type corrections to high energy asymptotics suggested by Kovchegov and Mueller in Ref.[4], using a method of solution proposed in Refs.[5][6]. Ross found in Ref.[7] that the conformal part of the NLO BFKL kernel crucially changes the diffusion in $\log k_{\perp}$ (k_{\perp} is a parton transverse momentum) which is a basic property of the LO BFKL Pomeron. We show how this affects the asymptotic behaviour of the scattering amplitude at high energy.

The running QCD coupling in the BFKL equation has been studied for some time, starting with the GLR paper [5], where the general solution to this problem was suggested. In Refs. [8][9][10] [11] [6][12][13][4] the different aspects of the problem were considered, but only now this can be done on solid basis of the NLO BFKL equation. The influence of the NLO on the BFKL diffusion was first pointed out and explored in Ref. [7].

Our paper deals with both problems on the same footing, and we give a simple and transparent derivation and discussion of the high energy asymptotics for the NLO BFKL equation. Our conclusion is rather pessimistic: we cannot avoid an oscillation in the total cross section at high energy in the NLO BFKL approach and, therefore, we do not think that the NLO BFKL equation can serve as a

basis for high energy phenomenology. However, we hope, that the experience with the NLO BFKL Pomeron will be useful for possible scenario of the high energy behaviour in QCD.

2 A general approach to the NLO BFKL equation

The BFKL equation governs the high energy asymptotics of a single-scale hard process. As an example of such a process we can consider the scattering of two virtual photons (with virtualities Q^2 and Q_0^2) at high energy in the kinematic region where $Q^2 \cong Q_0^2 \gg m^2$ [14], where m^2 a scale of the “soft” interaction. The cross section for this process can be written as

$$\sigma_{\gamma^*\gamma^*}(s) = \int \frac{d^2q}{2\pi q^2} \int \frac{d^2q_0}{2\pi q_0^2} \Phi(Q^2, q^2) \Phi(Q_0^2, q_0^2) \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{q q_0} \right)^\omega G_\omega(\vec{q}, \vec{q}_0), \quad (2.1)$$

where \vec{q} and \vec{q}_0 are transverse momenta of gluons, $\Phi(Q^2, q^2)$ and $\Phi(Q_0^2, q_0^2)$ are impact functions that has been calculated in Ref.[14]. It should be stressed that functions Φ provide $q^2 \cong Q^2$ and $q_0^2 \cong Q_0^2$. $G_\omega(\vec{q}, \vec{q}_0)$ obeys the BFKL equation

$$\omega G_\omega(\vec{q}, \vec{q}_0) = \delta^{(2)}(\vec{q} - \vec{q}_0) + \int d^2q' K(\vec{q}, \vec{q}') G_\omega(\vec{q}', \vec{q}_0). \quad (2.2)$$

Kernel $K(\vec{q}, \vec{q}')$ can be written as a sum of LO and NLO kernels

$$K(\vec{q}, \vec{q}') = K^{LO}(\vec{q}, \vec{q}') + K^{NLO}(\vec{q}, \vec{q}'). \quad (2.3)$$

This kernel has eigenfunction[3]

$$\varphi_f(q^2) = \frac{1}{\sqrt{q^2}} \left(\frac{q^2}{\sqrt{\alpha_S(q^2)}} \right)^f \quad (2.4)$$

and corresponding eigenvalues

$$\int d^2q' K(\vec{q}, \vec{q}') \varphi_f(q'^2) = \omega(f) \varphi_f(q^2). \quad (2.5)$$

$\omega(f)$ has a form

$$\omega(f) = \bar{\alpha}_S(\mu^2) (\chi^{LO}(f) + \bar{\alpha}_S(\mu^2) \chi^{NLO}(f)) - \frac{N_c \alpha_S^2(\mu^2)}{\pi} b \ln(q^2/\mu^2) \chi^{LO}(f), \quad (2.6)$$

where $\chi^{LO}(f) = 3\psi(1) - \psi(\frac{1}{2} + f) - \psi(\frac{1}{2} - f)$, the explicit form of χ^{NLO} is written in Ref. [3], which has also symmetry under transform $f \rightarrow -f$ as the LO

part, and where $b = \frac{11N_c - 2N_f}{12\pi}$ for number of colours N_c and number of flavours N_f . We use notation $\bar{\alpha}_S$ for $\bar{\alpha}_S = \frac{N_c \alpha_S}{\pi}$.

One can see that the first term has a conformal symmetry while this symmetry is broken in the NLO due to the second term in Eq. (2.6). μ^2 is the normalization point which is arbitrary in the NLO calculations. We take it to be equal to the value of the initial virtuality q_0^2 ($\mu^2 = q_0^2$) without losing any accuracy in the NLO approach.

Following Refs. [5] [6] we rewrite Eq. (2.6) in the form:

$$\omega(f) = \frac{r_0}{r} \bar{\alpha}_S(q_0^2) (\chi^{LO}(f) + \bar{\alpha}_S(q_0^2) \chi^{NLO}(f)) = \frac{r_0}{r} \omega_{conf}(f) . \quad (2.7)$$

Here we define $r = \ln \frac{q^2}{\Lambda^2} - \frac{1}{2} \ln \alpha_S(q^2)$ and the running QCD coupling constant in leading log is equal to ¹

$$\alpha_S(q^2) = \frac{1}{br} ,$$

where Λ^2 is the position of the infrared Landau pole in running α_S .

We want to stress that Eq. (2.7) coincides with Eq. (2.6) in the NLO approximation. However, this form of the kernel is much more convenient in searching of a solution and we firmly believe it corresponds more to the general incorporation of the effect of the running QCD coupling in the BFKL equation (see a discussion of this point of view in Ref. [6]).

To find a solution to Eq. (2.2) we expand $G_\omega(q^2)$ with respect to the complete set of eigenfunction of Eq. (2.4), namely

$$G_\omega(q^2) = \int_{a-i\infty}^{a+i\infty} \frac{df}{2\pi i} g(\omega, f) \varphi_f(q^2) = \frac{1}{\sqrt{q^2 q_0^2}} \int_{a-i\infty}^{a+i\infty} \frac{df}{2\pi i} g(\omega, f) e^{rf} , \quad (2.8)$$

where the contour of integration is situated to the right of all singularities of function $g(\omega, f)$.

Using Eq. (2.5) and Eq. (2.7) one obtains the following equation for function $g(\omega, f)$:

$$-\omega \frac{dg(\omega, f)}{df} = r_0 \omega_{conf}(f) g(\omega, f) + r_0 e^{-fr_0} . \quad (2.9)$$

The solution of homogeneous equation (Eq. (2.9) without the last term) can be easily found and it has the form (see Refs. [5] [6] for details):

$$g(\omega, f) = \tilde{g}(\omega) e^{-\frac{r_0}{\omega} \int_{f_0}^f \omega_{conf}(f') df'} . \quad (2.10)$$

¹We add factor $\frac{1}{2} \ln \alpha_S(q^2)$ for convenience but it does not affect the value of $\omega(f)$ in the NLO approximation.

Function $\tilde{g}(\omega)$ should be specified from initial or boundary conditions. The value of f_0 can be arbitrary since its redefinition is included in function $\tilde{g}(\omega)$. Unless it is specially stipulated $f_0 = 0$.

We will show that the small values of f will be dominant in Eq. (2.8) in the wide range of large $y = \ln \frac{s}{\sqrt{q^2 q_0^2}}$. Therefore, we can expand ω_{conf} at small f , namely,

$$\omega_{conf} = \omega_L + D f^2 + O(f^4), \quad (2.11)$$

where in the NLO ω_L and D are equal to (see Eq.(7) of Ref.[7]):

$$\omega_L = \bar{\alpha}_S(q_0^2) \{ 2.772 - 18.3\bar{\alpha}_S(q_0^2) \}; \quad (2.12)$$

$$D = \bar{\alpha}_S(q_0^2) \{ 16.828 - 322\bar{\alpha}_S(q_0^2) \}. \quad (2.13)$$

One can see from Eq. (2.12) how large and essential the NLO corrections are. They considerably diminish the value of ω_L which can be even negative for $\bar{\alpha}_S(q_0^2) > 0.152$ and change the sign of D at $\bar{\alpha}_S \approx 0.05$. Note, that the positive value of D corresponds to diffusion in $\ln q^2$.

To solve this problem we need to find $\tilde{g}(\omega)$ in the general solution of Eq. (2.10), which depends on the initial or boundary conditions. We find it very instructive to introduce two Green functions for the BFKL equation.

1.

The first one ($G_r(y, r)$) satisfies the following boundary condition:

$$\begin{aligned} G_r(y, r) : & \quad (2.14) \\ G_r(y, r = r_0) &= \delta(y - y_0) \end{aligned}$$

This Green function allows us to find us the solution of the BFKL equation for any boundary input distribution $G_{in}(y, q^2 = q_0^2)$ at $q^2 = q_0^2$ ($r = r_0$). Indeed, such a solution is equal to

$$G(y, r) = \int dy_0 G_r(y, r) G_{in}(y_0, q^2 = q_0^2). \quad (2.15)$$

Such a Green function is very useful for study of the boundary condition for the DGLAP evolution. Using $G_r(y, r)$ and Eq. (2.15), we can investigate the y -dependence at $q^2 \cong q_0^2$. We can distinguish two cases with different solutions:

1. the integral over y_0 depends mostly on properties of input function G_{in} ;
2. the integral over y_0 is sensitive to the Green function. In this case we can claim that the energy behaviour of our boundary condition is defined by the BFKL dynamics.

Therefore, this Green function ($G_r(y, r)$) can provide us with an educated guess for the energy dependence of the boundary condition in the DGLAP evolution equations [15].

In the LO BFKL approach $G_r(y - y_0, r, r_0)$ is equal to[1]

$$G_r^{LO}(y - y_0, r, r_0) = \sqrt{\frac{\pi(r - r_0)}{2D^{LO}(y - y_0)^3}} e^{\omega_L^{LO}(y - y_0) - \frac{(r - r_0)^2}{4D^{LO}(y - y_0)}}. \quad (2.16)$$

We call this expression Regge-BFKL asymptotics since it has a power - like behaviour, similar to the exchange of the Reggeon. We would like to recall that the second factor in the exponent is different from a Reggeon contribution, and it has an origin in the diffusion in the log of transverse momentum of partons which is a typical feature of QCD.

The question is how general is Eq. (2.16) and does it preserve the main characteristics like power behaviour and/or the diffusion in the log of transverse momentum in the NLO approximation.

2.

The initial condition for the second Green function ($G_y(y, r)$) can be written as follows:

$$\begin{aligned} G_y(y, r) : \\ G_y(y = y_0, r) = \delta(r - r_0) \end{aligned} \quad (2.17)$$

It is obvious that one can find a general solution using this Green function if we have an input from experiment and/or nonperturbative QCD, namely, the dependence on transverse momentum at fixed value of y . Indeed,

$$G(y, r) = \int dr_0 G_y(y, r) G_{in}(y = y_0, r_0). \quad (2.18)$$

From Eq. (2.18) one can see that $G_y(y, r)$ defines the asymptotic of the scattering amplitude for a single-scale process. For example, this Green function gives the asymptotic behaviour of $\sigma_{\gamma^* \gamma^*}$ [14].

The last remark in this general section: it is necessary to choose sufficiently large initial transverse momentum (q_0^2) in order to safely apply the pQCD methods. It has been discussed in many papers[8][16][17][18] [13][4] that at high energies the BFKL diffusion in log of transverse momenta inevitably leads to the fact that small values of the transverse momenta become important (Bartel's cigar [16]). It means that we cannot safely calculate the high energy asymptotics in the framework of pQCD. The criteria for being able to trust pQCD formulated in Ref.[17] (see also Ref. [4] for discussion in the case of running QCD coupling) is

$$y \leq \frac{\pi}{14 N_c \zeta(3) b^2} \times \frac{1}{\alpha_S^3(q_0^2)} \quad (2.19)$$

which suggests that we should take q_0^2 as large as possible

In the LO BFKL approach this Green function has a similar form as G_r , namely [1]

$$G_y^{LO}(y-y_0, r, r_0) = \sqrt{\frac{\pi}{2D^{LO}(y-y_0)}} e^{\omega_L^{LO}(y-y_0) - \frac{(r-r_0)^2}{4D^{LO}(y-y_0)}}. \quad (2.20)$$

One can see that both Green functions in the LO BFKL approach are similar and only differ in pre-exponential factors.

3 $\mathbf{G_r(y, r)}$

3.1 Solution

This Green function has been calculated in Ref.[6] but for completeness we will reproduce simple calculation to examine what happens to G_r in the NLO. Substituting Eq. (2.11) in Eq. (2.10) we find that the general solution of the homogeneous equation is

$$G(y, r) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \int_{f_0-i\infty}^{f_0+i\infty} \frac{df}{2\pi i} \tilde{g}(\omega) e^{\omega(y-y_0) + fr - \frac{r_0(\omega_L f + \frac{D}{3} f^3)}{\omega}}. \quad (3.1)$$

The integration over f leads to Airy function [6] $Ai\left(\left(\frac{\omega}{r_0 D}\right)^{\frac{1}{3}}\left[r - \frac{\omega_L}{\omega} r_0\right]\right)$. Therefore to satisfy the boundary condition of Eq. (2.14) we have to choose a function $\tilde{g}(\omega) = Ai^{-1}\left(\left(\frac{\omega}{r_0 D}\right)^{\frac{1}{3}}\left[r_0 - \frac{\omega_L}{\omega} r_0\right]\right)$.

Finally[6], $G_r(y-y_0, r, r_0)$ is equal to

$$G_r(y-y_0, r, r_0) = \sqrt{\frac{r}{r_0}} \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} e^{\omega(y-y_0)} \frac{Ai\left(\left(\frac{\omega}{r_0 D}\right)^{\frac{1}{3}}\left[r - \frac{\omega_L}{\omega} r_0\right]\right)}{Ai\left(\left(\frac{\omega}{r_0 D}\right)^{\frac{1}{3}}\left[r_0 - \frac{\omega_L}{\omega} r_0\right]\right)}. \quad (3.2)$$

3.2 Regge-BFKL asymptotics

To recover the Regge-BFKL asymptotics we assume that the arguments of both Airy functions in Eq. (3.2) are large:

$$\begin{aligned} \left(\frac{\omega}{r_0 D}\right)^{\frac{1}{3}}\left[r - \frac{\omega_L}{\omega} r_0\right] &= \left(\frac{\omega}{r_0 D}\right)^{\frac{1}{3}} r_0 \left[\frac{\delta r}{r_0} - \frac{\Delta}{\omega_L}\right] \gg 1; \\ \left(\frac{\omega}{r_0 D}\right)^{\frac{1}{3}}\left[r_0 - \frac{\omega_L}{\omega} r_0\right] &= \left(\frac{\omega}{r_0 D}\right)^{\frac{1}{3}} r_0 \left[\frac{\Delta}{\omega_L}\right] \gg 1; \end{aligned} \quad (3.3)$$

where $\delta r = r - r_0$ and $\Delta = \omega - \omega_L$.

Using the asymptotics of Airy function [19] $Ai(z)|_{z>0;|z|\gg 1} \rightarrow \frac{1}{2z^{\frac{3}{4}}} e^{-\frac{2}{3}z^{\frac{3}{2}}}$ we obtain for positive D ($D > 0$):

$$G_r(y-y_0, r, r_0) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} e^{\Psi(\Delta, y, r)}, \quad (3.4)$$

where Ψ is equal to

$$\Psi = \omega_L(y-y_0) + \Delta(y-y_0) - \frac{2}{3} \sqrt{\frac{\omega_L}{D}} r_0 \left(\left\{ \frac{\delta r}{r_0} + \frac{\Delta}{\omega_L} \right\}^{\frac{3}{2}} - \left\{ \frac{\Delta}{\omega_L} \right\}^{\frac{3}{2}} \right). \quad (3.5)$$

We can calculate this integral using the saddle point method, in which the value of saddle point (Δ^S) is defined from the equation:

$$\frac{d\Psi}{d\Delta} \Big|_{\Delta=\Delta^S} = 0. \quad (3.6)$$

In our case this equation reads

$$(y-y_0) - \sqrt{\frac{1}{D\omega_L}} r_0 \left(\left\{ \frac{\delta r}{r_0} + \frac{\Delta^S}{\omega_L} \right\}^{\frac{1}{2}} - \left\{ \frac{\Delta^S}{\omega_L} \right\}^{\frac{1}{2}} \right) = 0 \quad (3.7)$$

We can easily find a solution to Eq. (3.7), assuming that

$$\kappa = \frac{\delta r \omega_L}{r_0 \Delta} \ll 1. \quad (3.8)$$

Eq. (3.7) can be reduced to the form

$$\sqrt{\Delta^S} = \frac{\delta r}{2\sqrt{D}(y-y_0)} \left(1 - \frac{1}{4} \kappa^S - \frac{1}{8} (\kappa^S)^2 + O((\kappa^S)^3) \right) \quad (3.9)$$

In the leading order with respect to κ^S , the solution to Eq. (3.9) has the form:

$$\sqrt{\Delta_0^S} = \frac{\delta r}{2\sqrt{D}(y-y_0)}; \quad \Delta_0^S = \frac{(\delta r)^2}{4D(y-y_0)^2}. \quad (3.10)$$

However, we need to calculate the value of the saddle point with better accuracy to calculate the deviation from the Regge-BFKL behaviour. Considering $\kappa_0^S = \frac{4D\omega_L(y-y_0)^2}{r_0 \delta r}$ being small, we find

$$\sqrt{\Delta^S} = \frac{\delta r}{2\sqrt{D}(y-y_0)} \left(1 - \frac{1}{4} \kappa_0^S + O((\kappa_0^S)^3) \right). \quad (3.11)$$

The integrand (see Eq. (3.5)) for $\Delta = \Delta^S$ is equal to

$$\Psi(\Delta^S) = \omega_L(y-y_0) \left(1 - \frac{\delta r}{2r_0} \right) - \frac{(\delta r)^2}{4D(y-y_0)} + \frac{1}{12} \frac{D\omega_L^2}{r_0^2} (y-y_0)^3. \quad (3.12)$$

One can see that for positive D we have a good saddle point, and can evaluate the integral using the steepest descent method (SDM). The result is

$$G_r(y-y_0, r) = \frac{1}{qq_0} \sqrt{\frac{\pi(\delta r)^2}{2D(y-y_0)^3}} e^{\Psi(\Delta^S)}. \quad (3.13)$$

The first three terms in Eq. (3.12) yield normal Regge-BFKL asymptotic form which is the same as in the LO BFKL approach. The difference is only in the second term which fixes the scale of the running QCD coupling in the expression for ω_L (see Eq. (2.12)), namely in the BFKL diffusion, the running QCD coupling enters at the scale $q_{av}^2 = qq_0$. In other words we have to calculate ω_L^{run} using

$$\omega_L^{run} = \bar{\alpha}_S(q_{av}^2 = qq_0) \{ 2.772 - 18.3 \bar{\alpha}_S(q_0^2) \} \quad (3.14)$$

instead of Eq. (2.12). This result was first obtained in Refs. [9][4]. We will show below that the same scale should be incorporated in the calculation of D .

The last term in Eq. (3.12) is the most interesting one, because it shows the violation of the Regge-BFKL asymptotics due to the running QCD coupling. It was suggested by Kovchegov and Mueller in Ref. [4] using quite a different method and, we hope, that our derivation presented here is more transparent. Eq. (3.12) shows explicitly that Regge-BFKL asymptotics is only valid in the limited range of $y - y_0$, namely, $D(\alpha_S \omega_L b)^2 (y - y_0)^3 \ll 1$ or $(y - y_0) \ll \alpha_S^{-\frac{5}{3}}$ [6][4].

The question arises can we trust the term proportional to $(y - y_0)^3$. Indeed, we have made a lot of assumptions, in the derivation as well as using the steepest descent method (SDM). Therefore, we have to check whether all our assumptions are selfconsistent. They are:

$$\left(\frac{\omega_L}{D r_0} \right)^{\frac{1}{3}} \times \frac{\Delta^S r_0}{\omega_L} \gg 1 \quad \textit{asymptotic of Airy function} \quad (3.15)$$

$$\kappa^S = \frac{\delta r \omega_L}{r_0 \Delta^S} \gg 1 \quad \textit{assumption used to obtain } \Delta^S \quad (3.16)$$

$$\frac{1}{3!} \frac{d^3 \Psi}{d^3 \Delta} \Big|_{\Delta=\Delta^S} \times \left(\frac{\frac{d^2 \Psi}{d^2 \Delta} \Big|_{\Delta=\Delta^S}}{\frac{d^2 \Psi}{d^2 \Delta} \Big|_{\Delta=\Delta^S}} \right)^{\frac{3}{2}} \ll 1 \quad \textit{selfconsistency of the SDM} \quad (3.17)$$

$$(y - y_0) \leq \frac{r_0^2}{4D} \quad \textit{applicability of pQCD (see Ref.[4])} \quad (3.18)$$

In addition, we have to check that $\frac{d^2 \Psi}{d^2 \Delta} \Big|_{\Delta=\Delta^S}$ is positive, but it is easy to find out that it is the case for $D > 0$. However, it is an indication that $D < 0$ it has to be considered separately. It turns out that the second equation of Eq. (3.15) is the most restrictive. Taking Δ^S from Eq. (3.10) we see that it leads to

$$y - y_0 \leq \left(\frac{\delta r r_0}{\omega_L 4 D} \right)^{\frac{1}{2}} \propto \sqrt{\frac{\delta r}{\alpha_S^3}} \quad (3.19)$$

Substituting Eq. (3.19) in $(y - y_0)^3$ -term in Ψ (see Eq. (3.12)) we obtain the estimate on maximum value of this term which we can guarantee in our approximation:

$$\frac{D \omega_L^2}{r_0^2} (y - y_0)^3 \leq \sqrt{\frac{\omega_L}{D r_0}} \times (\delta r)^{\frac{3}{2}} \propto \delta r \sqrt{\frac{\delta r}{\alpha_S}}. \quad (3.20)$$

One can see that in wide region $\delta r > (\alpha_S)^{-\frac{1}{3}}$, it is legitimate to keep this term and, therefore, we can conclude, that the running QCD coupling does not satisfy the Regge-type asymptotics. However, we have to understand this result better since, at first sight, zeros of the denominator in Eq. (2.10) yield Regge asymptotics.

3.3 High energy asymptotics (a more detail analysis)

$D > 0$

First, Airy functions have zeros only at the negative values of the argument, and their position can be found with good accuracy from the simple equation[19]:

$$z = -\left(\frac{3\pi n}{2} - \frac{3\pi}{8}\right)^{\frac{2}{3}}, \quad (3.21)$$

where z is the argument of the Airy function and n is arbitrary integral number ($n = 0, 1, \dots$).

Taking the argument of the Airy function in the denominator of Eq. (3.2) we obtain [6][13]

$$\Delta_n = -\omega_L \left(\frac{D}{\omega_L r_0^2}\right)^{\frac{1}{3}} \times \left(\frac{3\pi n}{2} - \frac{3\pi}{8}\right)^{\frac{2}{3}}. \quad (3.22)$$

Here, $\Delta_n = \omega_n - \omega_L$ and one can see that all poles in ω are located to the left from ω_L ($\omega_n < \omega_L$ and $\Delta_n \propto \alpha_S^{\frac{2}{3}} \omega_L$). Therefore, we can legitimately calculate Δ_n in framework of our approach. The whole structure of singularities in the ω - plane is as follows [6][13]:

1. the rightmost pole ω_0 is located to the left of ω_L but, theoretically, very close to it. Namely, its position is

$$\omega_0 = \omega_L - \omega_L \left(\frac{D}{\omega_L r_0^2}\right)^{\frac{1}{3}} \times \left(\frac{3\pi}{8}\right)^{\frac{2}{3}}. \quad (3.23)$$

2. for large $n \rightarrow \infty$

$$\omega_n |_{n \gg 1} \rightarrow \left(\frac{D}{r_0^2}\right)^{\frac{1}{3}} \times \frac{2}{3\pi n}; \quad (3.24)$$

3. there fore, the solution has the infinite number of poles in ω -plane to the left of ω_0 , which accumulate at $\omega = 0$ at $n \rightarrow \infty$.

This picture of singularities, justifies our saddle point calculation, since the position of the saddle point turns out to be shifted to the right of ω_L . Note, that the contour of ω -integration is chosen to be located to the right of all singularities of the integrand.

We can evaluate the integral in a different way, namely, calculating each pole separately. In this case we have the asymptotic form

$$G_r(y - y_0, r, r_0) = \sum_{n=0}^{\infty} e^{\omega_n (y - y_0)} V(\omega_n, r), \quad (3.25)$$

where $V(\omega_n, r)$ is the ω_n - pole. This series was investigated in Ref.[13] where it was shown that the saddle point approximation effectively describes the sum over n at sufficiently large n . The right most singularity ω_0 gives a suppressed contribution because of smallness of its residue, while $\omega_n \rightarrow 0$ at large n , but the residue is rather big.

D < 0

In this case the general solution has the same form of Eq. (3.2) but the structure of singularities changes crucially:

1. all poles ω_n are located to the right of ω_L ;
2. the position of the leftmost pole is

$$\omega_0 = \omega_L + \omega_L \left(\frac{D}{\omega_L r_0^2} \right)^{\frac{1}{3}} \times \left(\frac{3\pi}{8} \right)^{\frac{2}{3}}; \quad (3.26)$$

3. at large n $\omega_n \rightarrow \infty$ accordingly the following expression:

$$\omega_n |_{n \gg 1} = \frac{|D|}{r_0^2} \times \left(\frac{3\pi n}{2} \right)^2. \quad (3.27)$$

In such a situation we cannot use the saddle point method and should rather analyze Eq. (3.25) with ω_n given by Eq. (3.27). It is easy to see that Eq. (3.25) can be reduced to the form:

$$G_r(y - y_0, r, r_0) = \sum_{n \gg 1}^{\infty} e^{\frac{|D|}{r_0^2} \left(\frac{3\pi n}{2} \right)^2 (y - y_0) \pm i \pi n \frac{\delta r}{r_0}}. \quad (3.28)$$

To evaluate this sum looks hopeless, at least, we do not see how it is possible to do. However, we have to go back to our derivation of solution (see Eq. (3.1)), namely, to the integration over f which led to the Airy function. The value of the typical f (f^S) in this integral can be evaluated using the saddle point approach and it is equal to

$$f^S = \pm \sqrt{\frac{\omega}{|D|} \left(r - r_0 \frac{\omega_L}{\omega} \right)}, \quad (3.29)$$

which is of the order of n for ω_n . Therefore, for large n we cannot trust the solution of Eq. (3.2) and have to generalize the solution including the next term in expansion of $\omega_{conf}(f)$ in Eq. (2.11), namely,

$$\omega_{conf} = \omega_L + D f^2 - B f^4, \quad (3.30)$$

where B was calculated in Ref.[7]

$$B = -\bar{\alpha}_S(q_0^2) \{ 64.294 - \bar{\alpha}_S(q_0^2) 2756 \} . \quad (3.31)$$

One can see that $B > 0$ everywhere, except of the region of very small α_S .

One can see that Eq. (3.30) can be written as

$$\omega_{conf} = \tilde{\omega}_L - B(f^2 - f_0^2)^2 , \quad (3.32)$$

where

$$\tilde{\omega}_L = \omega_L + B f_0^4$$

and

$$f_0^2 = -\frac{|D|}{2B} ; \quad f_0 = \pm i\sqrt{\frac{|D|}{2B}}$$

Introducing a new variable $f = f_0 + \nu$ and integrating in Eq. (2.10) from f_0 we obtain

$$G(y, r) = \quad (3.33)$$

$$\int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \int_{\hat{f}-i\infty}^{\hat{f}+i\infty} \frac{d\nu}{2\pi i} \tilde{g}(\omega) \cos\left(\delta r \sqrt{\frac{|D|}{2B}}\right) e^{\omega(y-y_0) + \nu r - \frac{r_0(\tilde{\omega}_L \nu + \frac{B f_0^2}{3} \nu^3)}{\omega}} .$$

Integration over ν in Eq. (3.33) gives the Airy function $Ai\left(\left(\frac{\omega}{r_0 D'}\right)^{\frac{1}{3}}\left[r - \frac{\tilde{\omega}_L}{\omega} r_0\right]\right)$ with

$$D' = 2|D| . \quad (3.34)$$

Finally, the solution is very similar to Eq. (3.2) with new D' of Eq. (3.34), namely

$$G_r(y - y_0, r, r_0) = \quad (3.35)$$

$$\sqrt{\frac{r}{r_0}} \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \cos\left(\delta r \sqrt{\frac{|D|}{2B}}\right) e^{\omega(y-y_0)} \frac{Ai\left(\left(\frac{\omega}{r_0 D'}\right)^{\frac{1}{3}}\left[r - \frac{\omega_L}{\omega} r_0\right]\right)}{Ai\left(\left(\frac{\omega}{r_0 D'}\right)^{\frac{1}{3}}\left[r_0 - \frac{\omega_L}{\omega} r_0\right]\right)} .$$

Therefore, Eq. (3.12) gives the answer in this case, for the saddle point value of Ψ with $D = D'$. The Green function $G_r(y - y_0, r, r_0)$ has the form:

$$G_r(y - y_0, r, r_0) = \frac{1}{q q_0} \sqrt{\frac{\pi(\delta r)^2}{2 D'(y - y_0)^3}} \cos\left(\delta r \sqrt{\frac{|D|}{2B}}\right) e^{\Psi(\Delta^S, D \rightarrow D')} . \quad (3.36)$$

The unpleasant fact which is a direct consequence of the NLO BFKL is the oscillation, which has first been found in Ref. [7]. This is a serious defect of the NLO BFKL, since G_r is proportional to the total cross section. Certainly, we cannot have faith in such an approach which leads to negative total cross section.

3.4 Asymptotics at ultra high energies

We can calculate the asymptotics of this Green function (G_r) even at very high values of energy. Indeed, Eq. (3.18) which restricts the value of $y - y_0$ does not affect G_r since the nonperturbative corrections have been absorbed in G_{in} in Eq. (2.15). Eq. (2.15) gives a nice example how we can separate the unknown nonperturbative input distribution (G_{in}) and perturbative BFKL Green function. Unfortunately, we are not able to do so in the case of the second Green function (G_y) for which Eq. (3.18) is a restriction (see Ref. [4] for details).

It is easy to see that at very large value of $y - y_0 \gg \alpha_S^{-\frac{5}{3}}$ the inequalities of Eq. (3.15) - Eq. (3.18) are violated and actually we cannot use the asymptotic expression for the Airy functions both in numerator and in denominator. Therefore, the only source of the asymptotics is the zero of the denominator at $\omega = \omega_0$, where (see Eq. (3.23))

$$\omega_0 = \omega_l - \omega_L \left(\frac{D}{\omega_L r_0^2} \right) \left(\frac{3\pi}{8} \right)^{\frac{2}{3}} . \quad (3.37)$$

Closing contour in ω - integration on this pole we obtain

$$G_r(y - y_0, r, r_0) = \quad (3.38)$$

$$\frac{8}{\left(\frac{3\pi\omega_L^2 r_0^4}{8D^2} \right)^{\frac{1}{6}}} Ai \left(\left(\frac{\omega_L}{r_0 D} \right)^{\frac{1}{3}} \delta r - \left(\frac{3\pi}{8} \right)^{\frac{2}{3}} \right) \times e^{\omega_0(y - y_0)} .$$

Therefore, we obtain the typical Regge asymptotics without any diffusion in the log of transverse momentum. The first factor is the Reggeon residue while the second one is the Reggeon propagator.

For the case $D < 0$ one can see that the asymptotics will be given by Eq. (3.37) with obvious substitute $\omega_L \rightarrow \tilde{\omega}$ (see Eq. (3.32)) and $D \rightarrow D' = 2|D|$ (see Eq. (3.34)).

One can see, that Regge behaviour depends only on initial virtualities $Q_0(r_0)$ and corresponds to the pole in the angular momentum with the position which is independent from any characteristics of the “hard” processes. We hope, that this example can be instructive for high energy phenomenology. At least, it gives an answer to the question: why and how the Regge-like behaviour could appear for the initial condition for the DGLAP evolution equations. Recall, that such a behaviour is heavily used in the solution of the DGLAP evolution equations.

4 $\mathbf{G}_y(\mathbf{y}, \mathbf{r})$

4.1 A general solution

To find $G_y(y, r)$ we return to a general equation of Eq. (2.9). The inhomogeneous term in it corresponds to Eq. (2.2) or, in other words, the inhomogeneous Eq. (2.9) is written for $G_y(y, r)$. One can easily find the solution to this equation, taking the solution of the homogeneous equation (see Eq. (2.10)) but considering \tilde{g} as a function of ω and f . Substituting it back to Eq. (2.9) we obtain the following equation for $\tilde{g}(\omega, f)$:

$$-\omega \frac{d\tilde{g}(\omega, f)}{df} = r_0 e^{-f r_0} e^{\frac{r_0}{\omega} \int_{f_0}^f \omega_{conf}(f') df'} . \quad (4.1)$$

Finally, a general solution can be reduced to the form:

$$G_y(y - y_0, r, r_0) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i} \int_{\hat{f}-i\infty}^{\hat{f}+i\infty} \frac{df}{2\pi i} \frac{r_0}{\omega} \int_f^\infty \frac{df'}{2\pi i} \quad (4.2)$$

$$e^{\omega(y-y_0) + f r - f' r_0 - \frac{r_0}{\omega} \int_f^{f'} \omega_{conf}(f'') df''} .$$

The integral over ω can be evaluated and we obtain the general solution in the form that we will use in this paper;

$$G_y(y - y_0, r, r_0) = r_0 \int \int \frac{df df'}{2\pi i} \theta(f' - f) e^{f r - f' r_0} \quad (4.3)$$

$$I_0 \left(2 \sqrt{(y - y_0) r_0} \int_f^{f'} \omega_{conf}(f'') df'' \right) .$$

It is easy to check that $G_y(y - y_0, r, r_0)$, given by Eq. (4.3), approaches $\delta(r - r_0)$ at $y \rightarrow y_0$.

Since at $y - y_0 \gg 1$ the argument of I_0 is large, we can use the asymptotics of the modified Bessel function, namely, $I_0(z) |_{z \gg 1} \rightarrow \frac{1}{\sqrt{2\pi z}} e^z$. To get a more compact answer we introduce: (i) new variables $f^- = f' - f$ and $f^+ = f' + f$; and (ii) the integral representation for θ - function

$$\theta(f^-) = \int_{\mu_0 - i\infty}^{\mu_0 + i\infty} \frac{d\mu}{2\pi i \mu} e^{\mu f^-} . \quad (4.4)$$

The final result is

$$G_y(y - y_0, r, r_0) = r_0 \int \int \int \frac{df df' d\mu}{(2\pi i)^2 \mu} \quad (4.5)$$

$$e^{-\left(\frac{r+r_0}{2} - \mu\right) f^- + \frac{r-r_0}{2} f^+ + 2 \sqrt{(y-y_0) r_0} \int_{\frac{f^+ - f^-}{2}}^{\frac{f^+ + f^-}{2}} \omega_{conf}(f'') df''} .$$

4.2 Regge-BFKL asymptotics

We assume that both f^- and f^+ are small enough to use Eq. (2.11) for $\omega_{conf}(f)$. We can see that the answer has a form:

$$G_y(y - y_0, r, r_0) = \int \int \int \frac{d f^+ d f^- d \mu}{(2 \pi i)^2 \mu} e^{\Psi(y, r, f^+, f^-)}, \quad (4.6)$$

and taking explicit integration over f'' , Ψ can be written as

$$\begin{aligned} \Psi(y, r, f^+, f^-) &= -\left(\frac{r+r_0}{2} - \mu\right) f^- + \frac{r-r_0}{2} f^+ + \\ &2\sqrt{\omega_L r_0 (y - y_0)} \sqrt{f^-} \cdot \left\{1 + \frac{D}{6\omega_L} \left[\frac{3}{4}(f^+)^2 + \frac{1}{4}(f^-)^2\right]\right\}. \end{aligned} \quad (4.7)$$

We evaluate the integrals over f^- and f^+ , using the steepest descent method in which we have the following equations for the saddle points :

$$\frac{\partial \Psi}{\partial f^-} \Big|_{f^- = f^-, S} = 0; \quad \frac{\partial \Psi}{\partial f^+} \Big|_{f^+ = f^+, S} = 0. \quad (4.8)$$

Eq. (4.8) gives

$$\frac{r+r_0}{2} - \mu + \frac{\sqrt{\omega_L r_0 (y - y_0)}}{\sqrt{f^-, S}} \times \left\{1 + \frac{D}{8\omega_L} [(f^+, S)^2 + \frac{5}{3}(f^-, S)^2]\right\} = 0 \quad (4.9)$$

$$\frac{r-r_0}{2} + \sqrt{\omega_L r_0 (y - y_0)} \sqrt{f^-} \times \frac{D}{2\omega_L} f^+, S = 0. \quad (4.10)$$

The leading order solutions of the above equations are

$$f_0^-, S = \left(\frac{\sqrt{\omega_L r_0 (y - y_0)}}{\frac{r+r_0}{2} - \mu}\right)^2; \quad (4.11)$$

$$f_0^+, S = -\frac{\frac{r-r_0}{2}}{\frac{D}{2\omega_L} \sqrt{\omega_L r_0 (y - y_0)} \sqrt{f_0^-, S}}. \quad (4.12)$$

Keeping the next order term we have

$$f^-, S = \left(\frac{\sqrt{\omega_L r_0 (y - y_0)}}{\frac{r+r_0}{2} - \mu}\right)^2 \times \left\{1 + \frac{D}{4\omega_L} [(f_0^+, S)^2 + \frac{5}{3}(f_0^-, S)^2]\right\}; \quad (4.13)$$

$$f^+, S = -\frac{\frac{r-r_0}{2}}{\frac{D}{2\omega_L} \sqrt{\omega_L r_0 (y - y_0)} \sqrt{f_0^-, S}} \left\{1 - \frac{D}{8\omega_L} [(f_0^+, S)^2 + \frac{5}{3}(f_0^-, S)^2]\right\}. \quad (4.14)$$

Substituting Eq. (4.13) and Eq. (4.14) in Eq. (4.7), we obtain for the saddle value of Ψ

$$\begin{aligned} \Psi(y, r, f^+, S, f^-, S) &= \omega_L \frac{2r_0}{r_0 + r} (y - y_0) - \frac{(\delta r)^2}{4D \frac{2r_0}{r_0 + r} (y - y_0)} + \\ &\frac{1}{12} D \frac{r+r_0}{2r_0} \left(\omega_L \frac{2r_0}{r_0 + r} \frac{r_0 + r}{2}\right)^2 (y - y_0)^3. \end{aligned} \quad (4.15)$$

Here, we have taken the integration over μ . Indeed, it is easy to see that the saddle point value of $\Psi(y, r, f^{+,S}, f^{-,S}) \rightarrow -\infty$ when $\mu \rightarrow -\infty$. Therefore, we can close contour of integration over μ and take a residue at $\mu = 0$.

In Eq. (4.15) one can see the power increase (Reggeon - like) with energy (the first term) and BFKL diffusion in log of transverse momenta (the second term). It is interesting to note that in both terms as well as in the third one, the running QCD coupling constant depends on the scale $q_{ev}^2 = q q_0 (r_{ev} = \frac{r_0+r}{2})$. Namely, coupling constant at this scale enters in calculation of both ω_L and D .

For both Green functions we obtain a term proportional to $(y - y_0)^3$ which was first found in Ref. [4]².

4.3 Regge-BFKL asymptotics for $D < 0$

One can see from Eq. (4.11) that the saddle point value of f^- does not depend on the value of D . It suggests the following logic of approach: we expand Ψ in a general solution of Eq. (4.5) in the region of small f^- while keeping f^+ to be arbitrary large. In such an approximation we have

$$\frac{d \int_{\frac{f^+ - f^-}{2}}^{\frac{f^+ + f^-}{2}} \omega_{conf}(f') df'}{df^-} = \frac{1}{2} \left\{ \omega_{conf} \left(\frac{f^+ + f^-}{2} \right) + \omega_{conf} \left(\frac{f^+ - f^-}{2} \right) \right\} =$$

$$\omega_{conf} \left(\frac{f^+}{2} \right) + \frac{1}{8} \omega''_{conf} \left(\frac{f^+}{2} \right) (f^-)^2, \quad (4.16)$$

where $\omega''_{conf} = \frac{d^2 \omega_{conf}(f)}{d^2 f}$.

The value of f^- in the saddle point is equal to

$$\sqrt{f^{-,S}} = \quad (4.17)$$

$$\frac{2}{r_0 + r} \sqrt{r_0 \omega_{conf} \left(\frac{f^+}{2} \right) (y - y_0) \left(1 + \frac{5}{48} \frac{\omega''_{conf} \left(\frac{f^+}{2} \right)}{\omega_{conf} \left(\frac{f^+}{2} \right)} (f_0^{-,S})^2 \right)},$$

where

$$\sqrt{f_0^{-,S}} = \frac{2}{r_0 + r} \sqrt{r_0 \omega_{conf} \left(\frac{f^+}{2} \right) (y - y_0)}. \quad (4.18)$$

Ψ for this saddle point is equal to

$$\Psi(y - y_0, r, f^+, f^{-,S}) = \quad (4.19)$$

²The numerical coefficient in front of this term ($\frac{1}{12}$) is different from the coefficient calculated in Ref. [4] ($\frac{1}{3}$) due to different definition of D ($D(our) = 4 D(Ref.[4])$).

$$\frac{\delta r}{2} + \frac{2r_0}{r_0 + r} \omega_{conf} \left(\frac{f^+}{2} \right) (y - y_0) \left[1 + \frac{1}{24} \frac{\omega''_{conf} \left(\frac{f^+}{2} \right)}{\omega_{conf} \left(\frac{f^+}{2} \right)} (f_0^{-,S})^2 \right].$$

The position of the saddle point for f^+ integration is located near to the minimum of $\omega_{conf}(\frac{f^+}{2})$. If we denote the position of minimum $f_0 = \text{Re}f_0 + i\text{Im}f_0$ we can obtain the result after taking the integral over f^+ using the saddle point method. In practice as we have mentioned in the NLO BFKL the minimum occurs at $f_0 = \pm i \sqrt{\frac{|D|}{2B}}$ (see Eq. (3.32)) for $D < 0$ and $f_0 = 0$ for positive D .

The result is

$$G_y(y - y_0, r, r_0) = \tag{4.20}$$

$$\frac{1}{q q_0} \sqrt{\frac{\omega_{conf} \left(\frac{f_0}{2} \right)}{\omega''_{conf} \left(\frac{f_0}{2} \right) \left(\frac{r_0+r}{2} \right)^3}} \times \cos(2\text{Im}f_0 \delta r) \times e^{\Psi(y-y_0, r, r_0, f^{+,S}, f^{-,S})},$$

where

$$\Psi(y - y_0, r, r_0, f^{+,S}, f^{-,S}) = \frac{\omega_{conf} \left(\frac{f_0}{2} \right) 2 r_0}{r_0 + r} (y - y_0) \tag{4.21}$$

$$- \frac{(\delta r)^2}{2\omega''_{conf} \left(\frac{f_0}{2} \right) (y - y_0)} + \frac{1}{24} \omega''_{conf} \left(\frac{f_0}{2} \right) \frac{\omega_{conf}^2 \left(\frac{f_0}{2} \right)}{\left(\frac{r_0+r}{2} \right)^2} (y - y_0)^3.$$

5 Summary

In this paper we analyze the prediction for high energy asymptotics that emerges from the NLO BFKL equation. Below we summarize the results of our analysis.

1. We found two Green functions (see Eq. (3.2) and Eq. (4.5)) that govern the asymptotic behaviour of the scattering amplitudes at high energy with two different initial conditions: at fixed virtuality $Q^2 = Q_0^2$ and at fixed energy (x).

2. For sufficiently small values of energy [6][4]

$$y = \ln(s/q q_0) \leq \alpha_S^{-\frac{5}{3}}$$

in both Green function we found the Regge - BFKL asymptotics

$$G \propto e^{\omega_a (y - y_0) - \frac{(r - r_0)^2}{4D_a (y - y_0)}} \tag{5.1}$$

However, parameters ω_a and D_a in Eq. (5.1) turns out to be different for different sign of D in the NLO expression for the BFKL kernel (see Eq. (2.11))

D > 0

$$\omega_a = \frac{2r_0}{r+r_0} \omega_L \quad (\text{see Eq. (2.11)})$$

$$D_a = \frac{2r_0}{r+r_0} D$$

D < 0

$$\omega_a = \frac{2r_0}{r+r_0} [\omega_L + \frac{D^2}{4B}] \quad (\text{see Eq. (3.32)})$$

$$D_a = 2 \frac{2r_0}{r+r_0} |D|$$

3. We confirm the result of Ref. [4] that the first corrections to the Regge-BFKL asymptotics lead to an additional term in the exponent of Eq. (5.1), which has a form:

$$\frac{1}{12} \frac{D_a \omega_a^2}{r_0^2} (y - y_0)^3 . \quad (5.2)$$

We showed that it is legitimate to consider such a term in the framework of our approach.

4. Unfortunately, we found the oscillation in $r - r_0$ for the asymptotics in the NLO BFKL in the huge region of initial virtualities $q_0^2(r_0)$ where the NLO generates $D < 0$ (see Eq. (3.32)). This oscillation was first suggested in Ref. [7]. Indeed, the asymptotics has a factor (see Eq. (4.20))

$$G_y \propto \cos \left(2 (r - r_0) \sqrt{\frac{|D|}{2B}} \right) . \quad (5.3)$$

We consider this result as a challenge for the experts in QCD since it is a strong indication that the NLO BFKL approach has a pathology. We do not think that an additional integration in Eq.(2.1) will rule out this oscillations. It should be stressed that the exchange of the BFKL Pomeron has a clear physical meaning as the asymptotics of the colour dipole - dipole scattering (see Ref.[20]). It is difficult to believe that anyone would cope with a negative total cross section for such a process ³.

5. Our solution of Eq. (3.2) suggests the asymptotic behaviour in the region of large values of Q^2 . Indeed, the Regge-BFKL asymptotics as well as the term which violates it come from the saddle point approximation in the region of $\omega \approx \omega_a$, but we find that ω_a falls at $r \gg 1$, while the contribution from the zeros of the denominator in Eq. (3.2) do not depend on virtuality $Q^2(r)$. Therefore, at large Q^2 the rightmost singularity in the ω integration will be the position of the zero

$$\omega_0 = \omega_a - \omega_a \left(\frac{D_a}{\omega_a r_0^2} \right) \left(\frac{3\pi}{8} \right)^{\frac{2}{3}} . \quad (5.4)$$

This singularity leads to Regge - behaviour

$$G_r \propto e^{\omega_0 (y - y_0)}$$

in spite of the fact that in the huge region of intermediate values of Q^2 we have a much more complicated behaviour.

³ It should be stressed that it was first demonstrated in Ref.[21] that the NLO BFKL corrections to the anomalous dimension lead to the negative probability at exceedingly small x . Perhaps, this result is correlated with ours, but the difference is that we observe our oscillations in the region of applicability of the BFKL approach, and, therefore, we cannot follow the “wise” advise of Dokshitzer [22]: “Let DIS structure functions in peace!”

This observation can be used as a justification of the Reggeon-like behaviour of the structure functions in the region of small x , used for the initial condition for the evolution equations.

6. We cannot trust the perturbative QCD calculation for the Green function G_y since at $y - y_0 \leq \frac{r_0^2}{4D}$ (see Ref. [4]) the contamination from the nonperturbative QCD region is rather big. We think, this is a principal difference between G_r and G_y , that in G_r all nonperturbative corrections can be included in the input distribution G_{in} in Eq. (2.15) while G_r can be calculated in the framework of perturbative QCD.

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