$K_{a,k}$ minors in graphs of bounded tree-width

Thomas Böhme*
Institut für Mathematik
Technische Universität Ilmenau
Ilmenau, Germany

John Maharry[†‡]
Department of Mathematics and Computing
Franklin College
Franklin, IN, USA

Bojan Mohar[§¶]
Department of Mathematics
University of Ljubljana
Ljubljana, Slovenia

October 17, 2000

Abstract

It is shown that for any positive integers k and w there exists a constant $N = N(k, w)$ such that every 7-connected graph of tree-width less than w and of order at least N contains $K_{3,k}$ as a minor. Similar result is proved for $K_{a,k}$ minors where a is an arbitrary fixed integer and the required connectivity depends only on a. These are the first results of this type where fixed connectivity forces arbitrarily large (nontrivial) minors.

1 Introduction

In this paper, all graphs are finite and may have loops and multiple edges. A graph H is a *minor* of a graph G, $H \leq_{\mathrm{m}} G$, if H can be obtained from a

* E-mail address: tboehme@theoinf.tu-ilmenau.de
[†] Supported in part part by the International Research Project SLO-US-007 on Graph Minors.
[‡] E-mail address: maharrj@franklincollege.edu
[§] Supported in part by the Ministry of Science and Technology of Slovenia, Research Project J1–0502–0101–98, and the International Research Project SLO-US-007 on Graph Minors.
[¶] E-mail address: bojan.mohar@uni-lj.si

subgraph of G by contracting connected subgraphs. There are many results concerning the structure of graphs that do not contain a certain graph as a minor. These excluded graphs include K_5 and $K_{3,3}$ [13], V_8 [8], the 3-cube [6] and the octahedron [7]. See also [2] and [12]. There are well-known structures which guarantee a certain minor exists for large graphs. For instance, any 5-connected graph on at least 11 vertices contains the 3-cube as a minor [6]. Any 5-connected non-planar graph on at least 8 vertices contains a V_8 minor [8]. In addition, there are Ramsey-type results similar to the fact that any sufficiently large connected graph contains either a k-path or a k-star. Oporowski, Oxley and Thomas [11] proved that any large 4-connected graph must have a large minor from a set of four families of graphs. Ding [3] has characterized large graphs that do not contain a $K_{2,k}$ minor. A corollary of his result is that any large 5-connected graph contains a $K_{2,k}$ minor.

Our results are a cross section of all of these types of results:

Theorem 1.1 *For any positive integers k and w there exists a constant $N = N(k, w)$ such that every 7-connected graph of tree-width less than w and of order at least N contains $K_{3,k}$ as a minor.*

Theorem 1.2 *There is a function $c : \mathbb{N} \to \mathbb{N}$ such that for any $a \geq 3$ the following holds. For any positive integers k and w there exists a constant $N = N(k, w)$ such that every $c(a)$-connected graph of tree-width less than w and of order at least N contains $K_{a,k}$ as a minor.*

Theorem 1.1 is sharp in the sense that the 7-connectivity condition cannot be relaxed. Moreover, the function $c(a)$ in Theorem 1.2 must be at least $2a + 1$. These facts follow from the following construction of a family of arbitrarily large $2a$-connected graphs (of tree-width $3a - 1$) none of which contain a $K_{a,2a+1}$-minor.

Let m and a be integers greater than 3. Define the graph $N_{m,a}$ as follows. Let the vertices be indexed $v_{x,y}$ where $1 \leq x \leq m$ and $1 \leq y \leq a$. The vertex $v_{x,y}$ is adjacent to another vertex $v_{w,z}$ if and only if $w \in \{x - 1, x, x + 1\}$ where $x \pm 1$ is considered modulo m.

Proposition 1.3 *For any integers $a \geq 3$ and $m \geq 3$, $K_{a,2a+1} \not\leq_{\mathrm{m}} N_{m,a}$.*

Proof. Suppose the theorem is false for some $a \geq 3$. Let m be the least integer such that $N_{m,a} \geq_{\mathrm{m}} K_{a,2a+1}$. Let the *clasps* of $N_{m,a}$ be defined as $CL_i = \{v_{i,y} \mid y = 1, 2, \ldots, a\}$ for $i = 1, 2, \ldots, m$.

As $N_{m,a} \geq_{\mathrm{m}} K_{a,2a+1}$, there is a set of $2a+1$ connected subgraphs, $\mathcal{S} = \{S_1, S_2, \ldots, S_{2a+1}\}$, and a set of a connected subgraphs of $N_{m,a}$, $\mathcal{T} = \{T_1, T_2, \ldots, T_a\}$, such that for every i, j there is an edge from some vertex in T_i to some vertex in S_j and such that all these subgraphs are pairwise disjoint. Assume that the S_i and T_i are chosen with $l := \sum_{i=1}^{2a+1} |V(S_i)| + \sum_{i=1}^{a} |V(T_i)|$. Then each of the subgraphs in $\mathcal{S} \cup \mathcal{T}$ is a path meeting each clasp in at most one vertex. Let $\mathcal{S}_1$ be the set of single vertex subgraphs contained in $\mathcal{S}$. It is easy to see that $\mathcal{T}$ cannot contain any single vertex subgraphs.

Claim 1: *For every $1 \leq i \leq m$, there is a subgraph $S_j \in \mathcal{S}_1$ such that $S_j \subseteq CL_i$.*

Suppose CL_i does not contain any of the subgraphs in $\mathcal{S}_1$. Then contracting a matching of isze a between CL_i and $CL_{i-1} \cup CL_{i+1}$ (indices taken modulo m) using as many edges of $\mathcal{S} \cup \mathcal{T}$ as possible gives a subgraph of $N_{m-1,a}$ that still contains $K_{a,2a+1}$ as a minor. This contradiction to the minimality of m proves the claim.

Claim 2: *If there is a subgraph in $\mathcal{S}$ that contains at least two vertices, then there is a clasp that contains no member of $\mathcal{S}_1$.*

Suppose S_1 (say) intersects CL_1 and CL_2. By the minimality of l, we may assume that $S_1 \cap CL_m = \emptyset$. Moreover, there is a subgraph T_j that does not intersect $CL_1 \cup CL_2 \cup CL_3$. Otherwise, the intersection of S_1 with CL_1 could be removed from S_1. Therefore, a single vertex subgraph $S_i \in \mathcal{S}_1$ contained in CL_2 would not be adjacent to T_j. Hence, the clasp CL_2 is as stated in the claim.

Claims 1 and 2 imply that all subgraphs in $\mathcal{S}$ are single vertices. To complete the proof, notice that if every clasp of $N_{m,a}$ contains one of the single vertex subgraphs of $\mathcal{S}_1$, then each T_j must must contain at least $m-2$ vertices in order to be adjacent to all of the subgraphs in $\mathcal{S}$. Hence $|V(\mathcal{S})| + |V(\mathcal{T})| \geq |\mathcal{S}| + (m-2)|\mathcal{T}| \geq 2a+1+(m-2)a > ma = |V(N_{m,a})|$. This contradiction completes the proof. □

In our proof of Theorem 1.2, $c(3) = 7$ and $c(a) = 264a + 1$ for $a \geq 4$, and we have no intention to find the best possible value for $c(a)$. However, the previous example shows that $c(a)$ must be at least $2a+1$ for $a \geq 3$. It is worth remarking that our proof of Theorem 1.2 works also for $c(a) = 3a - 1$ if we assume that the minimum degree is at least $264a + 1$.

2 Bounded tree-width structure

A *tree decomposition* of a graph G is a pair (T, Y), where T is a tree and Y is a family $\{Y_t \mid t \in V(T)\}$ of vertex sets $Y_t \subseteq V(G)$, such that the following two properties hold:

(W1) $\bigcup_{t\in V(T)} Y_t = V(G)$, and every edge of G has both ends in some Y_t.

(W2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The *width* of a tree decomposition (T, Y) is $\max_{t\in V(T)}(|Y_t| - 1)$. It was shown in [11] that if a graph G has a tree decomposition of width at most w then G has a tree decomposition of width at most w that further satisfies:

(W3) For every two vertices t, t' of T and every positive integer k, either there are k disjoint paths in G between Y_t and $Y_{t'}$, or there is a vertex t'' of T on the path between t and t' such that $|Y_{t''}| < k$.

(W4) If t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$.

(W5) If $t_0 \in V(T)$ and B is a component of $T - t_0$, then $\bigcup_{t\in V(B)} Y_t \setminus Y_{t_0} \neq \emptyset$.

In the rest of the paper we give the proof of Theorems 1.1 and 1.2. We let $a \geq 3$, k, and w be given positive integers. Let G be an $c(a)$-connected graph with a tree decomposition (T, Y) of width at most w that satisfies (W1)–(W5).

We will develop a structure that is similar to that used in [11]. First, we define the constants that will be used in the proofs.

$$
\begin{aligned}
n_5 &= r^{n_4}, \quad \text{where } r = (k-1)\binom{w+1}{a} \\
n_4 &= n_3^{w+1} \\
n_3 &= (2n_2)^p, \quad \text{where } p = 2^{w+1} \\
n_2 &= n_1^q, \quad \text{where } q = 2^{\binom{w+1}{2}} \\
n_1 &= \begin{cases} 2k(2w+3)^2 & \text{if } a = 3 \\ 2k(c(a)+2a+2) - 4a - 2 & \text{if } a \geq 4 \end{cases}
\end{aligned}
$$

We assume that $|V(G)| = N \geq (w+1)n_5$ and that G has no $K_{a,k}$-minor. By (W1) we have

Claim 2.1 $|V(T)| \geq n_5$.

Claim 2.2 *Every vertex of T has degree at most $r = (k-1)\binom{w+1}{a}$.*

Proof. Suppose $t_0 \in V(T)$ has degree at least $r+1$. Let $\mathcal{C}$ be the set of components of $G - Y_{t_0}$. By (W2) and (W5), it is clear that $|\mathcal{C}| \geq r+1$. For $C \in \mathcal{C}$, let $X(C)$ be the set of vertices of Y_{t_0} adjacent to some vertex of C. Clearly, $|X(C)| \geq a$ for every $C \in \mathcal{C}$ since G is $c(a)$-connected and $c(a) \geq a$. By the Pigeonhole Principle, there is a set $\mathcal{C}' \subseteq \mathcal{C}$ of k components for which $\bigcap_{C \in \mathcal{C}'} X(C)$ contains a (or more) vertices of Y_{t_0}. By contracting B to a vertex for each $B \in \mathcal{C}'$, we see that G contains a $K_{a,k}$ minor, a contradiction. □

From this it follows that

Claim 2.3 *T contains a path R of length $|E(R)| \geq n_4$.*

The proof of the following claim can be found in [11].

Claim 2.4 *There is a subsequence of length n_3 of the vertices of $V(R)$, $r_1, r_2, \ldots, r_{n_3}$, such that for some $s \geq 1$, $|Y_{r_i}| = s$ for $i = 1, 2, \ldots, n_3$ and for every vertex of R between r_1 and r_{n_3}, $|Y_{r_i}| \geq s$.*

From now on we replace R by the subpath from r_1 to r_{n_3}. Note that because of the $c(a)$-connectivity and (W5), $c(a) \leq s \leq w+1$.

By (W3) and Claim 2.4, there are s disjoint paths in G from Y_{r_1} to $Y_{r_{n_3}}$. Fix these paths, denote them by $P_1, P_2, \ldots, P_s$, and put $Z = P_1 \cup \cdots \cup P_s$. Since G is 3-connected, these paths can be chosen such that every Z-bridge in G is attached to at least two of the paths (cf., e.g., [4]), which we assume henceforth.

Notice that for any $t, t' \in \{r_1, \ldots, r_{n_3}\}$ and for every $j \in \{1, \ldots, s\}$ there is a unique subpath of P_j with one end in Y_t and the other end in $Y_{t'}$. Denote this subpath by $P_j(t, t')$.

The path P_j is said to be *trivial* if it consists of a single vertex, and it is said to be *everywhere nontrivial* (*almost nontrivial*) w.r.t. the sequence $r_1, \ldots, r_{n_3}$ if $P_j(r_i, r_{i+1})$ contains at least three (respectively, at least two) vertices for each $i = 1, \ldots, n_3 - 1$.

Claim 2.5 *There is a subsequence $q_1, q_2, \ldots, q_{n_2}$ of $r_1, \ldots, r_{n_3}$ of length n_2 such that for each $j = 1, \ldots, s$, $P_j(q_1, q_{n_2})$ is either trivial or everywhere nontrivial (w.r.t. the subsequence).*

Proof. Clearly, there is a subsequence of $r_1, \ldots, r_{n_3}$ of length $\sqrt{n_3}$ such that the corresponding segment of P_1 is either trivial or everywhere almost nontrivial with respect to the subsequence. By repeating this argument on the subsequence for $P_2, \ldots, P_s$, respectively, we end up with a sequence of length at least $2n_2$ such that every path is either trivial or everywhere almost nontrivial. By taking every second element of this sequence, the required subsequence $q_1, q_2, \ldots, q_{n_2}$ is obtained. $\square$

The paths P_j and P_l are said to be *everywhere bridge connected* (resp. *everywhere bridge disconnected*) with respect to a sequence $p_1, \ldots, p_n$ of vertices of R if for every $i = 1, \ldots, n-1$, there exists (resp. does not exist) a Z-bridge which has a vertex of attachment in $P_j(p_i, p_{i+1})$ and a vertex of attachment in $P_l(p_i, p_{i+1})$.

Claim 2.6 *There is a subsequence $p_1, p_2, \ldots, p_{n_1}$ of $q_1, \ldots, q_{n_2}$ of length n_1 such that for every distinct pair of indices $j, l \in \{1, \ldots, s\}$, $P_j(p_1, p_{n_1})$ and $P_l(p_1, p_{n_1})$ are either everywhere bridge connected or everywhere bridge disconnected (w.r.t. the new subsequence).*

Proof. The proof is similar to the proof of Claim 2.5 except that we have to repeat the subsequence argument $\binom{s}{2} \leq \binom{w+1}{2}$ times. $\square$

3 The auxiliary graph A

Our next goal is to examine the structure of the *auxiliary graph A* which contains information about which pairs of the paths are everywhere bridge connected. The graph A has vertex set $V(A) = \{P_1, \ldots, P_s\}$, and the paths P_j and P_l are adjacent vertices in A if they are everywhere bridge connected w.r.t. $p_1, \ldots, p_{n_1}$ (cf. Claim 2.6).

Claim 3.1 *Suppose that $U \subseteq V(A)$ contains only everywhere nontrivial paths. If the subgraph of A induced by U is connected, then $V(A) \setminus U$ contains at most $a-1$ vertices that are adjacent to U in A.*

Proof. Suppose that $P_1, \ldots, P_a$ are vertices in $V(A) \setminus U$ adjacent to U in A. Contract each path P_j ($j = 1, \ldots, a$) in G to a single vertex w_j. Next, for $i = 1, 3, 5, \ldots, 2k-1$, contract all segments $P_j(p_i, p_{i+1})$, where $P_j \in U$, and also contract all edges in bridges connecting these segments in G, to get k vertices $z_1, z_3, \ldots, z_{2k-1}$ in a minor of G. Clearly, $n_1 \geq 2k$, so

$z_1, z_3, \dots, z_{2k-1}$ exist. Since U is adjacent to $P_1, \dots, P_a$ in A, it is easy to see that vertices $w_1, \dots, w_a$ and $z_1, z_3, \dots, z_{2k-1}$ give rise to a $K_{a,k}$ minor of G.

□

We shall apply Claim 3.1 together with the help of the following lemma.

Lemma 3.2 *Let H be a connected graph. If H has at least $2a^2$ vertices of degree ≥ 3, then H contains a tree T with $\geq a$ vertices of degree 1.*

Proof. Let d be the maximum vertex degree in H, and let v_0 be a vertex of degree d. If $d \geq a$, then T is the star centered at v_0. So, suppose that $d < a$. Then it is sufficient to prove the following. Assuming that H has at least $2a^2 - (d-1)^2$ vertices of degree ≥ 3, we shall prove by induction on $a - d$ that the tree T exists. Let N_1 be the set of all vertices of degree ≥ 3 which can be reached from v_0 on paths whose internal vertices all have degree 2. Then $1 \leq |N_1| \leq d$. Let N_2 be the "second neighborhood" of v_0, consisting of vertices of degree ≥ 3 which are not in $N_1 \cup \{v_0\}$ and which can be reached from v_0 on paths for which exactly one internal vertex has degree ≥ 3. Similarly, let N_3 be the "third neighborhood" of v_0. Then $1 \leq |N_2| \leq d(d-1)$ and $|N_3| \geq 1$ since H is connected and $2a^2 - (d-1)^2 > 1 + d + d(d-1) \geq 1 + |N_1| + |N_2|$. Let $v_3 \in N_3$, and let W be a path from v_0 to v_3 which contains precisely two other vertices of degree ≥ 3. Now, contract W to a vertex $\tilde{v}_0$ and remove possible parallel edges. Denote the resulting graph by $\tilde{H}$. If a vertex of $\tilde{H}$ has degree smaller than in H, then it was adjacent to two (or three) vertices of W. This implies that $\tilde{H}$ has at least $2a^2 - (d-1)^2 - (2d-1) = 2a^2 - ((d+1)-1)^2$ vertices of degree ≥ 3. Since v_0 and v_3 have no common neighbors, $\tilde{v}_0$ is its vertex of maximum degree $\geq d+1$. By the induction hypothesis, $\tilde{H}$ contains a tree $\tilde{T}$ with at least a vertices of degree 1. Clearly, $\tilde{T}$ gives rise to the required tree T in H.

□

At least one of the paths is everywhere nontrivial, say P_1. Let A_1 be the induced subgraph of A on the everywhere nontrivial paths. Let A_0 be the induced subgraph of A consisting of the connected component of A_1 containing P_1 together with (at most $a-1$) trivial paths adjacent to that component.

From now on we shall assume that G is $c(a)$-connected, where $c(3) = 7$ and $c(a) = 264a + 1$ for $a \geq 4$.

Claim 3.3 *$A_0 \cap A_1$ has at least $\lceil \frac{c(a)-a+1}{2} \rceil$ vertices. If $a = 3$, A_0 is isomorphic to a path or a cycle on at least four vertices. If $a \geq 4$, then every* *vertex of $A_0 \cap A_1$ has degree at most $a-1$ and at most $2a^2$ of these vertices have degree more than 2 in $A_0 \cap A_1$.*

Proof. Let $U = V(A_0 \cap A_1)$, $x = |U|$, and $y = |V(A_0)| - x$. By Claim 3.1 we see that $y \le a-1$. Since the $2x+y$ endvertices of the paths in A_0 in Y_{p_1} and Y_{p_3} separate the graph G, we have $2x+y \ge c(a)$. This implies that $x \ge (c(a)-a+1)/2$, and proves the first part of the claim.

By Claim 3.1, every vertex in $A_0 \cap A_1$ has degree at most $a-1$ in A. If $a = 3$, this implies that $A_0 \cap A_1$ is a path or a cycle, and the trivial paths in $V(A_0)$ can be adjacent only to vertices of degree ≤ 1 in $A_0 \cap A_1$. This implies that A_0 is a path or a cycle. If $|V(A_0)| \le 3$, then the endpoints of the paths in $V(A_0)$ would give a ≤ 6-separator in G.

Suppose now that $a \ge 4$. By Claim 3.1 every vertex of $A_0 \cap A_1$ has degree at most $a-1$. Suppose that there are more than $2a^2$ vertices of degree ≥ 3. By Lemma 3.2, $A_0 \cap A_1$ contains a tree T with $\ge a$ vertices of degree 1. Let U be the set of vertices of degree ≥ 2 in T. The subgraph of A induced by U is connected, and Claim 3.1 yields a contradiction. This completes the proof. □

Denote by $Z'(i)$ the union of $P_j(p_i, p_{i+1})$ where $P_j \in V(A_0)$, $i = 1, 2, \dots, n_1 - 1$. Let Z_i be the subgraph of G obtained by taking the union of $Z'(i)$ and all those Z-bridges B that have all vertices of attachment in $Z'(i)$ such that there is no $i' < i$ for which B would have all its vertices of attachment in $Z'(i')$.

4 Finding $K_{3,k}$ minors

In this section we consider the case when $a = 3$ since the best possible connectivity 7 requires more elaborate techniques than the general case treated in the next section. For $i = 1, 2, \dots, n_1 - 2w - 2$, let $H_i = \bigcup_{k=0}^{2w} Z_{i+k}$. Let $R, R' \in V(A_0)$ be paths which are adjacent in A_0. For $i = 1, 2, \dots, n_1 - 2w - 2$ define the graph $D_i = D_i(R, R')$ as follows. First, take $S = (R \cup R') \cap H_i$ together with all Z-bridges in H_i that have vertices of attachment on R and on R'. Finally, add two edges e_1, e_2, where e_1 joins the "left" endvertices, λ in $R \cap H_i$ and λ' in $R' \cap H_i$, and e_2 joins the "right" endvertices, ρ and ρ', of these two paths. Then $S + e_1 + e_2 =: C$ is a cycle in D_i. If R (R') is everywhere trivial, then $\lambda = \rho$ ($\lambda' = \rho'$).

Claim 4.1 *Suppose that $a = 3$. Then for every i, there are adjacent vertices R, R' of A_0 such that $D_i(R, R')$ has no embedding in the plane where the vertices $\lambda, \lambda', \rho', \rho$ would lie on the outer face in the prescribed order.*

Proof. Suppose that H_i is a planar graph. Let v_j be the number of vertices of degree j in H_i. By Euler's formula and standard counting arguments it follows that

$$L := \sum_{j \geq 0} (6-j)v_j \geq 12. \tag{1}$$

Observe that H_i has at most $2s$ vertices of degree ≤ 6 since the minimum degree in G is at least 7 (by the 7-connectivity of G). On the other hand, since at least three of the paths in H_i are nontrivial, these paths contain at least $3(2(2w+1)-1) = 12w+3$ vertices of degree ≥ 7 in H_i. Therefore,

$$L \leq 6 \cdot 2s - (12w+3) \leq 12(w+1) - 12w - 3 = 9.$$

This contradiction to (1) shows that H_i is not planar. Recall that A_0 is a path or a cycle on at least 4 vertices, $R_1, \ldots, R_d$, $d \geq 4$. This implies, in particular, that no Z-bridge in H_i is attached to more than two of the paths (otherwise, there would be a 3-cycle in A_0, and so A_0 would be equal to the 3-cycle). Moreover, if every $D_i(R_j, R_{j+1})$ ($j = 1, \ldots, d$, indices taken modulo d) has an embedding in the plane with the corresponding cycle C_j being the outer cycle, then $\bigcup_{j=1}^{d} D_i(R_j, R_{j+1}) \supseteq H_i$ would be planar as well, contrary to the above. Hence, there is an index j such that $D_i(R_j, R_{j+1})$ has no such embedding. Since there are no local Z-bridges, $D_i(R_j, R_{j+1})$ neither has an embedding in the plane where the vertices $\lambda, \lambda', \rho', \rho$ are on the outer face in the prescribed order. $\square$

We shall need a result about crossing paths from from [9]. A *separation* of a graph G is a pair (A, B) of subraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and its *order* is $|V(A \cap B)|$. By a *society* we mean a pair (G, Ω), where G is a graph and Ω a cyclic permutation of a subset $\overline{\Omega}$ of $V(G)$. A *cross* in (G, Ω) is a pair of disjoint paths in G with ends s_1, t_1 and s_2, t_2, respectively, all in $\overline{\Omega}$, such that s_1, s_2, t_1, t_2 occur in Ω in that order (but not necessarily consecutive). The following formulation of a theorem of Robertson and Seymour [9] appears in [10].

Theorem 4.2 (Robertson and Seymour) *Let (G, Ω) be a society such that there is no separation (A, B) of G of order ≤ 3 with $\overline{\Omega} \subseteq V(A) \neq V(G)$. Then the following are equivalent:*

(a) *There is no cross in (G, Ω).*

(b) *G can be drawn in a disc with the vertices in $\overline{\Omega}$ drawn on the boundary of the disc in order given by Ω.*

Claim 4.3 *If $D_i(R, R')$ is nonplanar, then one of the following holds:*

(a) *$D_i(R, R')$ contains disjoint paths Q_1, Q_2 connecting λ with ρ' and λ' with ρ, respectively.*

(b) *$D_i(R, R')$ contains a path Q (resp., Q') disjoint from R' (resp., R) which connects λ and ρ (resp., λ' and ρ') such that after replacing R (resp., R') by Q (resp., Q'), there is a Z-bridge in H_i which is attached to more than two of the paths $P_1, \dots, P_s$.*

Proof. Let $H = D_i(R, R')$. Let C be the cycle of H defined before Claim 4.1. Let $\overline{\Omega}$ be the set of vertices of C which are incident with an edge in $E(G) \setminus E(H)$. The cyclic order of $\overline{\Omega}$ on C defines the society (H, Ω). Since G is 4-connected and no vertex in $V(H) \setminus \overline{\Omega}$ is incident with an edge in $E(G) \setminus E(H)$, there is no separation (A, B) of H of order ≤ 3 with $\overline{\Omega} \subseteq V(A) \neq V(G)$. Since H is nonplanar, Theorem 4.2 implies that there is a cross R_1, R_2 in (H, Ω). Let α_i, β_i be the endvertices of R_i $(i = 1, 2)$. We may assume that:

(i) None of the vertices $\lambda, \lambda', \rho, \rho'$ is an internal vertex of R_1 or R_2.

Subject to (i) choose the cross R_1, R_2 such that

(ii) $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ contains as many vertices in $\{\lambda, \lambda', \rho, \rho'\}$ as possible and, subject to (i) and (ii)

(iii) as few edges in $E(H) \setminus E(R \cup R')$ as possible.

If $\lambda, \lambda', \rho, \rho'$ are all endvertices of R_1, R_2, then we have (a). Hence we may assume that λ is not an endvertex of R_1, R_2. If $R \cap (R_1 \cup R_2) \neq \emptyset$, let v be the first vertex of $R_1 \cup R_2$ on R (starting at λ towards ρ). We may assume that $v \in V(R_1)$. Let $R_1 = R_1' \cup R_1''$ where $V(R_1') \cap V(R_1'') = \{v\}$. By replacing one of the segments R_1' or R_1'' in R_1 by a segment from v to λ on R, a new cross is obtained which contradicts (ii) or (iii), except when R_1' or R_1'' is the segment of R from v to ρ. In particular, three of the endvertices of R_1, R_2 are on R'. The above proof implies that λ' and ρ' are the endvertices of the paths. Since R_1, R_2 cross, R_1 joins a vertex $x \in V(R') \setminus \{\lambda', \rho'\}$ with ρ, and R_2 joins λ' and ρ', where R_2 is disjoint from R. It is easy to see, that this gives (b).

Suppose now that $R \cap (R_1 \cup R_2) = \emptyset$. Condition (ii) implies that λ' and ρ' are the endvertices of R_1 and R_2, respectively. There is a C-bridge B in H such that $E(R_1 \cup R_2) \cap E(B) \neq \emptyset$. Since B is not a local bridge, it is attached to R as well. Therefore, there is a path L in B from R to $R_1 \cup R_2$

(say to R_2) which is internally disjoint from $C \cup R_1 \cup R_2$. Let y be the vertex of R_1 which is as close as possible to ρ' on R'. Let R_2' be the segment of R_2 from $R_2 \cap L$ to the end of R_2 distinct from ρ'. By (iii), R_2' is disjoint from the segment Q'' of R' from y to ρ'. Therefore, the path Q' composed of the segment of R_1 from λ' to y and Q'' can be taken as the path Q' in (b). Note that, after replacing R' by Q', the Z-bridge containing $L \cup R_2'$ will be attached to at least three paths in $\{P_1, \ldots, P_s\}$. $\square$

We are ready to complete the proof of Theorem 1.1. Suppose that $a = 3$ and that A_0 is a path or a cycle on consecutive vertices $R_1, \ldots, R_d$, where $4 \leq d \leq w+1$. Let $D_i^j = D_i(R_j, R_{j+1})$, $j = 1, \ldots, d$. We shall only consider the indices i of the form $i = 1 + t(2w+2)$, $t = 0, 1, \ldots$, and we call them *admissible indices*.

Let us first assume that the case (b) of Claim 4.3 occurs less than $2kd$ times at admissible indices i. Since there are at least $4kd$ admissible indices, Claim 4.3(a) implies that there is an index $j \in \{1, \ldots, d\}$, and there are admissible indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n_1 - 2w - 2$ such that

(i) each of $D_{i_1}^j, D_{i_2}^j, \ldots, D_{i_k}^j$ contains paths as stated in Claim 4.3(a), and

(ii) for $l = 1, \ldots, k-1$, $i_{l+1} - i_l \geq 2w + 2$.

We can exchange the segments of the paths R_j and R_{j+1} in H_{i_l} by the two paths Q_1, Q_2 of Claim 4.3(a). In this way the new paths in $H_{i_l} \cup Z_{i_l+2w+2}$ would no longer satisfy the condition of Claim 3.1. Namely, if R_j and R_{j+1} have degrees d_1, d_2 in A_0, then they would be everywhere bridge connected (w.r.t. the sequence $p_{i_1-1}, p_{i_2-1}, \ldots, p_{i_k-1}$) with $d_1 + d_2 - 1$ other paths. If $d_1 = d_2 = 2$, this gives a $K_{3,k}$ minor in the same way as in the proof of Claim 3.1 (since one of R_j or R_{j+1} is everywhere nontrivial). If $d_1 = 1$ (say), then the path R_{j+2} has degree 2 in A_0 by Claim 3.3 and (in addition to R_{j+3}) it becomes everywhere bridge connected to the two new paths (w.r.t. the sequence $p_{i_1-1}, p_{i_2-1}, \ldots, p_{i_k-1}$). It is easy to see from the definition of A_0 that R_{j+2} cannot be trivial, so the proof of Claim 3.1 applies again.

Let us now assume that the case (b) of Claim 4.3 occurs $2kd$ or more times (for admissible indices i). Then there is an index $j \in \{1, \ldots, d\}$, and there are admissible indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n_1 - 2w - 2$ such that

(i) each of $D_{i_1}^j, D_{i_2}^j, \ldots, D_{i_k}^j$ contains a path Q (or each of $D_{i_1}^j, D_{i_2}^j, \ldots, D_{i_k}^j$ contains a path Q') as stated in Claim 4.3(b), and

(ii) for $l = 1, \ldots, k-1$, $i_{l+1} - i_l \geq 2w + 2$.

For any $D^j_{i_l}$ we replace the segment of R_j (resp., R_{j+1}) by the corresponding path Q (resp., Q') such that there is a Z-bridge (where Z is defined as the union of the new paths) attached to R_j, R_{j+1}, and R_{j+2} (or R_{j-1}). We may assume that k of these bridges, $B_1, \dots, B_k$ are attached to R_j, R_{j+1}, and R_{j+2}. Now, there is a $K_{3,k}$-minor obtained by contracting R_j, R_{j+1}, R_{j+2} into single vertices and adding paths in $B_1, \dots, B_k$ to these vertices. This completes the proof of Theorem 1.1.

5 Finding $K_{a,k}$ minors for $a \geq 4$

Suppose now that $a \geq 4$ and $c(a) = 264a + 1$. Let $r = 2c(a) + 2$. For $i = 1, 2, \dots, n_1 - r$, let $H_i = \bigcup_{j=0}^{r-1} Z_{i+j}$. We also write $S_i = Y_{p_i}$.

Claim 5.1 *For every $1 \leq i \leq n_1 - r$, the average degree of vertices in H_i is at least $c(a) - \frac{1}{2}$.*

Proof. Every vertex of G has degree at least $c(a)$. Let $s_0 = |V(A_0 \cap A_1)|$ be the number of everywhere nontrivial paths in $V(A_0)$. Then

$$|V(H_i)| \geq s_0(2r + 1) > 4s_0c(a). \tag{2}$$

Each trivial path in $V(A_0)$ is everywhere bridge connected to some nontrivial path. Hence, the degree of the corresponding vertex in H_i is at least $r/2 \geq c(a)$. Only the ends of nontrivial paths can have degree less than $c(a)$ in H_i. This fact and inequality (2) imply that

$$2|E(H_i)| \geq c(a)(|V(H_i)| - 2s_0) \geq (c(a) - \frac{1}{2})|V(H_i)|.$$

This completes the proof. $\square$

A graph L is said to be *q-linked* if it has at least $2q$ vertices and for any ordered q-tuples $(s_1, \dots, s_q)$ and $(t_1, \dots, t_q)$ of $2q$ distinct vertices of L, there exist pairwise disjoint paths $P_1, \dots, P_q$ such that for $i = 1, \dots, q$, the path P_i connects s_i and t_i. Such collection of paths is called a *linkage* of $(s_1, \dots, s_q)$ and $(t_1, \dots, t_q)$.

Claim 5.2 *For every $1 \leq i \leq n_1 - r$, there exists a subgraph L_i of H_i which is $3a$-linked.*

Proof. Mader [5] proved that every graph of average degree at least $4c$ contains a c-connected subgraph. Therefore, since H_i has average degree at least $c(a) - 1 \geq 264a$, H_i contains a $66a$-connected subgraph L_i. Bollobás and Thomason [1] have shown that every $22t$-connected graph is t-linked. Hence, the graph L_i is $3a$-linked. □

We will now construct a disjoint paths $\mathcal{P}_1^\circ, \ldots, \mathcal{P}_a^\circ$ by routing the paths $P_1, \ldots, P_s$ through L_i in at least k pairwise disjoint subgraphs H_i. In each graph L_i, there will also be an extra vertex linked to each of the a paths. Contracting these paths will then give a $K_{a,k}$-minor in G.

Claim 5.3 *In H_i, there exist $2a$ pairwise disjoint paths, $Q_1^{(i)}, \ldots, Q_a^{(i)}$ and ${Q'}_1^{(i)}, \ldots, {Q'}_a^{(i)}$ such that the following hold:*

(a) *For $l = 1, 2, \ldots, a$, the path $Q_l^{(i)}$ starts in L_i and ends in S_{i+r}.*

(b) *For $l = 1, 2, \ldots, a$, the path ${Q'}_l^{(i)}$ starts in S_i and ends in L_i.*

(c) *Every path $Q_l^{(i)}$ and ${Q'}_l^{(i)}$ $(l = 1, 2, \ldots, a)$ has only its endvertices in $S_i \cup S_{i+r} \cup V(L_i)$.*

Proof. Let $\Pi_0 = V(A_0) \setminus V(A_1)$ be the set of vertices of H_i corresponding to the trivial paths in A_0. Let $\mathcal{W} = \{W_1, \ldots, W_{2a}\}$ be a set of $2a$ pairwise disjoint paths joining $V(L_i)$ with $S_i \cup S_{i+r}$ such that:

(1) $W_l \subseteq H_i - \Pi_0$ for every $l = 1, 2, \ldots, 2a$.

(2) The number of edges in $\bigcup_{l=1}^{2a} E(W_l) \setminus \bigcup_{j=0}^{r-1} E(Z'(i+j))$ is minimum.

(3) Subject to (2), if n_L is the number of paths W_l ending in S_i, and n_R is the number of paths W_l ending in S_{i+r}, $|n_L - n_R|$ is minimum.

Disjoint paths satisfying (1) exist by large connectivity: Since $c(a) \geq 3a-1$, and $|V(L_i)| > 3a$, and $|S_i \cup S_{i+r}| \geq 3a-1$, there exist $3a-1$ disjoint paths from $V(L_i)$ to $S_i \cup S_{i+r+1}$ by Menger's theorem. Since there are at most $a-1$ vertices in Π_0, the removal of those paths which intersect Π_0 leaves at least $2a$ paths satisfying condition (1).

If at least two paths of $\mathcal{W}$ intersect a path P_j, then let W and W' be the paths that intersect P_j as close as possible (on P_j) to S_i and S_{i+r}, respectively. If $W = W'$, suppose that the intersection u of W with P_j nearest S_i (say) comes before the intersection nearest S_{i+r}. By (2), W ends at S_i, i.e., its segment from u to its end coincides with the segment $P_j(u, S_i)$

of P_j. This shows that $W \neq W'$. Then the path W (resp. W') must end at S_i (resp. S_{i+r}) by (2).

Suppose that precisely one path, say $W \in \mathcal{W}$, intersects a path P_j. In this case we can elect to have W ending at $P_j \cap S_i$ or at $P_j \cap S_{i+r}$ by following the path P_j. This implies that the value $|n_L - n_R|$ in (3) can be made to be zero. Then $n_L = n_R = a$.

Now let the a paths in $\mathcal{W}$ that end in S_i be called ${Q'_1}^{(i)}, {Q'_2}^{(i)}, \ldots, {Q'_a}^{(i)}$ and the a paths in $\mathcal{W}$ that end in S_{i+r} be called $Q_1^{(i)}, Q_2^{(i)}, \ldots, Q_a^{(i)}$. It is easy to see that (c) may be requested. This completes the proof. $\square$

Let T be a spanning tree of $A_0 \cap A_1$. By Claim 3.3, $|V(T)| \geq a$. This implies the following claim.

Claim 5.4 *There are vertices $t_1, t_2, \ldots, t_a$ of T such that for $l = 1, 2, \ldots, a$, the vertex t_l is a leaf of the subtree $T \setminus \{t_1, \ldots, t_{l-1}\}$.*

For each $i = 1, 2, \ldots, n_1 - r$ and each $l = 1, 2, \ldots, a$, let $J_l^{(i)} \in \{P_1, \ldots, P_s\}$ be the vertex of T such that $Q_l^{(i)}$ ends up on the corresponding path in G. Choose an enumeration of $Q_1^{(i)}, Q_2^{(i)}, \ldots, Q_a^{(i)}$ such that, for $l = 1, 2, \ldots, a$, the distance from $J_l^{(i)}$ to t_l in T is minimum (where smaller values of l have preference over the larger values).

Choose a similar enumeration of ${Q'_1}^{(i)}, \ldots, {Q'_a}^{(i)}$.

Define $\alpha = r + 4a + 2$ and for $t = 1, \ldots, k$ set $i_t = 1 + (t-1)\alpha$. Observe that $i_k = n_1 - r$.

To construct the path $\mathcal{P}_l^\circ$, we first link $Q_l^{(i_t)}$ to ${Q'_l}^{(i_{t+1})}$ for every $t = 1, \ldots, k-1$. Then each ${Q'_l}^{(i_t)}$ is linked to $Q_l^{(i_t)}$ inside L_{i_t} $(t = 1, \ldots, k)$. We do this as described below.

Let $i' = i + \alpha$. Link $Q_l^{(i)}$ with ${Q'_l}^{(i')}$ as follows: Follow the path $J_l^{(i)}$ from $J_l^{(i)} \cap S_{i+r}$ through $2l$ segments to the separator S_{i+r+2l}. Continue the path within Z_{i+r+2l} to the path t_l. This can be done by following the bridges between paths corresponding to the path in the spanning tree T from $J_l^{(i)}$ to t_l.

Construct a similar path from ${Q'_l}^{(i')}$ to t_l using bridges in $Z_{i'-2l}$. Then connect these paths along t_l, and denote by P_l^i the resulting path joining $Q_l^{(i)}$ with ${Q'_l}^{(i')}$.

Claim 5.5 *The constructed paths P_l^i $(l = 1, \ldots, a)$ are pairwise disjoint.*

Proof. Consider two of the paths, say P_l^i and P_m^i, where $l < m$. There are four possibilities where these two paths may intersect:

(1) P_l^i intersects $J_m^{(i)}$ inside Z_{i+r+2l}: This is not possible since $J_m^{(i)}$ would then be closer to t_l in T, and the path $Q_m^{(i)}$ would be indexed before $Q_l^{(i)}$.

(2) P_m^i intersects t_l inside Z_{i+r+2m}: This is not possible since t_l is a leaf in $T \setminus \{t_1, \ldots, t_{l-1}\}$.

The remaining cases, when P_l^i intersects P_m^i inside $Z_{i'-2l}$ or inside $Z_{i'-2m}$ (respectively) are handled similarly. This completes the proof. $\square$

Let v_l be the vertex of $Q_l'^{(i)}$ in L_i, and let u_l be the vertex of $Q_l^{(i)}$ in L_i. Choose u_l' to be a neighbor of u_l in $L_i \setminus \{v_1, \ldots, v_a, u_1, \ldots, u_a\}$. Since L_i is $3a$-linked, the minimum degree of L_i is at least $3a$, so such neighbors exist. The vertices u_l' may even be chosen so that they are pairwise distinct. Let $v_1' = u_1'$, and let $v_2', \ldots, v_a'$ be distinct neighbors of v_1' in L_i. We may assume that if $v_\alpha' = u_\beta'$, then $\alpha = \beta$.

Since L_i is $2a$-linked, there is a linkage from $(v_1, \ldots, v_a, v_1', \ldots, v_a')$ to $(u_1, \ldots, u_a, u_1', \ldots, u_a')$. The resulting paths joining v_l and u_l ($l = 1, \ldots, a$) are used to link $Q_l'^{(i)}$ and $Q_l^{(i)}$ inside L_i, for $i \in \{i_1, \ldots, i_k\}$. Together with the paths P_l^i, $i \in \{i_1, \ldots, i_{k-1}\}$, this determines the path $\mathcal{P}_l^\circ$. On the other hand, the paths in the linkage from $(v_1', \ldots, v_a')$ to $(u_1', \ldots, u_a')$ are disjoint from $\mathcal{P}_1^\circ, \ldots, \mathcal{P}_a^\circ$ and can be used to link v_1' to each of these paths.

Now, it can be shown that G contains a $K_{a,k}$ minor: For each $l = 1, \ldots, a$, contract the path $\mathcal{P}_l^\circ$ to a single vertex. For $i \in \{i_1, \ldots, i_k\}$, the vertex $v_1' \in V(L_i)$ is joined to $u_1', \ldots, u_a'$ and hence to each of the a paths $\mathcal{P}_1^\circ, \ldots, \mathcal{P}_a^\circ$. Since this is repeated k times, we get a $K_{a,k}$ minor in G.

The proof of Theorem 1.2 is complete.

6 Conclusion

Our more recent results show that the condition on bounded tree-width in Theorem 1.1 can be removed. The authors plan a second paper in which the large tree-width case is handled. This will prove the following, which was conjectured independently by Ding [3] and the authors:

There is a function $f : \mathbb{N} \to \mathbb{N}$ such that any 7-connected graph on at least $f(k)$ vertices contains a $K_{3,k}$ minor.

It seems reasonable to the authors that this result can be extended to $K_{4,k}$-minors and possibly even to $K_{a,k}$-minors. The logical conjectures would be the following:

Conjecture 6.1 *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that any 9-connected graph on at least $f(k)$ vertices contains a $K_{4,k}$ minor.*

Conjecture 6.2 *There are functions $f : \mathbb{N} \to \mathbb{N}$ and $c : \mathbb{N} \to \mathbb{N}$ such that any $c(a)$-connected graph on at least $f(k)$ vertices contains a $K_{a,k}$ minor.*

Our final remark is that the sequence of graphs $K_{a,k}$, where a is fixed and k tends to infinity, is essentially the only family of graphs for which a result like our Theorem 1.2 holds. More precisely:

Theorem 6.3 *Let c and $w \geq c$ be positive integers, and let H_k $(k \geq 1)$ be a sequence of graphs such that $\lim_{k\to\infty} |V(H_k)| = \infty$. Suppose that for any positive integer k there exists an integer $N(k)$ such that every c-connected graph of tree-width $\leq w$ and of order at least $N(k)$ contains H_k as a minor. Then $H_k \leq_{\mathrm{m}} K_{c,N(k)}$ for $k \geq 1$.*

Proof. Clearly, the graph $K_{c,N(k)}$ is c-connected and has tree-width $c \leq w$. By the assumption on the family H_k, $K_{c,N(k)}$ contains H_k as a minor. $\square$

References

[1] B. Bollobás, A. Thomason, Highly linked graphs, Combinatorica 16 (1996) 313–320.

[2] R. Diestel, Graph Decompositions – A Study in Infinite Graph Theory, Oxford University Press, Oxford, 1990.

[3] Guoli Ding, private communication.

[4] M. Juvan, J. Marinček, B. Mohar, Elimination of local bridges, Math. Slovaca 47 (1997) 85–92.

[5] W. Mader, Existenz n-fach zusammenhängender Teilgraphen in Graphen genügend grosser Kantendichte, Abh. Math. Sem. Univ. Hamburg 37 (1972) 86–97.

[6] J. Maharry, A characterization of graphs with no cube minor, J. Combin. Theory Ser. B, to appear.

[7] J. Maharry, An excluded minor theorem for the octahedron, J. Graph Theory 31 (1999) 95–100.

[8] N. Robertson, private communication.

[9] N. Robertson, P. D. Seymour, Graph minors. IX. Disjoint crossed paths, J. Combin. Theory Ser. B 49 (1990) 40–77.

[10] N. Robertson, P. D. Seymour, An outline of a disjoint paths algorithm, in: "Paths, Flows, and VLSI-Layout," B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver Eds., Springer-Verlag, Berlin, 1990, pp. 267–292.

[11] B. Oporowski, J. Oxley, R. Thomas, Typical subgraphs of 3- and 4-connected graphs, J. Combin. Theory Ser. B 57 (1993) 239–257.

[12] R. Thomas, Recent excluded minor theorems for graphs, in "Surveys in Combinatorics, 1999 (Canterbury)," Cambridge Univ. Press, Cambridge, 1999, pp. 201–222.

[13] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937) 570–590.