Using Resonant Filters for the Synthesis of Time-Varying Sinusoids

Jean Laroche
Joint E-mu/Creative Technology Center
1600 Green Hills Road
P.O. Box 660015
Scotts Valley, CA 95067
Email: jeanl@emu.com

Abstract

This paper discusses sinusoidal synthesis by means of resonant filters. Resonant recursive filters have long been used to synthesize exponentially damped sinusoids but surprisingly little has been written about stability issues when the filter coefficients are allowed to vary and interpolation problems. In this paper, we discuss some of the issues one faces when synthesizing sinusoids with time-varying coefficients by means of recursive resonant filters. It is shown that different topologies yield different results when the coefficients are allowed to move, and that stability can indeed be problematic with traditional topologies such as the direct form or the lattice form. Other structures that exhibit good behavior in terms of time-varying stability are shown to perform poorly under linear-interpolation of their coefficients.

1 Introduction

Synthesizing sinusoids by means of recursive resonant filters is a viable alternative to table-lookup or frequency-domain synthesis [1]. As is well known, the impulse response of a second-order recursive filter with complex-conjugate poles is an exponentially decaying sinusoid whose frequency and damping factor depend only on the filter's poles. By setting the filter coefficients appropriately, and by exciting the filter with an appropriate impulse-like signal, one can synthesize any arbitrary exponentially decaying sinusoid at the cost of only about 2 multiplies and one add per output samples (using a direct form second-order section and not counting the initial coefficient setup and impulse generation). By comparison, linear-interpolated table-lookup synthesis requires two multiplies and three adds (one multiply and two adds for the linear interpolation, one multiply for the gain and one add for the phase increment) with an additional (!) multiply for an exponentially decaying amplitude. In addition to being reasonably cost-effective, resonant filter synthesis also makes for interesting sound-effects when actual audio signals are fed into the synthesis filter, a process akin to long-term convolution.

The downside of resonant filter synthesis is that controlling the time-varying parameters of the sinusoids can be difficult. Because the frequency and the damping factor depend only on the filter coefficients, they can easily be controlled in real-time by appropriately altering the filter coefficients. Unfortunately, the sinusoidal phase and amplitude usually depend both on the filter's coefficients and current internal states. This means that modifying the frequency or damping factors of the sinusoid is likely to alter its amplitude and phase as well. Furthermore, when the coefficients are allowed to vary, the stability of the filter is usually no longer guaranteed even if the coefficients at each instant correspond to a stable time-invariant filter.

In this paper, we discuss some of the issues one faces when implementing resonant filter synthesis, namely filter stability and interpolation problems.

2 Time-varying filter stability

The stability of time-invariant filters is a well known topic and many simple criteria have been derived to guarantee the Bounded-Input Bounded-Output (BIBO) stability or many types of filters (see for example [2]). The literature on time-varying filters is much more scarce, and the theory more complicated. Several familiar properties of time-invariant filters no longer hold in the time-varying case: for example, time-varying filters do not necessarily commute and topologies that are equivalent in the time-invariant case no longer are in the
time-varying case. Surprisingly, a time-varying filter whose coefficients at any given time are those of a stable time-invariant filter is not necessarily stable. For example, a direct form second-order section whose coefficients are allowed to vary within the triangle of stability is not guaranteed to be stable unless the coefficients become invariant at some time. Further insight into this puzzling fact can be gained by looking at the state-variable representation of the filter, as suggested in [3] and [4, 5].

2.1 A sufficient stability criterion

For time-varying filters, the output signal \( y(n) \) and the input signal \( x(n) \) are related via a series of impulse-responses \( h(n, i) \):

\[
y(n) = \sum_{i=-\infty}^{\infty} h(n, i) x(i)
\]

(1)

where \( h(n, i) \) is the output of the time-varying filter observed at time \( n \) when excited by an impulse at time \( i \). In that case, the necessary and sufficient BIBO stability condition is

\[
\text{Filter BIBO stable } \iff \exists G, \forall n, \sum_{i=-\infty}^{\infty} |h(n, i)| < G
\]

(2)

This necessary and sufficient condition can be expressed in terms of the state-space representation of the filter: The state-space representation of a time-varying filter consists of two linear time-varying equations:

\[
\begin{align*}
X_{n+1} &= P_n X_n + Q_n x_n \\
y_n &= R_n X_n + S_n x_n
\end{align*}
\]

(3)

Here, \( X_n \) is a \( p \)-dimensional vector representing the state of the filter, \( P_n \) is a \( p \) by \( p \) matrix, \( Q_n \) is a \( p \) by \( 1 \) matrix, \( R_n \) is a \( 1 \) by \( p \) matrix, and \( S_n \) is a scalar (we assume the input \( x_n \) and the output \( y_n \) are scalar).

The impulse response of the filter excited at sample \( i \) and observed at sample \( n \) is easily obtained from the state-space representation Eq. (3):

\[
h(n, i) = \begin{cases} 
0 & \text{if } n < i \\
S_i & \text{if } n = i \\
R_n \left( \prod_{m=i+1}^{n} P_m \right) Q_i & \text{if } n > i
\end{cases}
\]

(4)

And the necessary and sufficient condition for BIBO stability can be re-written by using the expression for the impulse-responses Eq. (4) and the condition Eq. (2) for BIBO stability:

\[
\text{Filter BIBO stable } \iff \exists G \forall n, |S_n| + \sum_{i=n+1}^{\infty} |R_n \left( \prod_{m=i}^{n} P_m \right) Q_i| < G
\]

(5)

Although the above expression provides a necessary and sufficient condition for BIBO stability, it is not directly usable because of the infinite summation. We now proceed to derive a more readily usable but only sufficient criterion. In the following, the norm of a vector will be taken to be the standard Euclidian norm, while the norm of a matrix is the matrix norm induced by the Euclidian vector norm. It is the maximum amplification the matrix can bring to any unit-norm vector \( x \): \( ||M|| = \max_{||x||=1} ||Mx|| \) where \( ||x|| \) is also the standard Euclidian norm. The norm of a matrix \( M \) can be shown to be equal to its largest singular value, or to the positive square roots of the largest eigenvalue of matrix \( M^T M \). See [6] for details.

From now on, we will assume that \( S_n, R_n \) and \( Q_n \) are bounded in norm by \( K \): \( |S_n| < K, |R_n| < K, \) and \( ||Q_n|| < K \). This is a reasonable assumption in practice as the coefficients of a filter, time-varying or not, are usually bounded. Under this assumption, a sufficient condition for Eq. (5) to hold, and therefore a sufficient condition for time-varying BIBO stability is the following:

**Criterion 1** If there exist a real constant \( 0 \leq \gamma < 1 \) such that

\[
||P_m|| \leq \gamma \quad \forall m
\]

then the corresponding time-varying filter is BIBO stable.
To see that, we first note that

$$|R_n \left( \prod_{m=1}^{n-1} P_m \right) Q_i| \leq ||R_n|| \left( \prod_{m=1}^{n-1} ||P_m|| \right) ||Q_i|| \leq ||R_n|| \gamma^n i \leq K^2 \gamma^n i$$

because the norm of a product of matrices is smaller than the product of the norms. Inserting this inequality in Eq. (5) we have

$$|S_n| + \sum_{i=n}^{\infty} |R_n \left( \prod_{m=1}^{n-1} P_m \right) Q_i| \leq K + \sum_{i=n}^{\infty} K^2 \gamma^n i \leq K + K^2 \frac{1}{1 - \gamma}$$

the infinite sum converging because of the hypothesis $\gamma < 1$. This proves the filter’s BIBO stability.

Note that this simple criterion for BIBO stability is sufficient but clearly not necessary: for example, a stable time-invariant direct-form II biquad never satisfies criterion 1 because the norm of the (time-invariant) transition matrix $P$ is always larger than 1. For a direct-form II biquad, the transition matrix is

$$P = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}$$

and its norm is at least $1 + a_1^2 \geq 1$ because $P(1,0)^t = (-a_1,1)^t$. Yet a stable time-invariant direct-form II biquad is stable! We will see however, that this criterion 1 can be used to prove that certain topologies are guaranteed to be stable under time-varying conditions provided their coefficients correspond to stable time-invariant filters at all time.

### 2.2 A closer look at time-varying instability

To understand why a filter that is stable under time-invariant conditions can become unstable under time-varying conditions, it is useful to consider the simpler case of a filter whose hitherto constant coefficients are switched to a different henceforth constant set at time $n_0$. For simplicity we will consider a second-order, purely recursive filter whose poles are complex conjugate and lie inside the unit circle both before and after time $n_0$. We will see that at the time the coefficients are switched, the impulse response of the filter undergoes a change in amplitude and phase. First, we recognize that the frequency $\omega$ and the damping factor $\rho$ of the zero-input response are known because they are related to the magnitude and angle of the complex poles. When the frequency and damping factor of a real exponentially damped sinusoid are known, then its amplitude $A$ and phase $\phi$ can be determined from just two successive samples $y_{n-1}$ and $y_n$ in the following way: in a vector form, write

$$\begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} \rho \cos \omega & \rho \sin \omega \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A \cos \phi \\ A \sin \phi \end{bmatrix} \text{ or } Y = T Z$$

And $A$ can be obtained by $A = ||Z|| = ||T^{-1}Y||$. In that case, $A$ represents the sinusoid’s amplitude envelope at sample $n-1$. The amplitude at sample $n$ is simply $\rho A$ and so on. Since the output sinusoid is a function of the filter’s state, it is possible to determine its amplitude $A$ from the state vector in the following way: write

$$y_n = RX_n + S_n x_n = RX_n$$
$$y_{n-1} = RX_n \begin{bmatrix} 1 \\ \end{bmatrix} = RP^\dagger X_n$$

$$\begin{bmatrix} y_n \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} R \\ \end{bmatrix} \begin{bmatrix} \ldots \\ \end{bmatrix} \text{ or } X_n = V X_n$$

where we used the fact that $x_n = 0$ (zero-input case) and assumed that the filter coefficients are the same at samples $n-1$ and $n$. As a result, the amplitude $A$ can be calculated from the state-vector as

$$A = ||T^{-1} V X_n||$$

As is clear from Eq. (6), $A$ depends on the current values of the frequency and damping factor, which are functions of the filter parameters, as well as the current filter state-vector.
At sample $n_0$, before the coefficients are switched, the amplitude is $||T_0^{-1}V_0X_{n_0}||$ where $T_0$ and $V_0$ are derived from the matrices $R_0$ and $R_0$ corresponding to the filter coefficients before the switch. Still at sample $n_0$, but after the coefficients are switched, the amplitude corresponding to the new filter coefficients is $||T_1^{-1}V_1X_{n_0}||$. The resulting gain is:

$$
\alpha = \frac{||T_1^{-1}V_1X_{n_0}||}{||T_0^{-1}V_0X_{n_0}||} = \frac{||T_1^{-1}V_1V_0^{-1}T_0^{-1}Z_{n_0}||}{||Z_{n_0}||} \quad \text{with} \quad Z_{n_0} = T_0^{-1}V_0X_{n_0}
$$

To evaluate whether the gain factor $\alpha$ is large or small requires looking at the matrix $T_1^{-1}V_1V_0^{-1}T_0$ and more specifically at its singular values $\mu_i$. Indeed, it is a standard result of linear algebra [6] that

$$
\mu_{\min} \leq \alpha \leq \mu_{\max}
$$

where $\mu_{\min}$ is the smallest singular value, and $\mu_{\max}$ is the largest singular value of matrix $T_1^{-1}V_1V_0^{-1}T_0$. This formula provides upper and lower bounds on the amount of gain/attenuation that will be brought to the sinusoidal amplitude when the coefficients of the filter are switched.

Upper and lower bounds on the gain are useful, but say nothing about the average amplification the sinusoidal amplitude will undergo. Average gain values can be obtained in a statistical sense if we assume that the filter is switched at a random time, so that the phase $\phi$ of the zero-input sinusoid at switch time is a random variable uniformly distributed between $0$ and $2\pi$. In that case, we can write

$$
Y_{n_0} = T_0AU \quad \text{and} \quad X_{n_0} = V_0^{-1}Y_{n_0} = V_0^{-1}T_0AU \quad \text{with} \quad U = \begin{bmatrix} \cos \phi \\ \sin \phi \end{bmatrix}
$$

where $A$ denotes the amplitude of the sinusoid at sample $n_0-1$. Inserting this expression for $X_{n_0}$ into Eq. (2.2), we get:

$$
\alpha = ||T_1^{-1}V_1V_0^{-1}T_0U|| = ||MU|| \quad \text{with} \quad M = T_1^{-1}V_1V_0^{-1}T_0
$$

where $U$ is a random, unit-norm vector whose direction is uniformly distributed in the plane. The following result can be shown to hold in that case [7]:

$$
\bar{\alpha} = \exp(E(\log(\alpha))) = \frac{\mu_1 + \mu_2}{2} = \frac{1}{2} \sqrt{\text{trace}(M^TM) + 2 \det(M)}
$$

where $E(\cdot)$ denotes the expectation, $\text{trace}(M)$ is the trace of matrix $M$ (the sum of the diagonal elements) and $\det(M)$ is the determinant of $M$. This equation yields an estimate of the average amplification resulting from coefficient switching for a random switch time. Although Eq. (8) is not readily usable because it involves calculating a determinant and a trace, it provides an explanation for curious phenomena observed in practice.

For example, one surprising consequence of Eq. (8) is that randomly switching from filter coefficients $B$, and then back to coefficients $A$, results, in average, in an increase of the output amplitude. Indeed, it is easy to see that if switching from $A$ to $B$ involves matrix $M$ in Eq. (8), switching from $B$ to $A$ involves matrix $M^{-1}$, whose singular values are the inverse of those of $M$. Because the function $x \rightarrow 1/x$ is concave for positive $x$, the average of the inverse of two positive numbers is larger than the inverse of the average, which shows that

$$
\bar{\alpha}_{A \rightarrow B} = \frac{\mu_1 + \mu_2}{2} \geq \frac{2}{1/\mu_1 + 1/\mu_2} = \frac{1}{\bar{\alpha}_{B \rightarrow A}}
$$

As a result

$$
\bar{\alpha}_{A \rightarrow B} \bar{\alpha}_{B \rightarrow A} \geq 1
$$

with equality only when $\mu_1 = \mu_2$. This is observed in practice when one tries to synthesize a sinusoid with a null damping factor and a frequency that is randomized around a nominal frequency. When the a direct form filter is used, the result is a noisy sinusoid with an increasing amplitude which soon exceeds the signal’s available dynamics. This is a direct consequence of the previous remark.

2.3 Stability of various time-varying second-order filters

Using criterion 1 above, it is possible to show that some second-order topologies are inherently stable, provided their coefficients, at all time, are those of a stable time-invariant filter.
Coupled Form The coupled-form [8, 9] (see Fig. 1) has been shown to exhibit good properties in terms of roundoff noise, overflow oscillations [10] and time-varying stability [4]. For this structure, and for complex-conjugate poles \( \rho_n \exp(\pm j\omega_n) \), the state-transition matrix is simply

\[
P_n = \begin{bmatrix} c_n & s_n \\ -s_n & c_n \end{bmatrix}
\]

with \( c_n = \rho_n \cos \omega_n \) and \( s_n = \rho_n \sin \omega_n \). This is a so-called "minimum norm matrix" [10], in the sense that its norm is equal to \( \rho_n \), the magnitude of its largest eigenvalue (\( P_n^t P_n = |\rho_n|^2 I \) where \( I \) is the identity matrix). By virtue of criterion 1, a time-varying filter implemented with this topology is BIBO stable as long as the poles are restricted to lie within a disk strictly inside the unit circle at all time: \( \exists \gamma < 1 \) such that \( \forall n, \rho_n \leq \gamma \). Note that there are several other ways to prove this result [4].

Normalized Ladder The normalized ladder topology [11, 12] (see Fig. 1) has also been shown to have good roundoff noise properties [11]. For this structure, the state-transition matrix is

\[
P_n = \begin{bmatrix} -k_1(n) & -k_2(n) \sqrt{1 - k_1^2(n)} \\ \sqrt{1 - k_2^2(n)} & -k_1(n)k_2(n) \end{bmatrix}
\]

which can easily be shown to be of unit norm, because

\[
P_n^t P_n = \begin{bmatrix} 1 & 0 \\ 0 & k_2^2(n) \end{bmatrix}
\]

which means that the two singular values of \( P_n \) are 1 and \( |k_2(n)| \). \( k_1(n) \) and \( k_2(n) \) are the reflection coefficients and time-invariant stability requires \( |k_1(n)| < 1 \) and \( |k_2(n)| < 1 \) [2]. Because \( |P_n| = 1 \), we cannot apply criterion 1 to prove BIBO stability (this would require the norm of \( P_n \) to be strictly less than \( I \)). However, it can be shown [7] that \( |P_n^t P_n| \leq \gamma \) if \( P_n \) corresponds to a stable time-invariant filter. This proves that a time-varying filter implemented with the normalized-ladder topology is BIBO stable as long as its coefficients are modified no more often than every other sample.

It is also interesting to note that, in the limit \( \rho_n \to 1 \), the transition matrices of the normalized ladder and of the coupled form become equivalent, because \( k_2(n) = \rho_n^2 \) and \( k_1(n) = (-2\rho_n \cos \omega_n)/(1 + \rho_n^2) \approx -\cos \omega_n \) and \( \sqrt{1 - k_1^2(n)} \approx \sin \omega_n \).

Direct Form II Here, we assume that \( b_1 = b_2 = 0 \) and \( b_c(n) = b_c \). In this case, it is straightforward to show that matrix \( M = T^{-1}_1 V_1 T_0^{-1} \) is

\[
M = \begin{bmatrix} \rho_c \cos \omega_1 & \rho_c \cos \omega_1 & 0 \\ \rho_c \cos \omega_1 & \rho_c \cos \omega_1 & \rho_c \sin \omega_1 \\ \rho_c \cos \omega_1 & \rho_c \cos \omega_1 & \rho_c \sin \omega_1 \end{bmatrix}
\]

and the average gain given by Eq. (8) resulting from coefficient switching is

\[
\bar{\alpha} = 1 + \frac{\rho_0^2 + \rho_1^2 - 2\rho_0 \rho_1 \cos(\omega_1 + \omega_0)}{\rho_0^2 \sin^2 \omega_1}
\]

and if \( \rho_0 = \rho_1 \) this simplifies as:

\[
\bar{\alpha} = \frac{\sin \omega_1 + \omega_0}{\sin \omega_1} \frac{2}{\sin \omega_1}
\]

This shows for example, that moving the pole frequency of the filter toward zero is likely to cause an increase in the amplitude of the zero-input response, all the more as the pole frequency is close to 0 or Nyquist. The direct form II (in fact, none of the direct forms) is not necessarily stable under time-varying condition, even when the poles are constrained to lie within the unit circle at all times.
For the lattice filter, the matrix $M$ is

$$
M = \begin{bmatrix}
\rho_0 \rho_0 \cos \omega_1 & \rho_0 \rho_0 \cos \omega_0 & 0 \\
\rho_0 \rho_0 \sin \omega_1 & \rho_0 \rho_0 \sin \omega_0 & 0 \\
\rho_0 \rho_0 \sin \omega_1 & \rho_0 \rho_0 \sin \omega_0 & 0
\end{bmatrix}
$$

where $a = (1 - \rho^2)/(1 + \rho^2)$. When $\rho_0 = \rho_1$ the expression for $\alpha$ is

$$
\alpha = \frac{\sin \frac{\omega_0 + \omega_1}{2}}{\sin \omega_1} \sqrt{1 + (a^2 - 1) \sin^2 \frac{\omega_1 - \omega_0}{2}}
$$

For highly resonant filters, $\rho \approx 1$ and $a \approx 0$ so

$$
\alpha = \frac{\sin \frac{\omega_0 + \omega_1}{2} \cos \frac{\omega_1 - \omega_0}{2}}{\sin \omega_1}
$$

Again, we see that lowering the pole frequency causes an increase of the amplitude of the zero-input response by a factor quite close to the one observed in the direct form II case. The lattice filter is not necessarily stable under time-varying condition, even when the poles are constrained to lie within the unit circle at all times.

### 2.4 Stability: What to do in practice.

In conclusion, using a direct form filter or a non-normalized lattice or ladder filter to generate time-varying sinusoids can be dangerous if the filter's coefficients are allowed to vary in time. In practice, problems are encountered with highly resonant filters (used to synthesize slowly damped sinusoids), when the frequency is allowed to vary in time (for example, synthesizing a sweeping sinusoid). When the damping factor is large, instabilities are less likely to occur as the filter's damping counteracts the potential amplitude gain resulting from coefficient switching. This is why stability problems are very rarely encountered in speech synthesis even though lattice filters are routinely used with linearly interpolated coefficients, which are not guaranteed to be stable in time-varying conditions. For sinusoidal synthesis however, stability problems easily occur, for example when synthesizing a slowly damped sinusoid with a frequency jitter.

One way to circumvent the problem is to use a structure that guarantees stability, the coupled form or the normalized lattice, even though the cost in terms of multiplies and adds is higher than for most of the more standard topologies. These topologies are known for their good behavior for roundoff noise, coefficient quantization and overflow oscillations. We will see however, that they may not be ideal for sinusoidal synthesis, in spite of their inherent robustness to time-varying instability. Another way is to put constraints on the amount and rapidity by which filter coefficients can vary, according to Eq. (8). For the jittered frequency example above, limiting the speed and range of the jitter so that the average amplitude increase given by Eq. (8) is smaller than the intrinsic filter damping can be enough to insure stability. In some cases however, it is not possible to limit the amount of variation in filter coefficients (for example, when the frequency needs to sweep from $f_1$ to $f_2$ in a given time). In such cases, the damping factor of the sinusoid can be artificially increased to counterbalance the amplitude increase resulting from coefficients modifications, still according to Eq. (8). A third solution consists of modifying the state of the filter when the coefficients are switched, in order to prevent any amplitude increase, as shown in [4]. In other words, in Eq. (6), one can also modify $X_n$ such that $A$ is left unchanged when the filter coefficients are switched. Unfortunately, this requires additional computations, including a division.

### 3 Coefficient interpolation

One of the drawbacks of resonant-filter synthesis is that in order to control one of the sinusoidal parameters (e.g., the frequency or the damping factor), one often has to modify several filter coefficients at the same time. For example, in the direct form, the coefficients of the recursive part of the filter are given by:

$$
a_1 = -2\rho \cos \omega \quad \text{and} \quad a_2 = \rho^2
$$

where $\rho$ is the damping factor in sample 1 and $\omega$ is the frequency in sample 1. Setting $\rho$ to a different value requires modifying both $a_2$ and $a_1$. While this is easy in this case, it is more difficult in the lattice topology...
where a division would be required: for the lattice topology, the coefficients of the recursive part of the lattice filter are given by

$$k_2 = \rho^2 \quad \text{and} \quad k_1 = \frac{-2\rho \cos \omega}{1 + \rho^2}$$

and calculating $k_1$ for a new value of $\rho$ is somewhat cumbersome.

To avoid these costly calculations, a straightforward solution consists of calculating new sets of filter coefficients at a lower sampling rate (for example, every 32 samples) and linearly interpolating from one set to the next one for each output sample. This way, the complex calculations occur less frequently and the only calculations needed on a sample basis are the linear interpolations of the filter coefficients.

Depending on the filter topology, linear interpolation of the filter coefficients yields different results when used for sinusoidal synthesis. In the following, we will use the very simple example of a second-order recursive filter, whose poles migrate from $p_1 \exp \pm j\omega_1$ to $p_2 \exp \pm j\omega_2$. In other words, the coefficients of the filter are calculated for the source poles $p_1 \exp \pm j\omega_1$ and for target poles $p_2 \exp \pm j\omega_2$ and a sinusoid is generated by exciting the filter whose coefficients are linearly interpolated from set 1 to set 2. We will see that depending on the structure, the results can be very different:

**Direct Forms** As mentioned above, for the direct form second-order section, the two coefficients of the recursive part of the filter are given by $a_1 = -2\rho \cos \omega$ and $a_2 = \rho^2$. In other words, $a_2$ is the square of the pole magnitude, while $a_1/2$ is the projection of the pole location on the x-axis. The trajectory of the pole is shown in Fig. 3 as the filter's coefficients are linearly interpolated. An interesting point is that the square of the pole radius varies linearly with time, which means that the damping of the sinusoid varies monotonically from the initial damping to the final damping. Fig. 3 shows that the pole moves along a spiral, which is a desirable behavior.

**Lattice form** For the lattice form, we have $k_2 = \rho^2$ and $k_1 = -2\rho \cos \omega/(1 + \rho^2)$. As in the direct form case, the square of the pole radius varies linearly with time when the coefficients are linearly interpolated. The dependence of $\omega$ with time is slightly different. The trajectory of the pole is shown in Fig. 3 as the filter's coefficients are linearly interpolated. The results are very similar to the direct form case, the pole travels along the same spiral, but the frequency dependence with time is slightly different.

**Coupled form** For the coupled form, the two coefficients of the recursive part are $c = \rho \cos \omega$ and $s = \rho \sin \omega$. In other words, the filter coefficients are the projection of the pole location on the x-axis and the y-axis. This means that as the coefficients are interpolated, the pole moves along a straight line from the initial pole location to the target pole location, as shown in Fig. 3. A consequence of that fact is that the damping of the sinusoid no longer varies monotonically (some of the intermediate poles are much closer to the origin than either the source or target poles). In the example shown on the figure, the damping factor of the sinusoid is very large for some of the intermediate stages, leading to a large decrease in amplitude, solely due to coefficient interpolation. This is highly undesirable, and is one of the major drawbacks of the coupled form for synthesizing sweeping sinusoids. Of course, one could decide to linearly interpolate over $\cos \omega$, $\sin \omega$ and $\rho$ and then multiply the results to obtain values for $c$ and $s$, which would then make the damping factor vary linearly. Unfortunately, this requires an additional linear interpolation and two additional multiplication.

### 4 Conclusion

Resonant-filters offer a viable alternative to wave-table interpolation for the synthesis of exponentially damped sinusoids. However, practical implementations usually face stability problems when the sinusoidal parameters must vary in time, because the stability of time-varying filters is more difficult to guarantee than that of time-invariant filters. This is especially true for nearly-undamped sinusoids with frequencies that vary rapidly. For such cases, picking the right topology can solve the problem: the direct form topologies, and the non-normalized lattice structures (we gave the example of the regular lattice structure, but very similar results hold for the Kelly-Lochbaum topology) are not guaranteed to be stable under time-varying conditions even when the parameters are kept within the stability region for time-invariant filters. Two structures stand out in that respect, the coupled Gold-Rader structure and the normalized ladder, which are guaranteed to be stable when
the parameters are allowed to vary anywhere within a region strictly inside the time-invariant stability region. Unfortunately, these structures are also much more costly in terms of computation than the direct or even the lattice form. Moreover, when the filter coefficients are linearly interpolated, the coupled form and the normalized lattice form exhibit undesirable overdamping, which can cause the output sinusoid to die prematurely. There doesn't seem to be any clear-cut better topology for the purpose of synthesizing sinusoids, and any practical choice will have to be the result of some compromise between computational cost, robustness against instability and acceptable behavior for linearly interpolated coefficients.

References


Jean Laroche was born in Bordeaux (France) on July 20, 1963. He received his M.S. degree from the Ecole Polytechnique in 1986 and his Ph.D. degree from the Ecole Nationale Supérieure des Télécommunications, Paris in 1989 in Signal Processing. In 1990 he was a fellow of the ITT international grant at the Center for Music Experiment at the University of California at San Diego. In 1991, he became an assistant professor at Telecom Paris in the Signal Department, teaching audio and speech processing and acoustics. Since 1996, he has been a researcher at the Joint Emu/Creative Technology Center, helping design techniques for advanced music and audio processing.
Figure 1: Left: Gold-Rader, coupled second-order topology. Right: Second-order normalized ladder structure.

Figure 2: Left: Direct-Form II structure. Right: Second-order lattice topology.

Figure 3: Pole trajectories for linearly interpolated filter coefficients. The stars are for the direct form, the circles for the coupled form, and the crosses for the lattice topology.