A Highly Symmetric Four-Dimensional Quasicrystal*

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Abstract

A quasiperiodic pattern (or quasicrystal) is constructed in real four-dimensional Euclidean space, having the noncrystallographic reflection group [3,3,5] of order 14400 as its point group. It is obtained as a projection of the eight-dimensional lattice E_8, and has as a cross-section a three-dimensional quasicrystal with icosahedral symmetry.

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1. Introduction

We use a version of de Bruijn's projection method[(1)–(5)] (as developed especially in its group-theoretic sense by Kramer and Neri[(2)]) to construct a quasiperiodic pattern (or quasicrystal, for short) in four-dimensional Euclidean space, having the reflection group $G_1 = [3,3,5]$ as its point group. This group is a noncrystallographic group of order 14400, and is the symmetry group of the regular four-dimensional polytope known as the 600-cell, usually denoted by the Schläfli symbol $\{3,3,5\}$[(6),(7)]. G_1 is the largest finite real four-dimensional group.[(8)] The new quasicrystal is obtained as a projection of the 8-dimensional lattice E_8; it has as a cross-section a three-dimensional quasicrystal with icosahedral symmetry.

2. The lattice E_8 and the group $G_1 = [3,3,5]$

Let V be 8-dimensional (8-D) Euclidean space with an orthonormal basis $e_1, \ldots, e_8$. The lattice E_8 consists of all points $P = \sum_{i=1}^{8} v_i e_i$ where $\sum_{i=1}^{8} v_i$ is even and either all $v_i \in \mathbf{Z}$ or all $v_i \in \mathbf{Z} + \frac{1}{2}$.[(6),(9),(10)] (**Z** denotes the integers.) The point group G_0 of this lattice is the Weyl group $W(E_8)$, of order $696729600 = 2^{14}\, 3^5\, 5^2\, 7$. G_0 has a subgroup isomorphic to G_1; to make this subgroup visible we give a second definition of E_8 using quaternions.

The *unit icosians* consist of the 120 quaternions obtained from

$$(\pm 1, 0, 0, 0),\ \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1),\ \frac{1}{2}(0, \pm 1, \pm\sigma, \pm\tau)$$

by allowing any choice of signs and any even permutation of the coordinates, where $\sigma = \frac{1}{2}(1 - \sqrt{5})$, $\tau = \frac{1}{2}(1 + \sqrt{5})$, and (a, b, c, d) is an abbreviation for the quaternion $a + bi + cj + dk$. The set (of *icosians* consists of all finite sums of unit icosians.[10],[11],[12]

A quaternion $q = a + bi + cj + dk$ has *quaternionic conjugate* $\bar{q} = a - bi - cj - dk$. The *quaternionic norm* of an icosian $q = (a, b, c, d)$ is $Q(q) = q\,\bar{q} = a^2 + b^2 + c^2 + d^2$, which is a real number of the form $x + y\sqrt{5}$ where x, y are rational; the *Euclidean norm* of q is $N(q) = x + y$, which can be shown to be a nonnegative rational number. The unit icosians are precisely the icosians of quaternionic norm 1. The icosians equipped with the quaternionic norm lie in a real 4-D space, in which the unit icosians form the 120 vertices of the regular polytope $\{3,3,5\}$[6],[11]. But the icosians equipped with the Euclidean norm lie in a real 8-D space, and form a lattice isomorphic to the E_8 lattice.[10],[12] We shall use the particular isomorphism between (and E_8 defined by the mappings shown in Table I. There are 240 icosians of Euclidean norm 1, consisting of the unit icosians and σ times the unit icosians, and these correspond to the 240 minimal vectors of the E_8 lattice. This correspondence can be deduced from Table I, and is written out in full in Table 8.1 of Ref. 10 and Table I of Ref. 12.

We define the group $G_1 = [3,3,5]$ to consist of all transformations

$$q \to r\,q\,s \quad \text{and} \quad q \to r\,\bar{q}\,s,\ q \in (\ ,$$

of the icosians, where r and s are unit icosians.[11] These transformations preserve $Q(q)$ and $N(q)$. There are 120 choices for each of r and s, but $-r$, $-s$ and r, s produce the same transformations, so G_1 has order 14400. G_1 is generated by the particular transformations[10],[12]

$$\begin{array}{ll} L_i:\ q \to iq, & R_i:\ q \to qi, \\ L_\omega:\ q \to \omega q, & R_\omega:\ q \to q\omega, \\ B:\ q \to \bar{q}, & \end{array}$$

where $\omega = ½\,(-1, \tau, \sigma, 0) \in$ ((and $\omega^3 = 1$).

The identification of (with E_8 shows that G_1 acts on V as a subgroup of G_0. Each element $A \in G_1$ is represented by an 8×8 matrix (a_{ij}), where $A(e_i) = \sum_{j=1}^{8} a_{ij}\, e_j$; the a_{ij} may be obtained from Table I. Consider L_i for example. From Table I we find that

$$e_1 = \frac{1}{4}\,(2 - \sigma, -\sigma, -\sigma, -\sigma)\ ,$$

$$i\, e_1 = \frac{1}{4}\,(\sigma, 2 - \sigma, \sigma, -\sigma)$$

$$= \frac{1}{2}\,(e_2 + e_4 + e_6 - e_8)\ ,$$

which gives the first row of the matrix

$$L_i = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} .$$

There are two 4-D subspaces X and $\tilde{X}$ of V that are invariant under the action of G_1. The space X is spanned by the vectors f_1, f_2, f_3, f_4, and $\tilde{X}$ by f_5, f_6, f_7, f_8, where $f_j = \sum_{i=1}^{8} \phi_{ij}\, e_i$ and $\Phi = (\phi_{ij})$ is the 8×8 orthogonal matrix

$$\Phi = \begin{bmatrix} c(I + \sigma H) & \tilde{c}\,(I + \tau H) \\ c(I - \sigma H) & \tilde{c}\,(I - \tau H) \end{bmatrix},$$

where $I = I_4 = \mathrm{diag}\{1, 1, 1, 1\}$,

$$c = \frac{1}{\sqrt{4 + 2\sigma}} = 0.602... ,\ H = \frac{1}{2} \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix},$$

and tilde is the algebraic conjugation operation that changes the sign of $\sqrt{5}$ (i.e. interchanges σ and τ). Thus $\tilde{c} = (4 + 2\tau)^{-1/2} = 0.372...$.

To see that X, $\tilde{X}$ are invariant under G_1 we represent points of V in the f_i basis. If $P = \sum_{i=1}^{8} v_i e_i = \sum_{i=1}^{8} v'_i f_i \in V$, where $v = (v_1, ..., v_8)$, $v' = (v'_1, ..., v'_8)$, then $v' = v\,\Phi$. This changes $A = (a_{ij}) \in G_1$ to $A' = \Phi^{tr} A\,\Phi$. The generators for G_1 in this basis are

$$L'_i = \mathrm{diag}\{T, T, T, T\},\ R'_i = \mathrm{diag}\{T, -T, T, -T\},\ T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$L'_\omega = \mathrm{diag}\{U_1, \tilde{U}_1\}$, $R'_\omega = \mathrm{diag}\{U_2, \tilde{U}_2\}$, where

$$U_1 = -\frac{1}{2} \begin{bmatrix} 1 & -\tau & -\sigma & 0 \\ \tau & 1 & 0 & \sigma \\ \sigma & 0 & 1 & -\tau \\ 0 & -\sigma & \tau & 1 \end{bmatrix},\ U_2 = -\frac{1}{2} \begin{bmatrix} 1 & -\tau & -\sigma & 0 \\ \tau & 1 & 0 & -\sigma \\ \sigma & 0 & 1 & \tau \\ 0 & \sigma & -\tau & 1 \end{bmatrix},$$

and $B' = \mathrm{diag}\{+1, -1, -1, -1, +1, -1, -1, -1\}$. In this basis the matrices have the form $\begin{bmatrix} g' & 0 \\ 0 & \tilde{g}' \end{bmatrix}$, where the g' are a 4-D representation of G_1; this shows that X and $\tilde{X}$ are invariant subspaces.

The projection map π_X from V to X sends $P = \sum_{i=1}^{8} v_i e_i \in V$ to $Q = \sum_{i=1}^{8} w_i e_i \in X$, where

$$w = v\,\Pi,\quad \Pi = \Phi \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Phi^{tr} = -\frac{1}{\sqrt{5}} \begin{bmatrix} \sigma^{-1} I & H \\ H & \sigma I \end{bmatrix} .$$

In the f_i basis π_X simply sends $P = \sum_{i=1}^{8} v'_i f_i$ to $Q = \sum_{i=1}^{4} v'_i f_i \in X$. The projection map $\pi_{\tilde{X}}$ from V to $\tilde{X}$ is described by the matrix $\tilde{\Pi}$, and sends $P = \sum_{i=1}^{8} v'_i f_i$ to $Q = \sum_{i=5}^{8} v'_i f_i \in \tilde{X}$.

We note that E_8 has only the origin in common with either of the spaces X or $\tilde{X}$.

3. The Voronoi cell of E_8 **and its projection onto** X

The *Voronoi cell* Ω of E_8 is defined by

$$\Omega = \{ Q \in V : \|Q\| \le \|Q - P\| \text{ for all } P \in E_8 \} .$$

The definition of the new quasicrystal involves the 4-D polytope $\Psi = \pi_{\tilde{X}}(\Omega)$ obtained by projecting Ω onto the subspace $\tilde{X}$.

The Voronoi cell Ω is a convex 8-D polytope, described in Refs. 8, 10, 13. It has 19440 vertices, shown in the e_i basis in Table II. (All permutations of the coordinates are permitted. The third column indicates when *all* sign combinations are permitted, or if there must be an *even* or *odd* number of minus signs.)

The polytope Ψ is the convex hull of the projection of these vertices onto $\tilde{X}$. As discussed in §2, to project onto $\tilde{X}$ we postmultiply these vectors by Φ and take the last four coordinates.

From this it can be shown (we omit the details) that Ψ has 720 vertices. There are 120 vertices with coordinates

$$\frac{\tau^2 \tilde{c}}{2} (\pm 2, 0, 0, 0), \ \frac{\tau^2 \tilde{c}}{2} (\pm 1, \pm 1, \pm 1, \pm 1), \ \frac{\tau^2 \tilde{c}}{2} (0, \pm 1, \pm \tau, \pm \sigma)$$

(with all choices of signs and any even permutation of the coordinates), which are obtained from the projections of the first three rows of Table II. These are the 120 vertices of a copy of the polytope $\{3,3,5\}$, having edge-length $\tau \tilde{c} = 0.602...$, circumradius $R_0 = \tau^2 \tilde{c} = 0.973...$, inradius $R_3 = 2^{-3/2} \tau^4 \tilde{c} = 0.901...$, and in which the distance from the center to the midpoint of a 2-D face is $R_2 = 3^{-1/2} \tau^3 \tilde{c} = 0.909...$ (cf. Ref. 6, pp. 157, 293). The other 600 vertices arise from the projections of the last four rows of Table II, and have coordinates

$$\frac{\tau^2 \tilde{c}}{3} (\pm \tau^2, \pm \tau^{-2}, \pm 1, 0), \ \frac{\tau^2 \tilde{c}}{3} (\pm \tau^2, \pm \tau^{-1}, \pm \tau^{-1}, \pm \tau^{-1}),$$

$$\frac{\tau^2 \tilde{c}}{3} (\pm \sqrt{5}, \pm \tau^{-1}, \pm \tau, 0), \ \frac{\tau^2 \tilde{c}}{3} (\pm \sqrt{5}, \pm 1, \pm 1, \pm 1),$$

$$\frac{\tau^2 \tilde{c}}{3} (\pm 2, \pm 2, 0, 0), \ \frac{\tau^2 \tilde{c}}{3} (\pm 2, \pm 1, \pm \tau, \pm \tau^{-1}),$$

$$\frac{\tau^2 \tilde{c}}{3} (\pm \tau, \pm \tau, \pm \tau, \pm \tau^{-2})$$

(again with all choices of signs and any even permutation of the coordinates). These are the 600 vertices of a copy of the reciprocal polytope $\{5,3,3\}$ (the 120-cell), having edge-length $2\tilde{c}/3 = 0.248...$, circumradius $R_0 = 2^{3/2} 3^{-1} \tau^2 \tilde{c} = 0.918...$, and in which the distance from the center to the midpoint of an edge is $R_1 = 3^{-1/2} \tau^3 \tilde{c} = 0.909...$ (cf. Ref. 6, pp. 157, 293).

Thus Ψ is the convex hull of reciprocal (and concentric) polytopes $\{3,3,5\}$ and $\{5,3,3\}$, arranged so that the midpoints of the edges of the $\{5,3,3\}$ pass through the centers of the triangular 2-D faces of the $\{3,3,5\}$. Ψ is a 4-D analog of the triacontahedron encountered in the investigation of 3-D quasicrystals, the convex hull of concentric polyhedra $\{3,5\}$ (an icosahedron) and $\{5,3\}$ (a dodecahedron) arranged so that the midpoints of their edges coincide.[4,5]

4. The four-dimensional quasicrystal

The 4-D quasicrystal # is obtained by projecting the lattice E_8 onto the subspace X, subject to the requirement that the projection onto $\tilde{X}$ lies in the polytope Ψ:

$$\# = \{ \pi_X (P) : P \in E_8, \pi_{\tilde{X}} (P) \in \Psi \} .$$

Properties of the quasicrystal. (i) # is invariant under a point group (fixing the origin) isomorphic to $G_1 = [3,3,5]$.

To see this we work in the f_i basis. Any element of G_1 occurs as the top left corner of some matrix $\begin{bmatrix} g' & 0 \\ 0 & \tilde{g}' \end{bmatrix}$. If $u \in \#$ there is a vector $v \in \Psi = \pi_{\tilde{X}} (\Omega)$ such that $(u,v) \in E_8$. Then

$$(u,v) \begin{bmatrix} g' & 0 \\ 0 & \tilde{g}' \end{bmatrix} = (ug' , v\tilde{g}') \in E_8 .$$

But Ω has the same symmetries as E_8, so $v\tilde{g}' \in \Psi$, and hence $ug' \in \#$.

(ii) # is closed under multiplication by τ.

Proof. Consider the map from V to V defined by the matrix

$$S = \begin{bmatrix} I & -H \\ -H^{tr} & 0 \end{bmatrix}$$

in the e_i basis, and by $S' = \Phi^{tr} S \Phi = \mathrm{diag}\{\tau, \tau, \tau, \tau, \sigma, \sigma, \sigma, \sigma\}$ in the f_i basis. This map preserves E_8, S' commutes with the elements g' of G_1 in the 4-D representation defined in §2, and S, S' have determinant 1 and satisfy the equation $Z^2 - Z - I = 0$. The map acts as an ‘‘inflation’’ on X (increasing lengths) and as a ‘‘contraction’’ on $\tilde{X}$ (decreasing lengths). In particular, working in the f_i basis, suppose $u \in \#$ with $(u,v) \in E_8$. Then the map sends (u,v) to $(\tau u, \sigma v) \in E_8$, and so $\tau u \in \#$.

(iii) $\#$ is a discrete set of points. In fact if t,u are distinct points of $\#$ then $\| t - u \| \geq 1/(80 \tau^4 \tilde{c}^2) = 0.00504\ldots$.

Proof. Let t,u be the projections of distinct points $P,Q \in E_8$, and let $R = P - Q = \sum_{i=1}^{8} r_i e_i$, $R_1 = \pi_X(R)$, $R_2 = \pi_{\tilde{X}}(R)$. Then $R \neq 0$ (since $P \neq Q$), and $R_1 \neq 0$, $R_2 \neq 0$ (since no point of E_8 except 0 lies in X or $\tilde{X}$). We write $\alpha = (r_1, \ldots, r_4)$, $\beta = (r_5, \ldots, r_8)$, obtain R_1 as the left part of $(\alpha, \beta)\Phi$, and find

$$\begin{aligned} R_1 \cdot R_1 = c^2 \{ & 2\, \alpha \cdot \alpha + 2\, \beta \cdot \beta \\ & + \sigma(2(\alpha - \beta) H (\alpha + \beta)^{tr} + (\alpha - \beta) \cdot (\alpha - \beta)) \} . \end{aligned}$$

$R_2 \cdot R_2$ is the algebraic conjugate of $R_1 \cdot R_1$. Using the fact that inner products in E_8 are integers it follows that

$$R_1 \cdot R_1 \;\; R_2 \cdot R_2 = c^2 \tilde{c}^2 n ,$$

for some positive integer n. Therefore $R_1 \cdot R_1 \;\; R_2 \cdot R_2 \geq c^2 \tilde{c}^2 = 1/20$. Furthermore, $\pi_{\tilde{X}}(P)$ and $\pi_{\tilde{X}}(Q) \in \Psi$, so $R_2 \cdot R_2 \leq (2 \tau^2 \tilde{c})^2$, and the result follows.

No lattice has properties (i) or (ii), and (i)-(iii) show that # is a discrete nonperiodic set of points.

(iv) Finding which points are in #.

It is not easy to apply the definition of # directly, since it is hard to tell if a point is in Ψ. We know from §3 that Ψ contains a sphere of radius $2^{-3/2} \tau^4 \tilde{c} = 0.901...$, and is contained in a sphere of radius $\tau^2 \tilde{c} = 0.973...$. It follows that $\pi_X (P)$ is certainly in # if $\| \pi_{\tilde{X}} (P) \| < 0.901...$, and is certainly not in # if $\| \pi_{\tilde{X}} (P) \| > 0.973...$.

This test is enough to show for example that exactly 120 of the 240 minimal vectors of E_8 project into #, producing points $c^{-1} (0, \pm 1, \pm \sigma, \pm \tau)$, etc., forming a copy of $\{3,3,5\}$. Similarly exactly 120 of the 2160 vectors in E_8 of length 2 project into #, producing points $\tau c^{-1} (0, \pm 1, \pm \sigma, \pm \tau)$, etc., forming a slightly larger $\{3,3,5\}$ concentric with the first. (Of course we know from property (i) that # is highly symmetric when viewed from the origin. The neighborhoods of other points are not so symmetric.)

There is however a simple algorithm for determining precisely which points are in #. To decide whether a point $P = \sum_{i=1}^{8} v_i e_i \in E_8$ projects into #, we first express P in terms of the f_i as $P = \sum_{i=1}^{8} v'_i f_i$, where $(v'_1, ..., v'_8) = (v_1, ..., v_8) \Phi$. Let $\beta = (v'_5, ..., v'_8)$. We must test whether $\beta \in \Psi$, or equivalently whether there is an $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that (α, β) is in the Voronoi cell Ω. It is known[6,10,13] that Ω is the set of points $P \in V$ satisfying

$$P \cdot T^{(k)} \leq 1 \quad (k = 1, ..., 240),$$

where the $T^{(k)}$ are the 240 minimal vectors of E_8. Writing $T^{(k)} = \sum_{i=1}^{8} t_i^{(k)} f_i$,

$\gamma^{(k)} = (t_1^{(k)}, \ldots, t_4^{(k)})$, $\delta^{(k)} = (t_5^{(k)}, \ldots, t_8^{(k)})$, we conclude that a necessary and sufficient condition for $\pi_X(P)$ to be in # is that there exists a real vector $\alpha = (\alpha_1, \ldots, \alpha_4)$ such that

$$\alpha \cdot \gamma^{(k)} \le 1 - \beta \cdot \delta^{(k)}, \text{ for } k = 1, \ldots, 240 .$$

This is a question of the existence of a ‘‘feasible solution’’ to a system of 240 linear inequalities in four variables, a standard (and easy) problem in linear programming.

(v) # has a cross-section which is a 3-D quasicrystal with icosahedral symmetry.

Proof. We write elements of G_1 with respect to the f_i basis. The top left corner matrices (the matrices g', in the notation of §2) corresponding to the elements $L'_i\ R'_i$ and $L'_\omega\ R'_\omega$ are respectively $a = \mathrm{diag}\{-1, -1, 1, 1\}$ and

$$b = \frac{1}{2} \begin{bmatrix} -1 & -\tau & -\sigma & 0 \\ \tau & \sigma & 1 & 0 \\ \sigma & 1 & \tau & 0 \\ 0 & 0 & 0 & +2 \end{bmatrix} .$$

Then $a^2 = b^3 = (a\,b)^5 = I$, so a, b generate a subgroup of G_1 that is isomorphic to the alternating group A_5 and fixes the fourth (or f_4) coordinate. This 4-D representation of A_5 is the sum of 3-D and 1-D representations. A cross-section of # in which the coefficient of f_4 is constant therefore has the desired properties.

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List of Table Captions

Table I

Identification of icosians and E_8 vectors

$$(1, 0, 0, 0) \rightarrow e_1 + e_5, \quad (0, 1, 0, 0) \rightarrow e_2 + e_6,$$
$$(0, 0, 1, 0) \rightarrow e_3 + e_7, \quad (0, 0, 0, 1) \rightarrow e_4 + e_8,$$
$$(\sigma, 0, 0, 0) \rightarrow \tfrac{1}{2}(-e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 - e_8),$$
$$(0, \sigma, 0, 0) \rightarrow \tfrac{1}{2}(-e_1 - e_2 + e_3 - e_4 + e_5 + e_6 - e_7 + e_8),$$
$$(0, 0, \sigma, 0) \rightarrow \tfrac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + e_7 - e_8),$$
$$(0, 0, 0, \sigma) \rightarrow \tfrac{1}{2}(-e_1 + e_2 - e_3 - e_4 + e_5 - e_6 + e_7 + e_8).$$

Table II

Vertices of Ω

Components	Signs	Number
$(\pm 1, 0^7)$	all	16
$(\pm 1^4, 0^4)/2$	all	1120
$(3, \pm 1^7)/4$	odd	1024
$(1, \pm 1^7)/3$	odd	128
$(\pm 2, \pm 1^4, 0^3)/3$	all	8960
$(3^3, \pm 1^5)/6$	odd	7168
$(\pm 5, \pm 1^7)/6$	even	1024
		19440

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