Abstract

Pseudodifferential operators that are invariant under the action of a discrete subgroup \( \Gamma \) of \( SL(2, \mathbb{R}) \) correspond to certain sequences of modular forms for \( \Gamma \). Rankin–Cohen brackets are noncommutative products of modular forms expressed in terms of derivatives of modular forms. We introduce an analog of the heat operator on the space of pseudodifferential operators and use this to construct bilinear operators on that space which may be considered as Rankin–Cohen brackets. We also discuss generalized Rankin–Cohen brackets on modular forms and use these to construct certain types of modular forms.

Keywords: Modular forms; Pseudodifferential operators; Jacobi-like forms; Rankin–Cohen brackets; Jacobi forms

1. Introduction

It is well known that certain combinations of derivatives of modular forms can be used to obtain new modular forms, although derivatives of modular forms themselves are not modular forms in general. For example, in [9] Rankin described the polynomials in the derivatives of modular forms for a discrete subgroup \( \Gamma \) of \( SL(2, \mathbb{R}) \) that are again modular forms. As a special case of such polynomials, Cohen [3] studied certain bilinear operators on the graded ring of
modular forms, which may be considered as noncommutative products of modular forms. These noncommutative products are known as Rankin–Cohen brackets.

Pseudodifferential operators are formal Laurent series in the formal inverse $\partial^{-1}$ of the differential operator $\partial = d/dz$ on the complex plane $\mathbb{C}$. One of the natural ways of describing Rankin–Cohen brackets for modular forms is by way of pseudodifferential operators. Indeed, if the coefficients of pseudodifferential operators are holomorphic functions on the Poincaré upper half plane, there is a natural correspondence between the set of pseudodifferential operators invariant under the action of a discrete group $\Gamma \subset SL(2,\mathbb{R})$ and the set of certain sequences of modular forms for $\Gamma$, and Rankin–Cohen brackets can be constructed by using the fact that the product of two $\Gamma$-invariant pseudodifferential operators is again $\Gamma$-invariant (see [4,10]). It is also known that $\Gamma$-invariant pseudodifferential operators correspond to Jacobi-like forms for $\Gamma$. Jacobi-like forms generalize Jacobi forms introduced systematically by Eichler and Zagier [6], and they are certain formal power series satisfying a certain transformation formula relative to an action of $\Gamma$.

Rankin–Cohen brackets for Jacobi forms were introduced in [2] by using the heat operator, and they can be used to construct Rankin–Cohen brackets for Jacobi-like forms. One of the main goals of this paper is to study bilinear forms on the space of pseudodifferential operators which correspond to Rankin–Cohen brackets for Jacobi-like forms. We introduce an analog of the heat operator on the space of pseudodifferential operators and use this to construct bilinear operators on that space which may be considered as Rankin–Cohen brackets. We also discuss generalized Rankin–Cohen brackets on modular forms and use these to construct certain types of modular forms.

2. Basic correspondences

In this section we review basic correspondences among Jacobi-like forms, pseudodifferential operators, and sequences of modular forms studied by Cohen, Manin, and Zagier [4,10].

Let $\mathcal{H}$ be the Poincaré upper half plane, and let $R$ be the space of holomorphic functions on $\mathcal{H}$ which are bounded by a power of the function $b(z) = (|z|^2 + 1)/\Im z$. We denote by $R[[X]]$ the complex algebra of formal power series in $X$ with coefficients in $R$. Let $\Gamma$ be a discrete subgroup of $SL(2,\mathbb{R})$ of finite covolume, and let $\lambda \in \mathbb{Z}$ and $\mu \in \mathbb{R}$.

**Definition 2.1.** A Jacobi-like form for $\Gamma$ of weight $\lambda$ and index $\mu$ is a formal power series $F(z, X) \in R[[X]]$ satisfying the following conditions:

(i) For each $z \in \mathcal{H}$ and $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma$ we have

$$F(\gamma z, (cz + d)^{-2} X) = (cz + d)^{\lambda} e^{\mu X/(cz+d)} F(z, X),$$

where $\gamma z = (az + b)(cz + d)^{-1}$.

(ii) Each coefficient of $F(z, X)$ is holomorphic at the cusps of $\Gamma$.

We denote by $J\mathcal{L}_{\lambda,\mu}(\Gamma)$ the space of Jacobi-like forms of weight $\lambda$ and index $\mu$.

**Definition 2.2.** Given a nonnegative integer $w$, a modular form of weight $w$ for $\Gamma$ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$(f|_w \gamma)(z) := (cz + d)^{-w} f(\gamma z) = f(z)$$

(2.2)
for all \( z \in \mathcal{H} \) and \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \) as well as the usual condition of holomorphy at each cusp of \( \Gamma \) (see, e.g., [8]). We denote by \( M_w(\Gamma) \) the space of modular forms of weight \( w \) for \( \Gamma \).

**Remark 2.3.** Let \( F_\delta(z, X) = \sum_{k=0}^{\infty} \phi_k(z)X^{k+\delta} \) with \( \delta \geq 0 \) be an element of \( J_\lambda,0(\Gamma) \). Then the condition (2.1) implies that

\[
\sum_{k=0}^{\infty} (cz + d)^{-2k-2\delta} \phi_k(\gamma z)X^{k+\delta} = (cz + d)^\lambda \sum_{k=0}^{\infty} \phi_k(z)X^{k+\delta}
\]

for all \( z \in \mathcal{H} \) and \( \gamma \in \Gamma \). Comparing the coefficients of \( X^{k+\delta} \), we see that \( \phi_k \) satisfies the transformation formula (2.2) for a modular form belonging to \( M_{2(k+\delta)+\lambda}(\Gamma) \) for each \( k \geq 0 \). As was pointed out by Cohen, Manin and Zagier in [4, Section 1], the boundedness condition for \( R \) implies the cusp condition for each \( \phi_k \) as well, so that \( \phi_k \in M_{2(k+\delta)+\lambda}(\Gamma) \).

Let \( R \) be the ring of holomorphic functions on \( \mathcal{H} \) satisfying the boundedness condition described above. We recall that a pseudodifferential operator over \( R \) is a formal Laurent series in the formal inverse \( \partial^{-1} \) of \( \partial \) with coefficients in \( R \) of the form \( \sum_{k=-\infty}^{u} h_k(z)\partial^k \) with \( u \in \mathbb{Z} \) and \( h_k \in R \). We denote by \( \PsiDO \) the set of all pseudodifferential operators over \( R \). Then the group \( SL(2, \mathbb{R}) \) acts on \( \PsiDO \) by

\[
\gamma \cdot \sum_{k=-\infty}^{u} h_k(z)\partial^k = \sum_{k=-\infty}^{u} h_k(\gamma z)(j(\gamma, z)2\partial)^k
\]

for all \( \gamma \in SL(2, \mathbb{R}) \). If \( \Gamma \) is a discrete subgroup of \( SL(2, \mathbb{R}) \), we denote by \( \PsiDO^\Gamma \) the set of pseudodifferential operators in \( \PsiDO \) that are fixed by each element of \( \Gamma \). The following proposition states correspondences among Jacobi-like forms, pseudodifferential operators, and modular forms.

**Proposition 2.4.** Let \( F_\delta(z, X) = \sum_{k=0}^{\infty} \phi_k(z)X^{k+\delta} \in R[[X]] \) for some nonnegative integer \( \delta \). Given a nonnegative integer \( \lambda \) and a nonzero real number \( \mu \), the following conditions are equivalent:

(i) The formal power series \( F_\delta(z, X) \) is a Jacobi-like form belonging to \( J_\lambda,\mu(\Gamma) \).

(ii) The coefficients of \( F_\delta(z, X) \) satisfy

\[ (\phi_k \mid_{2(k+\delta)+\lambda} \gamma)(z) = \sum_{r=0}^{k} \frac{1}{r!} \left( \frac{c\mu}{cz + d} \right)^{r} \phi_{k-r}(z) \]

for all \( k \geq 0 \) and \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \).

(iii) Each coefficient of \( F_\delta(z, X) \) can be written in the form

\[ \phi_k = \sum_{r=0}^{k} \frac{\mu^{r}}{r!(2(k+\delta)+\lambda-r-1)!} f^{(r)}_{k+\delta+r} \]

for \( k \geq 0 \), where \( f_w \) is a modular form belonging to \( M_{2w+\lambda}(\Gamma) \) for each \( w \geq 0 \).

(iv) For each \( k \geq 0 \) we have

\[ \sum_{r=0}^{k} (-\mu)^{r} \frac{(2(k+\delta)+\lambda-r-2)!}{r!} \phi_{k-r}^{(r)} \in M_{2(k+\delta)+\lambda}(\Gamma). \]
Furthermore, if $\lambda$ is even with $\eta = \lambda/2$, then the above conditions are also equivalent to the following condition:

(v) The pseudodifferential operator

$$\psi(z) := \sum_{k=0}^{\infty} (-1)^{k+\delta+\eta}(k+\delta+\eta)!(k+\delta+\eta-1)!\phi_k(z)\mu^{-k-\delta-\eta}\partial^{-k-\delta-\eta}$$

is an element of $\Psi DO_\Gamma$, that is, $\psi(z)$ is $\Gamma$-invariant.

Proof. The proof of this proposition can be carried out by modifying the proof of Proposition 2 in [4], where the case of $\delta = 1$, $\lambda = 0$, and $\mu = 1$ was considered. \qed

3. Pseudodifferential operators and heat operators

Rankin–Cohen brackets on the space of Jacobi forms have been studied in [1] by using the heat operator. This heat operator can be extended to the space of Jacobi-like forms. In this section we introduce an analog of the heat operator on the space of pseudodifferential operators that corresponds to the heat operator on the space of Jacobi-like forms. We then use this operator to construct Rankin–Cohen brackets on the space of pseudodifferential operators that are compatible with those on the space of Jacobi-like forms.

We first introduce certain linear maps on the space of pseudodifferential operators as well as on the space Jacobi-like forms. Given a formal power series

$$F_\delta(z, X) = \sum_{k=0}^{\infty} f_k(z)X^{k+\delta} \in R[[X]]$$

and a pseudodifferential operator

$$\Phi_\delta(z) = \sum_{k=0}^{\infty} \phi_k(z)\partial^{-k-\delta} \in \Psi DO$$

with $\delta \geq 0$, we set

$$\left(F_\delta\right)_\mu^\Psi(z) = \sum_{k=0}^{\infty} (-1)^{k+\delta}(k+\delta)!\mu^{-k-\delta}f_k(z)\partial^{-k-\delta}, \quad (3.1)$$

$$\left(\Phi_\delta\right)_\mu^X(z, X) = \sum_{k=0}^{\infty} (-1)^{k+\delta}\mu^{k+\delta}\phi_k(z)(k+\delta)!X^{k+\delta}.$$  (3.2)

Thus, for fixed $\delta \geq 0$, these formulas determine the linear maps

$$\left(\cdot\right)_\mu^\Psi : R[[X]] \rightarrow \Psi DO, \quad \left(\cdot\right)_\mu^X : \Psi DO \rightarrow R[[X]],$$

which are in fact isomorphisms. In fact, by using Proposition 2.4 we see easily that

$$\left(F_\delta\right)_\mu^\Psi = F_\delta, \quad \left(\Phi_\delta\right)_\mu^X = \Phi_\delta$$  (3.3)

for all $F_\delta \in R[[X]]$ and $\Phi_\delta \in \Psi DO$. 

Next, we consider a differential operator on the space of formal power series, which may be regarded as the heat operator on that space. Given \( \mu \in \mathbb{R} \), we define \( L_\mu \) to be the formal differential operator given by

\[
L_\mu = \mu \frac{\partial}{\partial z} - \frac{1}{2} \frac{\partial}{\partial X} - X \frac{\partial^2}{\partial X^2},
\]

which obviously determines a linear map \( L_\mu : R[[X]] \rightarrow R[[X]] \).

**Remark 3.1.**

(i) The operator \( L_\mu \) may be considered as the heat operator on \( R[[X]] \) for the following reason. If \( w = \sqrt{\frac{X}{2\pi i}} \), we have

\[
\frac{\partial^2}{\partial w^2} = 2\sqrt{\frac{X}{2\pi i}} \frac{\partial}{\partial X} \left( 2\sqrt{\frac{X}{2\pi i}} \frac{\partial}{\partial X} \frac{1}{2\pi i} \right) = 4\pi i \left( \frac{\partial}{\partial X} + 2X \frac{\partial^2}{\partial X^2} \right);
\]

hence we obtain

\[
L_\mu = \frac{1}{8\pi i} \left( 8\pi i \mu \frac{\partial}{\partial z} - \frac{\partial^2}{\partial w^2} \right),
\]

which is a heat diffusion operator with respect to the variables \( z \) and \( w \). Such a heat operator has already been used in the construction of Rankin–Cohen brackets on the space of Jacobi forms (see [1,2]).

(ii) In general, the operator \( L_\mu \) on \( R[[X]] \) does not preserve the Jacobi-like property, that is, the image \( L_\mu(F) \) of an element \( F \in J_\lambda,\mu \subset R[[X]] \) under \( L_\mu \) is not necessarily an element of \( J_\LL_\lambda,\mu \).

Let \( A : \Psi \text{DO} \rightarrow \Psi \text{DO} \) be a formal integration operator with respect to the symbol \( \partial \), that is, an operator given by

\[
A \left( \sum \phi_k(z) \partial^{-k-\delta} \right) = \sum \frac{\phi_k(z)}{1-k-\delta} \partial^{-k-\delta+1}.
\]

The next proposition suggests that the operator

\[
R_\mu := \mu \left( \partial - \frac{A}{2} \right) : \Psi \text{DO} \rightarrow \Psi \text{DO}
\]

plays the role of the heat operator on the space \( \Psi \text{DO} \).

**Theorem 3.2.** If \( \partial = \partial/\partial z \) as usual, we have

\[
(L_\mu(F_\delta))^\Psi_\mu = R_\mu((F_\delta)^\Psi_\mu)
\]

for all \( F_\delta \in R[[X]] \). More generally, we have

\[
(L_\mu^\ell(F_\delta))^\Psi_\mu = R_\mu^\ell((F_\delta)^\Psi_\mu)
\]

for each positive integer \( \ell \), where \( L_\mu^\ell = L_\mu \circ \cdots \circ L_\mu \) denotes the \( \ell \)-fold composite of the linear endomorphism \( L_\mu \) of \( R[[X]] \).
Proof. Consider the formal power series \( F_\delta = \sum_{k=0}^{\infty} f_k(z)X^{k+\delta} \in R[[X]] \) for some nonnegative integer \( \delta \). Using (3.4), we have

\[
\mathcal{L}_\mu(F_\delta) = \sum_{k=0}^{\infty} \left( \mu f'_k(z)X^{k+\delta} - \frac{k+\delta}{2} f_k(z)X^{k+\delta-1} - (k+\delta)(k+\delta-1) f_k(z)X^{k+\delta-1} \right)
\]

\[
= \sum_{k=0}^{\infty} \left( \mu f'_k(z)X^{k+\delta} - (k+\delta)(k+\delta-1/2) f_k(z)X^{k+\delta-1} \right)
\]

\[
= \sum_{k=0}^{\infty} \left( \mu f'_k(z) - (k+\delta+1)(k+\delta+1/2) f_{k+1}(z) \right)X^{k+\delta} - \delta(\delta - 1/2) f_0(z)X^{\delta-1}.
\]

Using this and (3.1), we obtain

\[
(\mathcal{L}_\mu(F_\delta))^\Psi = \sum_{k=0}^{\infty} (-1)^{k+\delta}(k+\delta)!(k+\delta-1)! \mu^{-k-\delta} \mu f'_k(z)\partial^{-k-\delta} 
\]

\[
\quad \times \left( \mu f'_k(z) - (k+\delta+1)(k+\delta+1/2) f_{k+1}(z) \right)\partial^{-k-\delta} - (-1)^{\delta-1}(\delta-1)! \delta(\delta - 1/2) \mu^{-\delta+1} f_0(z)\partial^{-\delta+1}. \tag{3.9}
\]

On the other hand, noting that

\[
(F_\delta)^\Psi = \sum_{k=0}^{\infty} (-1)^{k+\delta}(k+\delta)!(k+\delta-1)! \mu^{-k-\delta} f_k(z)\partial^{-k-\delta},
\]

and using (3.5) and (3.6), we see that

\[
\mathcal{R}_\mu((F_\delta)^\Psi) = \sum_{k=0}^{\infty} (-1)^{k+\delta} \mu^{-k-\delta}(k+\delta)!(k+\delta-1)!(k+\delta+1)! f_k(z)\partial^{-k-\delta+1} 
\]

\[
\quad \times \mu \left( f'_k(z)\partial^{-k-\delta} + f_k(z)\partial^{-k-\delta+1} - \frac{f_k(z)}{2(1-k-\delta)}\partial^{-k-\delta+1} \right)
\]

\[
= \sum_{k=0}^{\infty} (-1)^{k+\delta} \mu^{-k-\delta+1}(k+\delta)!(k+\delta-1)! f'_k(z)\partial^{-k-\delta}
\]

\[
- \sum_{k=0}^{\infty} (-1)^{k+\delta-1} \mu^{-k-\delta+1}(k+\delta)!(k+\delta-2)!(k+\delta-1/2) f_k(z)\partial^{-k-\delta+1}.
\]

By rewriting the last infinite sum in the previous equation in the form

\[
\sum_{k=0}^{\infty} (-1)^{k+\delta} \mu^{-k-\delta}(k+\delta+1)!(k+\delta-1)!(k+\delta+1/2) f_{k+1}(z)\partial^{-k-\delta}
\]

\[
+ (-1)^{\delta-1} \mu^{-\delta+1} \delta!(\delta-2)!(\delta-1/2) f_0(z)\partial^{-\delta+1},
\]

we obtain
\[
R_\mu \left( (F_\delta)^\Psi \right) = \sum_{k=0}^{\infty} (-1)^{k+\delta} \mu^{-k-\delta} (k+\delta)! (k+\delta-1) \\
\times \left( \mu f_k^\mu(z) - (k+\delta+1) (k+\delta+1/2) f_{k+1}(z) \right) \partial^{k+\delta} \\
- (-1)^{\delta-1} \mu^{-\delta+1} (\delta)! (\delta-2)! (\delta-1/2) f_0(z) \partial^{-\delta+2}.
\]
Comparing this with (3.9), we obtain (3.7). Then, for each \( \ell \geq 1 \), relation (3.8) follows easily from this by induction. □

**Corollary 3.3.** If \( \Psi_\delta \in \Psi_{DO} \), then we have
\[
\left( R_\mu^{\ell} (\Psi_\delta) \right)_\mu^X = L_\mu^{\ell} \left( (\Psi_\delta)_\mu^X \right)
\]
for each positive integer \( \ell \).

**Proof.** Given \( \Psi_\delta \in \Psi_{DO} \), using (3.3) and (3.8), we have
\[
\left( R_\mu^{\ell} (\Psi_\delta) \right)_\mu^X = \left( R_\mu^{\ell} \left( (\Psi_\delta)_\mu^X \right) \right)_\mu^X = \left( (L_\mu^{\ell} \left( (\Psi_\delta)_\mu^X \right)) \right)_\mu^X = L_\mu^{\ell} \left( (\Psi_\delta)_\mu^X \right),
\]
which proves the corollary. □

Although there is a natural multiplication operation on \( \Psi_{DO} \), we now introduce a new bilinear operation. If \( \Psi \) and \( \Phi \) are elements of \( \Psi_{DO} \) given by
\[
\Psi_\delta = \sum_{k=0}^{\infty} \psi_k(z) \partial^{-(k+\delta)}, \quad \Phi_\epsilon = \sum_{\ell=0}^{\infty} \phi_\ell(z) \partial^{-(\ell+\epsilon)}
\]
with \( \delta, \epsilon \geq 0 \), we set
\[
\Psi_\delta \odot \Phi_\epsilon = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+\delta+\epsilon}{k+\delta} \binom{n+\delta+\epsilon-1}{k+\delta-1} (n+\epsilon-k) \psi_k(z) \phi_{n-k}(z) \partial^{-(n+\delta+\epsilon)}.
\]
We now use this operation to define a set of more general bilinear operators on \( \Psi_{DO} \).

**Definition 3.4.** Let \( v \) be a positive integer, and let \( \Psi_\delta \) and \( \Phi_\epsilon \) be elements of \( \Psi_{DO} \) given by (3.11). Then we define a bilinear operator
\[
[[ , ]]_v : \Psi_{DO} \times \Psi_{DO} \to \Psi_{DO}
\]
on \( \Psi_{DO} \) by
\[
[[ \Psi_\delta, \Phi_\epsilon ]]_v := \sum_{\ell=0}^{v} (-1)^{\ell} \binom{v+\delta_1-1}{v-\ell} \binom{v+\epsilon_1-1}{\ell} R_\mu^{\ell} (\Psi_\delta) \odot R_\mu^{v-\ell} (\Phi_\epsilon),
\]
where \( R_\mu \) is as in (3.6) and \( \delta_1 = \delta - 1/2, \epsilon_1 = \epsilon - 1/2 \).

By using the Rankin–Cohen brackets for Jacobi forms described in [2] we can write Rankin–Cohen brackets defined on the space of Jacobi-like forms as follows.
Definition 3.5. Let \( F_\delta, G_\epsilon \in J_L^{\lambda, \mu} \), and let \( \nu \) be a nonnegative integer. Then the \( \nu \)th Rankin–Cohen bracket of \( F_\delta \) and \( G_\epsilon \) is defined by

\[
[F_\delta, G_\epsilon]_\nu = \sum_{\ell=0}^{\nu} (-1)^\ell \binom{v + \delta_1 - 1}{\nu - \ell} \binom{v + \epsilon_1 - 1}{\ell} L_\mu^\ell (F_\delta) L_{\nu - \ell}^\mu (G_\epsilon),
\]

(3.14)

where \( L_\mu^\ell \) is the differential operator in (3.4) and \( \delta_1 = \delta - 1/2, \epsilon_1 = \epsilon - 1/2 \).

The next theorem states that the bilinear operators \([,]_\nu\) defined on \( \Psi DO^\Gamma \) is compatible with the Rankin–Cohen brackets \([,]_\nu\) defined on \( J_L^{\lambda, \mu} \) when the weight \( \lambda \) is equal to 0.

Theorem 3.6. Let \( \Psi_\delta, \Phi_\epsilon \in \Psi DO \) and \( F_\delta, G_\epsilon \in R\�bra\ket_\nu \).

(i) For each positive integer \( \nu \) we have

\[
[[\Psi_\delta, \Phi_\epsilon]]_\nu^X = [[(\Psi_\delta)^X_\mu, (\Phi_\epsilon)^X_\mu]]_\nu^X, \quad ([F_\delta, G_\epsilon]_\nu)_\Psi = [[[F_\delta]_\Psi, [G_\epsilon]_\Psi]]_\nu^X.
\]

(ii) The bilinear operator \([,]_\nu\) in (3.13) carries \( \Gamma \)-invariant pseudodifferential operators to \( \Gamma \)-invariant pseudodifferential operators, that is, \([\Psi_\delta, \Phi_\epsilon]_\nu \in \Psi DO^\Gamma \) whenever \( \Psi_\delta, \Phi_\epsilon \in \Psi DO^\Gamma \).

Proof. Let \( \Psi_\delta, \Phi_\epsilon \in \Psi DO^\Gamma \) be as in (3.11). Then by (3.1) we have

\[
(\Psi_\delta^X_\mu) = \sum_{k=0}^{\infty} \frac{(-1)^k \mu^k \psi_k(z)}{(k + \delta_1)!(k + \delta - 1)!} X^{k+\delta}, \quad (\Phi_\epsilon^X_\mu) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell \mu^\ell \phi_\ell(z)}{(\ell + \epsilon_1)!(\ell + \epsilon - 1)!} X^{\ell+\epsilon}.
\]

The formal product of these two power series can be written in the form

\[
(\Psi_\delta^X_\mu) \cdot (\Phi_\epsilon^X_\mu) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n \mu^n \psi_k(z) \phi_{n-k}(z)}{(k + \delta_1)!(k + \delta - 1)!(n-k+\epsilon_1)!(n-k+\epsilon-1)!} X^{n+\delta+\epsilon}.
\]

On the other hand, using (3.12) and (3.2), we obtain

\[
(\Psi_\delta \odot \Phi_\epsilon)^X_\mu = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n \mu^n (n + \epsilon - k)}{(n + \delta + \epsilon)!(n + \delta + \epsilon - 1)!} \binom{n + \delta + \epsilon}{k + \delta} \times \binom{n + \delta + \epsilon - 1}{k + \delta - 1} \psi_k(z) \phi_{n-k}(z) X^{n+\delta+\epsilon}.
\]

This implies that

\[
(R_\mu(\Psi_\delta) \odot R_\mu(\Phi_\epsilon))^X_\mu = (R_\mu(\Psi_\delta)^X_\mu) \odot (R_\mu(\Phi_\epsilon)^X_\mu).
\]

(3.16)

Using this and (3.3), we see that

\[
(F_\delta \cdot G_\epsilon)^\Psi_\mu = \left(\left(\left(F_\delta\right)^\Psi_\mu \cdot \left(G_\epsilon\right)^\Psi_\mu\right)^\Psi_\mu\right) = \left(\left(F_\delta\right)^\Psi_\mu \odot \left(G_\epsilon\right)^\Psi_\mu\right)^\Psi_\mu.
\]

We now use (3.13), (3.14) and (3.16) to obtain
\[
\left( [\Psi_{\delta}, \Phi_{\epsilon}]_v \right)_X^\mu = \sum_{\ell = 0}^\nu (-1)^\ell \binom{\nu + \delta_1 - 1}{\nu - \ell} \binom{\nu + \epsilon_1 - 1}{\ell} \left( R_{\mu}^\ell (\Psi_{\delta}) \otimes R_{v-\ell}^\nu (\Phi_{\epsilon}) \right)_X^\mu
\]
\[
= \sum_{\ell = 0}^\nu (-1)^\ell \binom{\nu + \delta_1 - 1}{\nu - \ell} \binom{\nu + \epsilon_1 - 1}{\ell} \left( R_{\mu}^\ell (\Psi_{\delta}) \right)_X^\mu \cdot \left( R_{v-\ell}^\nu (\Phi_{\epsilon}) \right)_X^\mu
\]
\[
= \sum_{\ell = 0}^\nu (-1)^\ell \binom{\nu + \delta_1 - 1}{\nu - \ell} \binom{\nu + \epsilon_1 - 1}{\ell} L_{\mu}^\ell (\Psi_{\delta})_X^\mu \cdot L_{v-\ell}^\nu (\Phi_{\epsilon})_X^\mu
\]
\[
= \left[ (\Psi_{\delta})_X^\mu, (\Phi_{\epsilon})_X^\mu \right]_v,
\]
where we also used (3.10) and (3.15). Using this and applying (3.3) repeatedly, we obtain
\[
\left( [F_{\delta}, G_{\epsilon}]_v \right)_X^\mu = \left( \left( \left( [F_{\delta}]_X^\mu, (G_{\epsilon})_X^\mu \right)_v \right) \right)_X^\mu = \left( \left( \left( [F_{\delta}]_X^\mu, (G_{\epsilon})_X^\mu \right)_v \right) \right)_X^\mu
\]
which proves (i). As for (ii), first we note that Rankin–Cohen operators send Jacobi forms to Jacobi forms (cf. [2]); hence it follows that the Rankin–Cohen brackets in (3.14) send Jacobi-like forms to Jacobi-like forms. Using this and the correspondence between and given in Proposition 2.4, we see that the Rankin–Cohen brackets also preserve the \( \Gamma \)-invariance property. □

4. Modular forms associated to products of Jacobi-like forms

From the results of Cohen, Manin, and Zagier [4] (see also [10]) as described in Proposition 2.4, we see that there is a one-to-one correspondence between Jacobi-like forms of the form \( \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta} \) and sequences \( \{ f_k \}_{k=0}^{\infty} \) of modular forms for \( \Gamma \). In this section, using the fact that the product of Jacobi-like forms is a Jacobi-like form, we construct sequences of modular forms from the products of certain types of modular forms.

Let \( F_{\delta}(z, X) \) and \( G_{\epsilon}(z, X) \) be Jacobi-like forms for \( \Gamma \) belonging to \( JL_{\lambda, \mu}(\Gamma) \) and \( JL_{\lambda', \mu'}(\Gamma) \), respectively, given by
\[
F_{\delta}(z, X) = \sum_{r=0}^{\infty} \phi_r(z) X^{r+\delta}, \quad G_{\epsilon}(z, X) = \sum_{\ell=0}^{\infty} \psi_{\ell}(z) X^{\ell+\epsilon}
\]
for some nonnegative integers \( \delta \) and \( \epsilon \). We set
\[
f_k(z) = \sum_{r=0}^{k} \mu_r^{r+\delta} (\mu_r')^{k-r+\epsilon} \phi_r(z) \psi_{r-k}(z),
\]
\[
g_k(z) = \sum_{r=0}^{k} \sum_{\ell=0}^{k-r} \sum_{s=0}^{r} \frac{(2(k+\delta+\epsilon)+\lambda+\lambda'-r-2)!}{s!(r-s)!} (-1)^r (\mu_r + \mu_r')^r \phi_{\ell}^{(s)}(z) \psi_{r-k-r}(z)
\]
for all \( z \in \mathcal{H} \) and \( k \geq 0 \).

**Proposition 4.1.** Let \( F_{\delta}(z, X) \in JL_{\lambda, \mu}(\Gamma) \) and \( G_{\epsilon}(z, X) \in JL_{\lambda', \mu'}(\Gamma) \) be as in (4.1). Then the functions \( f_k, g_k: \mathcal{H} \to \mathbb{C} \) given by (4.2) and (4.3) are modular forms belonging to \( M_{2(k+\delta+\epsilon)+\lambda+\lambda'}(\Gamma) \) for each \( k \geq 0 \).
Proof. Using Definition 2.1, we see that

\begin{align*}
F_\delta(z, -X/\mu)G_\varepsilon(z, X/\mu) &\in \mathcal{J}_{\lambda+\lambda', 0}(\Gamma), \\
F_\delta(z, X)G_\varepsilon(z, X) &\in \mathcal{J}_{\lambda+\lambda', \mu+\mu'}(\Gamma).
\end{align*}

On the other hand, from (4.1) we obtain

\begin{align*}
F_\delta(z, -X/\mu)G_\varepsilon(z, X/\mu) &= \sum_{r=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^r}{\mu^{r+\delta}(\mu')^{\ell+\varepsilon}} \phi_r(z)\psi_\ell(z)X^{r+\ell+\delta+\varepsilon} \\
&= \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{(-1)^r}{\mu^{r+\delta}(\mu')^{k-r+\varepsilon}} \phi_r(z)\psi_{k-r}(z)X^{k+\delta+\varepsilon}.
\end{align*}

(4.4)

Similarly, we have

\begin{align*}
F_\delta(z, X)G_\varepsilon(z, X) &= \sum_{k=0}^{\infty} \sum_{r=0}^{k} \phi_r(z)\psi_{k-r}(z)X^{k+\delta+\varepsilon}.
\end{align*}

(4.5)

Hence by (4.4) and Remark 2.3 the function \(f_k: \mathcal{H} \rightarrow \mathbb{C}\) given by (4.2) is an element of \(M_{2(k+\delta+\varepsilon)+\lambda+\lambda'}(\Gamma)\) for each \(k \geq 0\). On the other hand, applying Proposition 2.4 to (4.5), we see that the function \(\tilde{g}_k(z) = \sum_{r=0}^{k} (-1)^r (\mu + \mu')^r (2(k+\delta+\varepsilon)+\lambda+\lambda'-r-2)! \sum_{\ell=0}^{k-r} (\phi_\ell\psi_{k-r-\ell})(r)(z)\)

is also an element of \(M_{2(k+\delta+\varepsilon)+\lambda+\lambda'}(\Gamma)\) for each \(k \geq 0\). From this and the relation

\begin{align*}
(\phi_\ell(z)\psi_{k-r-\ell})(r) &= \sum_{s=0}^{r} \frac{r!}{s!(r-s)!} \phi_\ell(s)(z)\psi_{k-r-\ell-s}(z),
\end{align*}

we obtain \(g_k(z) = \tilde{g}_k(z)\), and therefore the proposition follows. \(\square\)

5. Rankin–Cohen brackets on modular forms

In this section we use the results of Section 4 to determine two types of noncommutative products of modular forms which generalize the usual Rankin–Cohen brackets for modular forms.

Given nonnegative integers \(\delta\) and \(\varepsilon\), let \(f\) and \(g\) be modular forms for \(\Gamma\) with

\begin{align*}
f \in M_{2\delta}(\Gamma), \quad g \in M_{2\varepsilon}(\Gamma).
\end{align*}

We consider the associated sequences \(\{f_k\}_{k=0}^{\infty}\) and \(\{g_k\}_{k=0}^{\infty}\) given by

\begin{align*}
f_k(z) = \begin{cases} 
  f(z) & \text{if } k = \delta, \\
  0 & \text{otherwise,}
\end{cases} \quad g_k(z) = \begin{cases} 
  g(z) & \text{if } k = \varepsilon, \\
  0 & \text{otherwise,}
\end{cases}
\end{align*}

for all \(k \geq 0\). By Proposition 2.4 these sequences correspond to Jacobi-like forms. Indeed, the formal power series

\begin{align*}
F_\delta(z, X) &= \sum_{j=0}^{\infty} \frac{f^{(j)}(z)}{j!(j+2\delta-1)!} X^{j+\delta}, \quad G_\varepsilon(z, X) = \sum_{\ell=0}^{\infty} \frac{g^{(\ell)}(z)}{\ell!(\ell+2\varepsilon-1)!} X^{\ell+\varepsilon}
\end{align*}
are Jacobi-like forms belonging to $\mathcal{J}L_{0,1}(\Gamma)$ (see [4]). Using the transformation property for Jacobi-like forms in (2.1), we see that the product $F_\delta(z, X)G_\varepsilon(z, -X)$ is a Jacobi-like form of weight 0 and index 0, and therefore by Remark 2.3 its coefficients are modular forms. Thus we obtain a sequence $\{[f, g]^{(\delta, \varepsilon)}_n\}_{n=0}^\infty$ of modular forms with $[f, g]^{(\delta, \varepsilon)}_n \in M_{2n+2\delta+2\varepsilon}(\Gamma)$ given by

$$
[f, g]^{(\delta, \varepsilon)}_n(z) = \sum_{j=0}^n \frac{(-1)^j f^{(j)}(z) g^{(n-j)}(z)}{j! (n-j)! (j+2\delta-1)! (n-j+2\varepsilon-1)!}
$$

for all $z \in \mathcal{H}$. The bilinear maps $(f, g) \mapsto [f, g]^{(\delta, \varepsilon)}_n$, or their constant multiples, for various $n$ define noncommutative products on the space of modular forms known as the Rankin–Cohen brackets (see [4,10]).

We now generalize the Rankin–Cohen brackets described above by using sequences of powers of modular forms to introduce other types of noncommutative products on the space of modular forms. Let $\delta$ and $\varepsilon$ be nonnegative integers as above, and let $f \in M_{2\delta}(\Gamma)$ and $g \in M_{2\varepsilon}(\Gamma)$. We define the sequences $\{[f, g]^A_n\}_{n=0}^\infty$ and $\{[f, g]^B_n\}_{n=0}^\infty$ of holomorphic functions on $\mathcal{H}$ by

$$
[f, g]^A_n(z) = \sum_{r=0}^{n-1} \sum_{\alpha=0}^{[r-1]/(2\delta)} \frac{(-1)^r (f^\alpha)^{(r-2\delta\alpha)}(z) (g^\beta)^{(n-r-2\beta\varepsilon)}(z)}{(r-2\alpha\delta)! (r+2\alpha\delta-1)!} 
$$

$$
\times (n-r-2\beta\varepsilon)! (n-r+2\beta\varepsilon-1)! \tag{5.1}
$$

$$
[f, g]^B_n(z) = \sum_{r=0}^{n-2} \sum_{u=0}^{n-r-1} \sum_{s=0}^{r} \frac{(-1)^r 2^r (2n-2-r)!}{s! (r-s)! (2\delta\alpha-r)!} 
$$

$$
\times (f^\alpha)^{(u-2\alpha\delta+s)}(z) (g^\beta)^{(n-u-2\beta\varepsilon+s)}(z) 
$$

$$
\frac{(n-r-u-2\beta\varepsilon)! (n-r+u+2\beta\varepsilon-1)!}{(u+2\delta\alpha-1)!} \tag{5.2}
$$

for all $z \in \mathcal{H}$, where $[\cdot]$ denotes the floor function.

**Theorem 5.1.** If $f$ and $g$ are modular forms with $f \in M_{2\delta}(\Gamma)$ and $g \in M_{2\varepsilon}(\Gamma)$, then the functions $[f, g]^A_n$ and $[f, g]^B_n$ on $\mathcal{H}$ given by (5.1) and (5.2) are modular forms for $\Gamma$ of weight $2n+2\delta+2\varepsilon$ for each $n \geq 1$.

**Proof.** Given $f \in M_{2\delta}(\Gamma)$ and $g \in M_{2\varepsilon}(\Gamma)$, we first define the sequences $\{f_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ of holomorphic functions on $\mathcal{H}$ by

$$
f_n(z) = \begin{cases} 
  f(z)^\alpha & \text{if } n = \delta \alpha \text{ with } \alpha \geq 1, \\
  0 & \text{otherwise}, \tag{5.3}
\end{cases}
$$

$$
g_m(z) = \begin{cases} 
  g(z)^\beta & \text{if } m = \varepsilon \beta \text{ with } \beta \geq 1, \\
  0 & \text{otherwise}, \tag{5.4}
\end{cases}
$$

for all $z \in \mathcal{H}$ and $n, m \geq 1$, and set

$$
\phi_n(z) = \sum_{r=0}^{n-1} \frac{1}{r! (2n-r-1)!} f^{(r)}_{n-r}(z),
$$

where $f^{(r)}_{n-r}(z)$ denotes the $r$-th derivative of $f_{n-r}(z)$. Then we can express $f_n(z)$ and $g_m(z)$ using $\phi_n(z)$ and $\phi_m(z)$ as follows:

$$
f_n(z) = \sum_{r=0}^{n-1} \frac{1}{r! (2n-r-1)!} f^{(r)}_{n-r}(z), \tag{5.5}
$$

$$
g_m(z) = \sum_{r=0}^{m-1} \frac{1}{r! (2m-r-1)!} g^{(r)}_{m-r}(z), \tag{5.6}
$$

for all $n, m \geq 1$. Using these expressions, we can show that $[f, g]^A_n$ and $[f, g]^B_n$ are modular forms of weight $2n+2\delta+2\varepsilon$ for each $n \geq 1$.
\[
\psi_m(z) = \sum_{s=0}^{m-1} \frac{1}{s!(2m-s-1)!} g_{m-s}^{(s)}(z)
\]
for all \(n, m \geq 1\). Then, using (5.3), we obtain
\[
\phi_n(z) = \sum_{r=0}^{n-1} \frac{1}{(n-r)!(n+r-1)!} f_{r}^{(n-r)}(z)
= \sum_{\alpha=1}^{\lfloor (n-1)/(2\delta) \rfloor} \frac{1}{(n-2\delta \alpha)!(n+2\delta \alpha-1)!} f_{2\delta \alpha}^{(n-2\delta \alpha)}(z).
\]
(5.5)

Similarly, by using (5.4) we have
\[
\psi_m(z) = \sum_{\beta=1}^{\lfloor (m-1)/(2\epsilon) \rfloor} \frac{1}{(m-2\epsilon \beta)!(m+2\epsilon \beta-1)!} g_{2\epsilon \beta}^{(m-2\epsilon \beta)}(z).
\]
(5.6)

By Proposition 2.4 the formal power series
\[
F_0(z, X) = \sum_{n=0}^{\infty} \phi_n(z)X^n, \quad G_0(z, X) = \sum_{m=0}^{\infty} \psi_m(z)X^m
\]
are elements of \(\mathcal{J}_{0,1}(\Gamma)\). Thus, using (5.5), (5.6) and \(\mu = \mu' = 1\), we see that the functions \(f_k\) in (4.2) and \(g_k\) in (4.3) reduce to \([f, g]_n^A\) in (5.1) and \([f, g]_n^B\) in (5.2), respectively; hence the theorem follows. \(\square\)

6. Theta functions

In this section we provide an application of the results in Section 4. We consider a Jacobi-like form whose coefficients are certain theta functions and use this Jacobi-like form to determine modular forms. Such Jacobi-like forms were studied in [5] (see also [7]).

Let \(w\) be a positive integer, and let \(\xi\) be an element of \(\mathbb{C}^{2w}\) considered as a column vector. Let \(A\) be a symmetric positive definite integral \(2w \times 2w\) matrix whose diagonal entries are even. For each nonnegative integer \(k\), we define the theta function \(\theta_{\xi,k}(z) : \mathcal{H} \to \mathbb{C}\) by
\[
\theta_{\xi,k}(z) = \sum_{\eta \in \mathbb{Z}^{2w}} (\xi^t A \eta)^k e^{\pi i (\eta^t A \eta) z} \quad (6.1)
\]
for all \(z \in \mathcal{H}\).

Now we define the formal power series \(\Theta(z, X) = R[[X]]\) associated to the sequence \(\{\theta_{\xi,k}(z)\}_{k=0}^{\infty}\) of theta functions by
\[
\Theta_{\xi}(z, X) = \sum_{n=0}^{\infty} \frac{2^n (2\pi i)^n}{(2n)!} \theta_{\xi,2n}(z) X^n \quad (6.2)
\]
for all \(z \in \mathcal{H}\).

Let \(N\) be the smallest positive integer such that \(NA^{-1}\) is an integral matrix with even diagonal entries, and let \(\Gamma_0(N) \subset SL(2, \mathbb{Z})\) be the associated congruence subgroup given by
\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.
\]
Let $\chi$ be the character on the set of positive integers defined by the Jacobi symbol 
\[
\chi(n) = \left(\frac{-1}{n} \right) \det A
\]
for each positive integer $n$.

**Theorem 6.1.** The formal power series $\Theta_\xi(z, X) \in R[[X]]$ given by (6.2) satisfies the transformation property 
\[
\Theta_\xi(\gamma z, (cz + d)^{-2} X) = \chi(d)(cz + d)^w \exp \left[ c\xi^t A\xi X / (cz + d) \right] \cdot \Theta_\xi(z, X)
\]
for all $z \in \mathcal{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

**Proof.** See [5, Theorem 1]. □

We consider another element $\xi'$ of $\mathbb{C}^{2w}$ and denote by $\theta_{\xi',k}(z)$ with $k \geq 0$ the associated theta function given by (6.1). For each nonnegative integer $k$, we set
\[
p_k(z) = \sum_{r=0}^{k} \frac{(-1)^r \theta_{\xi,2r}(z) \theta_{\xi',2k-2r}(z)}{(\xi^t A\xi)^r (\xi'^t A\xi')^{k-r} (2r)! (2k-2r)!},
\]
\[
q_k(z) = \sum_{r=0}^{k-r} \sum_{\ell=0}^{r} \sum_{s=0}^{r} \frac{(-1)^r (2k + 2w - r - 2)! \theta_{\xi,2\ell}(z) \theta_{\xi',2k-2r-2\ell}(z)}{(\xi + \xi')^r A(\xi + \xi') \phi_{r-s}(z) (r-s)! (2\ell)! (2k-2r-2\ell)!},
\]
for all $z \in \mathcal{H}$.

**Theorem 6.2.** The functions $p_k, q_k : \mathcal{H} \to \mathbb{C}$ given by (6.3) and (6.4) are modular forms for $\Gamma_0(N)$ of weight $2(k + w)$ for all $k \geq 0$.

**Proof.** Let $\Theta_\xi(z, X)$ and $\Theta_{\xi'}(z, X)$ be the power series in (6.2) associated to the elements $\xi$ and $\xi'$ of $\mathbb{C}^{2w}$, and set
\[
\vartheta_1(z, X) = \Theta_\xi(z, -X / (\xi^t A\xi)) \Theta_{\xi'}(z, X / (\xi'^t A\xi')),
\]
\[
\vartheta_2(z, X) = \Theta_{\xi'}(z, X) \Theta_{\xi'}(z, X),
\]
then by Remark 2.3 and Theorem 6.1 we see that $\vartheta_1(z, X)$ and $\vartheta_2(z, X)$ are Jacobi-like forms for $\Gamma_0(N)$ with
\[
\vartheta_1(z, X) \in \mathcal{J} \mathcal{L}_{2w,0}(\Gamma_0(N)), \quad \vartheta_2(z, X) \in \mathcal{J} \mathcal{L}_{2w,(\xi^t A\xi + \xi'^t A\xi')}(\Gamma_0(N)).
\]
Thus the functions $f_k(z)$ and $g_k(z)$ in (4.2) and (4.3) reduce to $p_k(z)$ and $q_k(z)$, respectively, if we set
\[
\delta = \varepsilon = 0, \quad \lambda = \lambda' = w, \quad \mu = \xi^t A\xi, \quad \mu' = \xi'^t A\xi',
\]
\[
\phi_{r}(z) = \frac{2^r (2\pi i)^r}{(2\ell)!} \theta_{\xi',2\ell}(z), \quad \psi_{r}(z) = \frac{2^r (2\pi i)^r}{(2r)!} \theta_{\xi',2r}(z).
\]
Hence the theorem follows by using this and Proposition 4.1. □
References