14 Shi's local and global derivative estimates

Let's suppose the norm of the curvatures is bounded by K at the initial time. The following estimate says that the **maximum of the curvatures cannot grow too fast too soon** as time increases.

Lemma 1 (Doubling time estimate) *Let $(M^n, g(t))$ be a solution of the Ricci flow on a closed manifold. If $|\mathrm{Rm}(g(0))| \leq K$, then*

$$|\mathrm{Rm}(g(t))| \leq 2K$$

for all $0 \leq t \leq 1/(16K)$.

Proof. From the evolution equation for Rm one can show that that

$$\frac{\partial}{\partial t} |\mathrm{Rm}|^2 \leq \Delta |\mathrm{Rm}|^2 - 2 |\nabla \mathrm{Rm}|^2 + 16 |\mathrm{Rm}|^3 . \tag{1}$$

Let $\rho(t)$ be the solution to the corresponding ODE

$$\frac{d}{dt}\rho^2 = 16\rho^3, \quad \rho(0) = K.$$

By the maximum principle we conclude

$$|\mathrm{Rm}(g(t))| \leq \rho(t) = \frac{K}{1 - 8Kt}.$$

■

The following result says that the solution exists as long as the curvature remains bounded.

Proposition 2 (Long time existence) *If $(M^n, g(t))$ is a solution of the Ricci flow on a closed manifold on a maximal time interval $[0, T)$, where $T < \infty$, then*

$$\lim_{t \to T} \max_{M^n} |\mathrm{Rm}(\cdot, t)| = \infty. \tag{2}$$

Remark 3 *In particular, if $|\mathrm{Rm}(g(0))| \leq K$, then $T > 1/(16K)$.*

The idea of the proof is to obtain **higher derivative of curvature estimates**. Solutions with a given bound on the curvature may have covariant derivatives that are initially arbitrarily large. However, for short time, one can obtain bounds for the higher covariant derivatives of the curvature which improve with time on a short enough initial time interval (see [2], p. 223-4).

Example 4 (Bumpy metrics) *Consider the 1-parameter family of initial metrics on $\mathbb{R} \times S^1$*

$$g_\varepsilon \doteqdot dr^2 + \left(1 + \varepsilon \sin\left(r/\sqrt{\varepsilon}\right)\right)^2 d\theta^2.$$

We have

$$K(g_\varepsilon) = \frac{\sin(r/\sqrt{\varepsilon})}{1 + \varepsilon \sin(r/\sqrt{\varepsilon})} = 1 - \frac{1}{1 + \varepsilon \sin(r/\sqrt{\varepsilon})}$$

which is uniformly bounded for $\varepsilon \in (0, 1/2]$. *Note that*

$$\left|\nabla K\left(g_{\varepsilon}\right)\right| = \left|\frac{\partial}{\partial r} K\left(g_{\varepsilon}\right)\right| = \frac{1}{\sqrt{\varepsilon}} \frac{\left|\cos\left(r/\sqrt{\varepsilon}\right)\right|}{\left(1+\varepsilon \sin\left(r/\sqrt{\varepsilon}\right)\right)^{2}}$$

so that

$$\sup_{\mathbb{R}\times S^{1}} \left|\nabla K\left(g_{\varepsilon}\right)\right| \geq \frac{1}{\sqrt{\varepsilon}}$$

which tends to infinity as $\varepsilon \to 0$.

Theorem 5 (Global derivative of curvature estimates) *If* $\left(M^{n}, g(t)\right), t \in [0, T)$, *is a solution of the Ricci flow on a closed manifold, then for each* $\alpha > 0$ *and every* $m \in \mathbb{N}$, *there exists a constant* $C(m, n, \alpha)$ *depending only on* m, *and* n, *and* $\max\{\alpha, 1\}$ *such that if*

$$|\operatorname{Rm}(x, t)|_{g(t)} \leq K \qquad \text{for all } x \in M^{n} \text{ and } t \in [0, \frac{\alpha}{K}] \cap [0, T),$$

then

$$\left|\nabla^{m} \operatorname{Rm}(x, t)\right|_{g(t)} \leq \frac{C(m, n, \alpha) K}{t^{m/2}} \qquad \text{for all } x \in M^{n} \text{ and } t \in (0, \frac{\alpha}{K}] \cap [0, T). \tag{3}$$

We call these estimates **Bernstein-Bando-Shi estimates**. For $m = 1$, we can prove (3) by considering the gradient quantity

$$F = t\left|\nabla \operatorname{Rm}\right|^{2} + \beta\left|\operatorname{Rm}\right|^{2},$$

and showing

$$\frac{\partial F}{\partial t} \leq \Delta F + C(n) \beta K^{3}, \tag{4}$$

for β depending only on $\max\{\alpha, 1\}$ and n, and where the constant $C(n)$ depends only on n. Since $F \leq \beta K^{2}$ at $t = 0$, by applying the maximum principle to (4), we obtain $t\left|\nabla \operatorname{Rm}\right|^{2} \leq F \leq C(n, \alpha) K^{2}$ for $t \in [0, \alpha/K]$, where $C(n, \alpha)$ depends only on $\max\{\alpha, 1\}$ and n. The first derivative estimate immediately follows. The higher derivative estimates can be proved using analogous quantities involving higher derivatives of curvature. We refer the reader to [2], Theorem 7.1 on p. 223-4.

Exercise 6 *Fill in the details of the proof of (4) for* $m = 1$. Hint*: show that (see (1))*

$$\frac{\partial}{\partial t}\left|\operatorname{Rm}\right|^{2} \leq \Delta\left|\operatorname{Rm}\right|^{2} - 2\left|\nabla \operatorname{Rm}\right|^{2} + 16\left|\operatorname{Rm}\right|^{3}$$

and

$$\frac{\partial}{\partial t}\left|\nabla \operatorname{Rm}\right|^{2} \leq \Delta\left|\nabla \operatorname{Rm}\right|^{2} + C\left|\operatorname{Rm}\right| \cdot \left|\nabla \operatorname{Rm}\right|^{2}$$

where we have dropped the good $-2\left|\nabla^{2} \operatorname{Rm}\right|^{2}$ *term from the* RHS.

We now give the statement of the following estimate of W.-X. Shi.

Theorem 7 (Local derivative of curvature estimates) *For any α, K, r, n and $m \in \mathbb{N}$, there exists C depending only on α, K, r, n and m such that if M^n is a manifold, $p \in M$, and $g(t)$, $t \in [0, T_0]$, $0 < T_0 \leq \alpha/K$, is a solution to the Ricci flow on an open neighborhood U of p containing $\bar{B}_{g(0)}(p, r)$ as a compact subset, and if*

$$|\mathrm{Rm}(x,t)| \leq K \text{ for all } x \in U \text{ and } t \in [0, T_0],$$

then

$$|\nabla^m \mathrm{Rm}(y,t)| \leq \frac{C(\alpha, K, r, n, m)}{t^{m/2}}$$

for all $y \in B_{g(0)}(p, r/2)$ and $t \in (0, T_0]$.

The proof of Proposition 2 is based on this and the following elementary result which gives a sufficient condition for the metrics to remain uniformly equivalent to the initial metric (see Lemma 14.2 of [3] and [2], p. 203).

Lemma 8 (Uniform equivalence of the metrics) *Let $g(t)$, $t \in [0, T)$, where $T \leq \infty$, be a smooth 1-parameter family of metrics on a manifold M^n. If there exists a constant $C < \infty$ such that*

$$\int_0^T \sup_{x \in M^n} \left| \frac{\partial g}{\partial t}(x,t) \right|_{g(t)} dt \leq C, \tag{5}$$

then for any $x_0 \in M^n$ and $t_0 \in [0, T)$, we have

$$e^{-C} g(x_0, 0) \leq g(x_0, t_0) \leq e^C g(x_0, 0). \tag{6}$$

Moreover, the metrics $g(t)$ converge uniformly as $t \to T$ to a continuous metric $g(T)$ with $e^{-C} g \leq g(T) \leq e^C$.

Applying the first part of this to a solution to the Ricci flow, we have

Corollary 9 (Bounded Ricci implies uniform equivalence) *If*

$$\sup_{M \times [0,T)} |\mathrm{Rc}| \leq K,$$

then

$$e^{-2KT} g(x,0) \leq g(x,t) \leq e^{2KT} g(x,0),$$

for all $x \in M^n$ and $t \in [0, T)$.

Proof of the lemma. Given $V \in TM_x$ and times $0 \leq t_1 \leq t_2 < T$, we have

$$\left| \log \frac{g_{(x,t_2)}(V,V)}{g_{(x,t_1)}(V,V)} \right| = \left| \int_{t_1}^{t_2} \frac{\frac{\partial}{\partial t} g_{(x,t)}(V,V)}{g_{(x,t)}(V,V)} dt \right| \leq \int_{t_1}^{T} \left| \frac{\partial g}{\partial t}(x,t) \right|_{g(t)} dt \doteqdot C(t_1) \leq C, \tag{7}$$

and $\lim_{t\to T} C(t) = 0$. Taking the exponential of this estimate with $t_1 = 0$ yields (6). By (7), the continuity of the metrics $g(t)$, and the formula

$$g(t)(V,W) = \frac{1}{4}\left(|V+W|^2_{g(t)} - |V-W|^2_{g(t)}\right),$$

we see that for every $V, W \in TM$

$$\lim_{t\to T} g(t)(V,W) \doteqdot g(T)(V,W)$$

exists; the convergence is uniform on compact sets, and $g(T)(V,W)$ is a metric continuous in V and W. ■

Exercise 10 *Prove the following. If* $\mathrm{Rc} \geq -K$, *then* $g(x,t) \leq e^{2KT} g(x,0)$. *If* $\mathrm{Rc} \leq K$, *then* $g(x,t) \geq e^{-2KT} g(x,0)$.

To prove long time existence we suppose that the curvature remains bounded on a time interval $[0,T)$ where $T < \infty$ and show that the metrics $g(t)$ converge to a smooth metric g_T as $t \to T$. In order to do this we need to bound the derivatives of the metric. This is possible using the Ricci flow equation and the bounds on the curvature and its derivatives (see also Proposition 6.48 on p. 203 of [2].) This shows that

$$\sup_{M^n\times[0,T)} |\mathrm{Rm}| = \infty. \tag{8}$$

With a little more work we can show (2).

References

[1] Chow, Bennett; Chu, Sun-Chin; Glickenstein, David; Guenther, Christine; Isenberg, Jim; Ivey, Tom; Knopf, Dan; Lu, Peng; Luo, Feng; Ni, Lei. *The Ricci flow: techniques and applications.* In preparation.

[2] Chow, Bennett; Knopf, Dan. *The Ricci flow: An introduction,* Mathematical Surveys and Monographs, AMS, Providence, RI, 2004.

[3] Hamilton, Richard S. *Three-manifolds with positive Ricci curvature.* J. Differential Geom. **17** (1982), no. 2, 255–306.

[4] Shi, Wan-Xiong. *Deforming the metric on complete Riemannian manifolds.* J. Differential Geom. **30** (1989), no. 1, 223–301.

[5] Shi, Wan-Xiong. *Ricci deformation of the metric on complete noncompact Riemannian manifolds.* J. Differential Geom. **30** (1989), no. 2, 303–394.