A REMARK ON WEIGHTED REPRESENTATION FUNCTIONS

Zhenhua Qu

Abstract. Let $G$ be a finite abelian group, and $k_1, k_2$ be two integers. For any subset $A \subset G$, let $r_{k_1,k_2}(A, n)$ denote the number of solutions of $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. In this paper, we generalize a result of Q.-H. Yang and Y.-G. Chen to finite abelian groups. More precisely, we characterize all subsets $A \subset G$ such that $r_{k_1,k_2}(A, n) = r_{k_1,k_2}(G\setminus A, n)$ for all $n \in G$.

1. INTRODUCTION

Let $N$ be the set of nonnegative integers. For any subset $A \subset N$ and $n \in N$, let $R_1(A, n)$, $R_2(A, n)$ and $R_3(A, n)$ denote the number of solutions of $n = a + a'$ with $a, a' \in A$, $n = a + a'$ with $a, a' \in A$, $a < a'$, and $n = a + a'$ with $a, a' \in A$, $a \leq a'$ respectively. These representation functions are studied by Erdős, Sárközy and Sós in a series of papers [7, 8, 11, 9, 10]. Since then, representation functions have been extensively studied by many authors.

Sárközy asked, for each $i = 1, 2, 3$, whether there exist sets $A$ and $B$ with infinite symmetric difference such that $R_i(A, n) = R_i(B, n)$ for all sufficiently large integers $n$. Dombi [5] observed that the answer is negative for $i = 1$, and is affirmative for $i = 2$. Chen and Wang [3] gave an example of a subset $A \subset N$ such that $R_3(A, n) = R_3(N\setminus A, n)$ for all $n \geq 1$. For $i = 2, 3$, Lev, Sándor and Tang [6, 12, 13] characterized all subsets $A$ with the property that $R_i(A, n) = R_i(N\setminus A, n)$ for all $n \geq 2N + 1$. Some asymptotic results of the representation functions of these sets are obtained in [1, 2].

For any two positive integers $k_1 \leq k_2$, $A \subset N$ and $n \in N$, one can also define the weighted representation function $r_{k_1,k_2}(A, n)$ as the number of solutions of the equation $n = k_1a_1 + k_2a_2$ with $a_1, a_2 \in A$. If $k_2 \geq k_1 \geq 2$, Cillrelo and Ruè [4] proved that $r_{k_1,k_2}(A, n)$ can not be eventually constant. Yang and Chen [14] proved that there...
exists a set \( A \subset \mathbb{N} \) such that \( r_{k_1,k_2}(A,n) = r_{k_1,k_2}(\mathbb{N}\setminus A,n) \) for all sufficiently large \( n \) if and only if \( k_1 \mid k_2 \) and \( k_1 < k_2 \).

In a recent paper [15], Yang and Chen studied weighted representation functions on \( \mathbb{Z}_m \), the cyclic group of order \( m \). For any two integers \( k_1, k_2, A \subset \mathbb{Z}_m \) and \( n \in \mathbb{Z}_m \), define \( r_{k_1,k_2}(A,n) \) to be the number of solutions of the equation \( n = k_1a_1 + k_2a_2 \) with \( a_1, a_2 \in A \). For \( d \mid m \), \( A \) is said uniformly distributed modulo \( d \) if

\[
\{x : x \in A, x \equiv i \pmod{d}\} = \lfloor A/d \rfloor
\]

for all \( i = 0, 1, \ldots, d - 1 \). They proved the following theorem.

**Theorem A.** Let \( m, k_1, \) and \( k_2 \) be three integers with \( m \geq 2 \), \( A \subseteq \mathbb{Z}_m \). Then \( r_{k_1,k_2}(A,n) = r_{k_1,k_2}(\mathbb{Z}_m\setminus A,n) \) for all \( n \in \mathbb{Z}_m \) if and only if \( |A| = m/2 \) and \( A \) is uniformly distributed modulo \( d'_1 \) and \( d'_2 \), respectively, where \( d'_1 = (k_1,m) \) and \( d'_2 = (k_2,m)/(k_1,k_2,m) \).

In this paper, we generalize their results to finite abelian groups. We fix some notation first. Let \( G \) be a finite abelian group of order \( m \). For any two integers \( k_1, k_2, A \subset G \) and \( n \in G \), we define similarly the weighted representation function \( r_{k_1,k_2}(A,n) \) to be the number of solutions of the equation \( n = k_1a_1 + k_2a_2 \) with \( a_1, a_2 \in A \). For any integer \( k \), let \( kG \) denote the subgroup \( kG = \{kg : g \in G\} \), and \( G_k \) denote the subgroup \( G_k = \{g : g \in G, kg = 0\} \). For \( i = 1, 2 \), let \( d_i = (k_i,m) \), \( d_3 = (d_1,d_2) = (k_1,k_2,m) \), \( d_i = d'_1d_3 \). Then \( d'_1 \) and \( d'_2 \) are coprime. Let \( H_i = d'_1G + G_{d_3}, i = 1, 2 \). For a subgroup \( H < G \), we say that \( A \) is uniformly distributed modulo \( H \) if \( |A \cap (g + H)| \) is independent of \( g \in G \). The following results are proved.

**Theorem 1.** Let \( G \) be a finite abelian group of order \( m \), and \( k_1, k_2 \) be two integers, \( A \subset G \). With other notations introduced as above, then \( r_{k_1,k_2}(A,n) = r_{k_1,k_2}(G\setminus A,n) \) for all \( n \in G \) if and only if \( |A| = m/2 \) and \( A \) is uniformly distributed modulo \( H_1 \) and \( H_2 \), respectively.

**Corollary 1.** Notations as in Theorem 1. Then there exists a set \( A \subset G \) such that \( r_{k_1,k_2}(A,n) = r_{k_1,k_2}(G\setminus A,n) \) for all \( n \in G \) if and only if \( |H_1| \) and \( |H_2| \) are both even.

**Remark.** For \( G = \mathbb{Z}_m \), \( G_{d_3} = \frac{m}{d_3}\mathbb{Z}_m \). Since \( d'_i = d_i/d_3 \) is a divisor of \( m/d_3 \), \( \frac{m}{d_3}\mathbb{Z}_m \subset d'_i\mathbb{Z}_m \), hence \( H_i = d'_i\mathbb{Z}_m + \frac{m}{d_3}\mathbb{Z}_m = d'_i\mathbb{Z}_m \) for \( i = 1, 2 \). Theorem 1 is consistent with Theorem A in this case. In general, \( d'_iG \) may be a proper subset of \( H_i \). For example, \( G = \mathbb{Z}_{60} \oplus \mathbb{Z}_2, m = 120, k_1 = d_1 = 12, k_2 = d_2 = 10 \). Then \( d_3 = 2, d'_1 = 6, d'_2G = 6\mathbb{Z}_{60} \oplus 2\mathbb{Z}_2, G_{d_3} = 30\mathbb{Z}_{60} \oplus \mathbb{Z}_2 \), and \( G_{d_3} \) is not a subset of \( d'_1G \), thus \( d'_1G \) is a proper subset of \( H_1 = d'_1G + G_{d_3} \).
2. Proof of the Results

For any subsets $S, T \subset G$ and $n \in G$, let $r_{k_1,k_2}(S,T,n)$ denote the number of solutions of $n = k_1s + k_2t$ with $s \in S$ and $t \in T$. Let $\Phi_i(n) = \{g: g \in G, n - k_3i g \in k_iG \}$ for $i = 1, 2$. We need the following lemma.

**Lemma 1.** Let $i = 1, 2$. If $n \notin d_3G$, then $\Phi_i(n) = \emptyset$. If $n \in d_3G$, then $\Phi_i(n)$ is a coset of $H_i$ and $|\Phi_i(n)| = |H_i|$. For any $A \subset G$ and $n \in G$,

$$r_{k_1,k_2}(G,A,n) = |A \cap \Phi_1(n)| \cdot |G_{d_1}|,$$

and

$$r_{k_1,k_2}(A,G,n) = |A \cap \Phi_2(n)| \cdot |G_{d_2}|.$$

**Proof.** If $\Phi_i(n) \neq \emptyset$, say $g \in \Phi_i(n)$, then $n \in k_3i g + k_1G \subset d_3G$ since $d_3 \mid (k_1, k_2)$. Thus $n \notin d_3G$ implies $\Phi_i(n) = \emptyset$.

Suppose $n \in d_3G$. Since $d_3 = (k_1, k_2, m)$, write $d_3 = k_1u + k_2v + mw$ for some $u, v, w \in Z$. For any $g \in G$,

$$d_3g = (k_1u + k_2v + mw)g = k_1(ug) + k_2(vg) \in k_1G + k_2G,$$

therefore $k_1G + k_2G \supseteq d_3G$. On the other hand, $k_1G + k_2G \subset d_3G$, thus we conclude that $k_1G + k_2G = d_3G$. In particular, $n = k_1g_1 + k_2g_2$ for some $g_1, g_2 \in G$, therefore $g_3i \in \Phi_i(n)$ and $\Phi_i(n) \neq \emptyset$. Assume $g \in \Phi_i(n)$, then $h \in \Phi_i(n)$ if and only if $(g - h)k_3i \equiv k_1G = d_3G$. Since $(k_3i, d_3) = d_3$, it is equivalent to $g - h \in d_3G + G_{d_3} = H_i$, thus $\Phi_i(n)$ is a coset of $H_i$. In particular, $|\Phi_i(n)| = |H_i|$. If $a \in A$ and $g \in G$ satisfy $n = k_1a + k_2g$, then $n - k_1a \in k_2G$, thus $a \in A \cap \Phi_2(n)$. On the other hand, for any $a \in A \cap \Phi_2(n)$, there exists $g_0 \in G$ such that $n = k_1a + k_2g_0$ by the definition of $\Phi_2(n)$. Since

$$\{g: g \in G, n = k_1a + k_2g\} = g_0 + G_{k_2} = g_0 + G_{d_2},$$

which is a coset of $G_{d_2}$, we have $|\{g: g \in G, n = k_1a + k_2g\}| = |G_{d_2}|$. Therefore

$$r_{k_1,k_2}(A,G,n) = |\{(a, g): a \in A, g \in G, k_1a + k_2g = n\}| \cdots \cdot |G_{d_2}|,$$

which is a coset of $G_{d_2}$, we have $|\{g: g \in G, n = k_1a + k_2g\}| = |G_{d_2}|$. Therefore

$$r_{k_1,k_2}(A,G,n) = |\{(a, g): a \in A, g \in G, k_1a + k_2g = n\}| \cdots |G_{d_2}|.$$
is equivalent to
\[(r_{k_1,k_2}(A, A, n) + r_{k_1,k_2}(B, A, n)) + (r_{k_1,k_2}(A, A, n) + r_{k_1,k_2}(A, B, n))\]
\[= (r_{k_1,k_2}(A, B, n) + r_{k_1,k_2}(B, B, n)) + (r_{k_1,k_2}(B, A, n) + r_{k_1,k_2}(B, B, n)),\]
that is,
\[(2) \quad r_{k_1,k_2}(G, A, n) + r_{k_1,k_2}(A, G, n) = r_{k_1,k_2}(G, B, n) + r_{k_1,k_2}(B, G, n).\]

By Lemma 1, equality (2) is equivalent to
\[(3) \quad \sum_{i=1}^{2} |A \cap \Phi_i(n)| \cdot |G_{d_i}| = \sum_{i=1}^{2} |B \cap \Phi_i(n)| \cdot |G_{d_i}|.\]

If \(n \notin d_3G\), then \(\Phi_i(n) = \emptyset\) by Lemma 1, hence both sides of (3) are zero. Assume now \(n \in d_3G\), by Lemma 1, \(|\Phi_i(n)| = |H_i|\), hence
\[|A \cap \Phi_i(n)| + |B \cap \Phi_i(n)| = |\Phi_i(n)| = |H_i|.
\]

Adding both sides of equation (3), we see that (3) is equivalent to
\[(4) \quad \sum_{i=1}^{2} |A \cap \Phi_i(n)| \cdot |G_{d_i}| = \frac{1}{2} \sum_{i=1}^{2} |H_i| \cdot |G_{d_i}|\]
for all \(n \in d_3G\).

We now prove the sufficiency part. Assume \(|A| = m/2\), and \(A\) is uniformly distributed modulo \(H_1\) and \(H_2\), respectively. We shall show that equality (4) holds for all \(n \in d_3G\). By Lemma 1, \(\Phi_i(n)\) is a coset of \(H_i\), therefore \(|A \cap \Phi_i(n)| = |H_i|/2\), and
\[
\sum_{i=1}^{2} |A \cap \Phi_i(n)| \cdot |G_{d_i}| = \frac{1}{2} \sum_{i=1}^{2} |H_i| \cdot |G_{d_i}|.
\]

Next we prove the necessity part. Assume that equalities (1)-(4) are satisfied. Since
\[(5) \quad |A|^2 = \sum_{n \in G} r_{k_1,k_2}(A, n) = \sum_{n \in G} r_{k_1,k_2}(B, n) = |B|^2,
\]
we have \(|A| = |B|\), hence \(|A| = m/2\). Note that the right hand side of equality (4) is fixed. If \(n \in k_1G\), then
\[\Phi_1(n) = \{g : g \in G, k_2g \in k_1G\} = H_1,
\]
which is independent of \(n \in k_1G\). Consequently, \(|A \cap \Phi_2(n)|\) is independent of \(n \in k_1G\). When \(n\) runs through all elements of \(k_1G\), \(\Phi_2(n)\) runs through all cosets of \(H_2\) as we immediately see that \(g \in \Phi_2(k_1g)\) for all \(g \in G\). It follows that \(|A \cap (g + H_2)|\)
is independent of \( g \in G \), hence \( A \) is uniformly distributed modulo \( H_2 \). By similar arguments, \( A \) is also uniformly distributed modulo \( H_1 \).

Proof of Corollary 1. Assume there exists a subset \( A \subset G \) such that

\[
r_{k_1, k_2}(A, n) = r_{k_1, k_2}(G \setminus A, n)
\]

for all \( n \in G \). By Theorem 1, \( |A| = m/2 \) and \( A \) is uniformly distributed modulo \( H_1 \) and \( H_2 \) respectively. Therefore \( |A \cap H_i| = |H_i|/2 \), hence \( |H_i| \) is even for \( i = 1, 2 \).

Conversely, assume \( |H_i| \) is even for \( i = 1, 2 \). Since

\[
H_1 + H_2 \supseteq d_1'G + d_2'G = (d_1', d_2', m)G = G,
\]

we have \( H_1 + H_2 = G \). Let \( X_1, X_2, \ldots, X_s \) and \( Y_1, Y_2, \ldots, Y_t \) denote the cosets of \( H_1 \) and \( H_2 \) respectively. Put \( H = H_1 \cap H_2 \). By Chinese Remainder Theorem, we have \( G/H \cong G/H_1 \times G/H_2 \). Therefore

\[
X_i \cap Y_j, \quad 1 \leq i \leq s, 1 \leq j \leq t
\]

are all the cosets of \( H \), and \( |X_i| = |H_1| = t|H| \), \( |Y_j| = |H_2| = s|H| \) for \( 1 \leq i \leq s, 1 \leq j \leq t \).

Case 1. \( |H| \) is even. We take

\[
A = \bigcup_{1 \leq i \leq s} A_{ij},
\]

where \( A_{ij} \subset X_i \cap Y_j \) is any subset with \( |A_{ij}| = |H|/2 \). Then

\[
|A \cap X_i| = \left| \bigcup_{1 \leq j \leq t} (A \cap X_i \cap Y_j) \right| = \sum_{j=1}^{t} |A_{ij}| = t|H|/2 = |H_1|/2,
\]

that is, \( A \) is uniformly distributed modulo \( H_1 \). Similarly, \( A \) is uniformly distributed modulo \( H_2 \).

Case 2. \( |H| \) is odd. Since \( |H_2| \) is even and \( |H_2| = s|H| \), we see that \( s \) is even. Similarly, \( t \) is also even. Write \( s = 2k, t = 2l \), and we take

\[
A = \left( \bigcup_{1 \leq i \leq k} \left( X_i \cap Y_j \right) \right) \bigcup \left( \bigcup_{k+1 \leq i \leq 2k} \left( X_i \cap Y_j \right) \right).
\]

For \( 1 \leq i \leq k \),

\[
|A \cap X_i| = \sum_{j=1}^{l} |X_i \cap Y_j| = l|H| = |X_i|/2.
\]
For $k + 1 \leq i \leq 2k$,

$$|A \cap X_i| = \sum_{j=l+1}^{2l} |X_i \cap Y_j| = l|H| = |X_i|/2.$$ 

Thus $A$ is uniformly distributed modulo $H_1$. Similarly, $A$ is uniformly distributed modulo $H_2$. By Theorem 1, we have

$$r_{k_1,k_2}(A, n) = r_{k_1,k_2}(G \setminus A, n)$$

for all $n \in G$. This completes the proof of the corollary.

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REFERENCES


Zhenhua Qu
Department of Mathematics
Shanghai Key Laboratory of PMMP
East China Normal University
500 Dongchuan Rd.
Shanghai 200241
P. R. China
E-mail: zhqu@math.ecnu.edu.cn