POSITIVELY CURVED MANIFOLDS WITH LARGE CONJUGATE RADIUS

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Abstract. Let M denote a complete simply connected Riemannian manifold with all sectional curvatures ≥ 1. The purpose of this paper is to prove that when M has conjugate radius at least $\pi/2$, its injectivity radius and conjugate radius coincide. Metric characterizations of compact rank one symmetric spaces are given as applications.

1. Introduction

The Rauch-Berger-Klingenberg sphere theorem has inspired many results in the study of complete, connected, and positively curved manifolds [Rch51, Ber60, Kli61].

Theorem (Rauch-Berger-Klingenberg)**.** *If M is a simply connected manifold with $1 \leq \sec < 4$, then M is homeomorphic to a sphere.*

The simply connected compact rank one symmetric spaces satisfy $1 \leq \sec \leq 4$. Consequently, the above hypothesis that sectional curvatures are positive and *strictly* quarter pinched cannot be relaxed to allow quarter pinched positive sectional curvatures. Berger's rigidity theorem [Ber60] classifies manifolds with positive quarter pinched sectional curvatures.

Theorem (Berger)**.** *If M is a simply connected manifold with $1 \leq \sec \leq 4$, then M is homeomorphic to a sphere or isometric to a compact rank one symmetric space.*

The problem of improving homeomorphism to diffeomorphism in the above two theorems has very recently been resolved using Ricci flow (c.f. [BöWi08, BrSc09, BrSc08, NiWo07, PeTa09]). An important step in the original proofs involves estimating the injectivity radii of manifolds with quarter pinched positive sectional curvatures. In [Kli61, ChGr80, KlSa80], it is proved that if M satisfies the hypotheses of Berger's theorem, then its injectivity radius $\mathrm{inj}(M)$ and conjugate radius $\mathrm{conj}(M)$ are equal. By the Rauch comparison theorem, manifolds with $\sec \leq 4$ satisfy $\mathrm{conj}(M) \geq \pi/2$. In this paper, the upper curvature bound assumption is replaced by a lower bound on the conjugate radius.

Proposition 1. *Let M be a simply connected manifold with* $\sec \geq 1$. *If* $\mathrm{conj}(M) \geq \pi/2$*, then* $\mathrm{inj}(M) = \mathrm{conj}(M)$*.*

Proposition 1 is applied to prove three rigidity results. The first generalizes Berger's rigidity theorem.

Theorem 1. *Let M be a simply connected manifold with* $\sec \geq 1$. *If* $\mathrm{conj}(M) \geq \pi/2$*, then M is homeomorphic to a sphere or isometric to a compact rank one symmetric space.*

Theorem 1 is an easy consequence of propostion 1 and earlier generalizations of the sphere and Berger rigidity theorems [GrSh77, GrGr87, Wi01]. As with the injectivity radius, conjugate radius, and diameter, the radius of a manifold $\mathrm{rad}(M)$ has played an important role in rigidity results for positvely curved manifolds (c.f. [GrPe93, ShYa89, Wa04, Wi96, Xia09]). The second application of propositon 1 generalizes the following theorem due to Xia [Xia06].

Theorem (Xia)**.** *If M satisfies* $\sec \geq 1$ *and* $\mathrm{rad}(M) > \pi/2$*, then* $\mathrm{conj}(M) \leq \mathrm{rad}(M)$ *with equality if and only if M is isometric to a constant curvature sphere.*

Note that manifolds M with $\sec \geq 1$ and $\mathrm{rad}(M) > \pi/2$ are homeomorphic to a sphere by [GrSh77]. In particular, such manifolds are simply connected.

Theorem 2. *If M is simply connected and satisfies* $\sec \geq 1$ *and* $\mathrm{rad}(M) \geq \pi/2$*, then* $\mathrm{conj}(M) \leq \mathrm{rad}(M)$ *with equality if and only if M is isometric to a compact rank one symmetric space.*

A final application is motivated by the Shankar-Spatzier-Wilking spherical rank rigidity theorem [ShSpWi05].

Theorem (Shankar-Spatzier-Wilking)**.** *Let M be a simply connected manifold with* $\sec \leq 1$. *If for each unit speed geodesic $\gamma : \mathbb{R} \to M$, $\gamma(\pi)$ is the first conjugate point to $\gamma(0)$ along γ, then M is isometric to a compact rank one symmetric space.*

Theorem 3. *Let M be a simply connected manifold with* $\sec \geq 1$. *If for each unit speed geodesic $\gamma : \mathbb{R} \to M$, $\gamma(\pi/2)$ is the first conjugate point to $\gamma(0)$ along γ, then M is isometric to a compact rank one symmetric space.*

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2. Preliminaries

This section collects together preliminary material and notation. General references about Riemannian manifolds include [ChEb75], and [doCa92] though our notation differs in parts. Throughout, M denotes a closed Riemannian manifold and $\pi : TM \to M$ its tangent bundle. Let T_pM denote the fiber of TM above a point $p \in M$ and for $r > 0$, let $B_r(0) \subset T_pM$ denote the open r-ball centered at 0. Let $S_pM \subset T_pM$ denote the unit sphere.

The Riemannian metric induces an exponential map $\exp : TM \to M$. Its restriction to the tangent space T_pM at a point p will be denoted by $\exp_p : T_pM \to M$. The metric also induces a Riemannian connection denoted by ∇. This connection is used to define the curvature tensor R by

$$R(X, Y, Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

where X, Y, and Z are vector fields on M. Finally, let $\mathrm{d} : M \times M \to \mathbb{R}$ denote the distance function on M.

2.1. **Metric invariants.** A critical point $v \in T_pM$ of $\exp_p$ is defined to be a *conjugate vector*. Its multiplicity is defined to be the dimension of the kernel of the derivative map of $\exp_p$ at the vector v. The point $q = \exp_p(v)$ is said to be a *conjugate point* to p along the geodesic $\gamma(t) = \exp_p(tv)$ and its multiplicity as a conjugate point is defined to be the multiplicity of v as a conjugate vector. Equivalently, q is a conjugate point to p along γ if there is a nonzero normal *Jacobi field* $J(t)$ along γ with $J(0) = J(1) = 0$. Jacobi fields are vector fields along γ satisfying the second order differential equation

$$J'' + R(\dot{\gamma}, J)\dot{\gamma} = 0.$$

They are determined by their initial value $J(0)$ and initial derivative $J'(0)$.

Let $\mathrm{TConj}(p) \subset T_pM$ denote the locus of conjugate vectors in T_pM. The *conjugate radius of M at a point p* is defined by

$$\begin{aligned} \mathrm{conj}(p) &= \sup\{r > 0 \,|\, \mathrm{TConj}(p) \cap B_r(0) = \emptyset\} \\ &= \sup\{r > 0 \,|\, \exp_p|_{B_r(0)} \text{ has full rank}\} \end{aligned}$$

and the *conjugate radius* of M is defined by

$$\mathrm{conj}(M) = \inf\{\mathrm{conj}(p) \,|\, p \in M\}.$$

By the Cartan-Hadamard theorem, manifolds with a point p satisfying $\mathrm{conj}(p) = \infty$ have universal covers diffeomorphic to Euclidean space. By the Bonnet-Myers theorem, manifolds with $\sec \geq 1$ have compact universal covers and therefore finite conjugate radius at each point.

The *radius of M at a point p* is defined by $\mathrm{rad}(p) = \max_{x\in M} \mathrm{d}(p,x)$. The *radius* of M is defined by $\mathrm{rad}(M) = \min_{p\in M} \mathrm{rad}(p)$ and the *diameter* of M is defined by $\mathrm{diam}(M) = \max_{p\in M} \mathrm{rad}(p)$.

For a unit vector $v \in S_pM$, the geodesic $\gamma(t) = \exp_p(tv)$ satisfies $\mathrm{d}(p, \exp_p(tv)) = t$ for small $t > 0$. Define $\mu : S_pM \to \mathbb{R}$ by

$$\mu(v) = \max\{t > 0 \,|\, \mathrm{d}(p, \gamma(t)) = t\}.$$

The vector $\mu(v)v \in T_pM$ is said to be a *cut vector*. The function $v \mapsto \mu(v)$ is a continuous function on the unit sphere S_pM. It achieves its minimum value, the *injectivity radius of M at the point p*, denoted by $\mathrm{inj}(p) > 0$. Equivalently,

$$\mathrm{inj}(p) = \max\{r > 0 \,|\, \exp_p|_{B_r(0)} \text{ is a diffeomorphism onto its image}\}.$$

The *injectivity radius* of M is defined by

$$\mathrm{inj}(M) = \min\{\mathrm{inj}(p) \,|\, p \in M\}.$$

The inequalites $\mathrm{inj}(M) \leq \mathrm{rad}(M) \leq \mathrm{diam}(M)$ always hold. Geodesics do not minimize beyond their first conjugate point so that $\mathrm{inj}(M) \leq \mathrm{conj}(M)$. In fact, the injectivity radius equals the minimium of the conjugate radius and half the length of a shortest closed geodesic in M.

Denote the locus of cut vectors in T_pM by $\mathrm{TCut}(p)$. Its image in M under $\exp_p$, denoted by $\mathrm{Cut}(p)$, is the *cut locus of p in M*. Much of the topology of M is contained in the cut locus of a point $p \in M$. To be more precise, note that each point $q \in M \setminus \{p\}$ has a unique expression as $q = \exp_p(t_q v_q)$ for some $v_q \in S_pM$ and $0 < t_q < \mu(v_q)$. Then the map

$$r : M \setminus \{p\} \times [0,1] \to M \setminus \{p\}$$

defined by

$$(q,s) \mapsto \exp_p([t_q + s(\mu(v_q) - t_q)]v_q)$$

is a strong deformation retraction of $M \setminus \{p\}$ to $\mathrm{Cut}(p)$. In particular, $\mathrm{Cut}(p)$ is homotopy equivalent to $M \setminus \{p\}$.

2.2. **Indices of geodesics.** A reference for this section is [BaThZi82, Section 1], though our notation differs slightly.

Let $\Lambda = \Lambda(M)$ denote the path space of M. It consists of piecewise smooth curves $c : [0,1] \to M$. The energy function $E : \Lambda \to \mathbb{R}$ is defined by

$$E(c) = \int_0^1 \langle \dot{c}, \dot{c} \rangle \, dt$$

for $c \in \Lambda$. Critical points of E are point maps and geodesic segments parameterized proportionally to arc length.

Given a geodesic $\gamma \in \Lambda$, let $T_\gamma\Lambda$ denote the space of piecewise smooth normal vector fields $V(t)$ along γ with $V(0) = V(1) = 0$. The index form is the symmetric bilinear $I_\Lambda : T_c\Lambda \times T_c\Lambda \to \mathbb{R}$ defined by

$$I_\Lambda(X, Y) = \int_0^1 \langle X', Y' \rangle - \langle R(\dot\gamma, X)\dot\gamma, Y \rangle \, dt$$

for vector fields $X, Y \in T_\gamma\Lambda$.

For a fine enough subdivision $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0, 1]$, the index form I_Λ is positive definite on the subspace $P \subset T_\gamma\Lambda$ consisting of normal vector fields $V(t)$ with $V(t_i) = 0$ for each $i = 0, \ldots, n$. The orthogonal complement of P in $T_\gamma\Lambda$ with respect to I_Λ is the finite dimensional space of *piecewise Jacobi fields*: vector fields $V \in T_\gamma\Lambda$ such that the restriction of V to $[t_i, t_{i+1}]$ is Jacobi for each $i = 0, \ldots, n-1$. Thus, the index and nullity of a I_Λ are both finite.

The index of γ as a geodesic segment, denoted by $\mathrm{ind}_\Lambda(\gamma)$, is defined to be the index of the form I_Λ and the nullity of γ as a geodesic segment, denoted by $\mathrm{null}_\Lambda(\gamma)$, is defined to be the nullity of I_Λ. By the Morse index theorem, $\mathrm{ind}_\Lambda(\gamma)$ is equal to the number of points $\gamma(t)$ conjugated to $\gamma(0)$ along γ with $t \in (0, 1)$ and counted with multiplicities.

The index of a smoothly closed geodesic is defined analogously by considering variations of the closed geodesic in the free loop space of M. Let $\Omega = \Omega(M)$ denote the free loop space of M. It consists of piecewise smooth curves $c : [0, 1] \to M$ with $c(0) = c(1)$. The energy function defined on Λ restricts to the energy function $E : \Omega \to \mathbb{R}$. Its critical points are point maps and smoothly closed geodesics parameterized proportionally to arc length.

Given a closed geodesic $\gamma \in \Omega$, let $T_\gamma\Omega$ denote the space of piecewise smooth normal vector fields $V(t)$ along γ with $V(0) = V(1)$. The index form $I_\Omega : T_\gamma\Omega \times T_\gamma\Omega \to \mathbb{R}$ is defined by the same formula as I_Λ. As above, the index and nullity of I_Ω are both finite. The index of γ as a closed geodesic, denoted by $\mathrm{ind}_\Omega(\gamma)$, is defined to be the index of the form I_Ω and the nullity of γ as a closed geodesic, denoted by $\mathrm{null}_\Omega(\gamma)$, is defined to be the nullity of I_Ω.

The nullspace of I_Ω

$$\mathrm{null}(I_\Omega) = \{V \in T_\gamma\Omega \mid I_\Omega(V, X) = 0 \,\text{for all}\, X \in T_\gamma\Omega\}$$

consists of *periodic* normal Jacobi fields along γ. As Jacobi fields are determined by their initial value and derivative, these are normal Jacobi fields $J(t)$ along γ satisfying $J(0) = J(1)$ and $J'(0) = J'(1)$.

Note that since $T_\gamma\Lambda \subset T_\gamma\Omega$, $\mathrm{ind}_\Lambda(\gamma) \leq \mathrm{ind}_\Omega(\gamma)$. The difference $\mathrm{ind}_\Omega(\gamma) - \mathrm{ind}_\Lambda(\gamma)$ is known as the *concavity* of γ and its calculation involves the *Poincare map*. Let $L = \dot{\gamma}(0)^\perp$. The Poincare map is the linear map

$$\Theta : L \times L \to L \times L$$

defined by $\Theta(v, w) = (J(1), J'(1))$, where $J(t)$ is the normal Jacobi field along γ with initial conditions $J(0) = v$ and $J'(0) = w$. When $\Theta = \mathrm{Id}$ the concavity is zero so that $\mathrm{ind}_\Lambda(\gamma) = \mathrm{ind}_\Omega(\gamma)$ (c.f. [BaThZi82, pg. 219]).

This section concludes with the statement of a simple lemma. Let V be a finite dimensional real vector space and $I : V \times V \to \mathbb{R}$ a symmetric bilinear form.

Lemma 1. *Suppose that I has index one and that X is a subspace of V such that $I(x, x) \leq 0$ for each $x \in X$. Then*

$$\dim(X \cap \mathrm{null}(I)) \geq \dim(X) - 1.$$

A proof is readily obtained after diagonalizing I.

2.3. **Sublevels of E.** For $a > 0$, let $\Omega^{<a}$ (resp. $\Omega^{\leq a}$) denote the subset of curves $c \in \Omega$ with $E(c) < a$ (resp. $E(c) \leq a$). This section contains two lemmas from [ChGr80] concerning the components of these sublevel sets. Some notation is needed first.

A continuous curve $\tilde{c} : [0, 1] \to TM$ is said to be *vertical* if $c = \pi \circ \tilde{c}$ is constant. Let P denote the set of all continuous curves $c : [0, 1] \to M$ and $P_0 \subset P$ the subset of closed curves. Let $\tilde{P}$ denote the set of all vertical curves in TM emanating from the zero section and $\tilde{P}_0 \subset \tilde{P}$ the subset of closed curves. Endow P, $\tilde{P}$, and their subsets with the uniform topology. Let $\mathrm{Exp} : \tilde{P} \to P$ denote the continuous map induced by $\exp : TM \to M$. Its image is defined to be the set of *liftable* curves in M and a curve $\tilde{c} \in \tilde{P}$ is said to be a *lift* of the curve $c = \mathrm{Exp}(\tilde{c})$.

A lift $\tilde{c} \subset T_pM$ is said to be a *regular lift* if $\exp_p$ has full rank at $\tilde{c}(t)$ for each $t \in [0, 1]$. Let $\tilde{Q}$ denote the open subset of $\tilde{P}$ consisting of regular lifts. The restriction of Exp to $\tilde{Q}$ is a homeomorphism onto its image $Q = \mathrm{Exp}(\tilde{Q})$. Let $\tilde{Q}_0 = \tilde{P}_0 \cap \tilde{Q}$ denote the set of closed regular lifts. The map Exp maps $\tilde{Q}_0$ onto an open subset Q_0 of P_0.

Finally, for a curve $c \in P$, let $c_{\frac{1}{2}}$ and $c_{-\frac{1}{2}}$ be the curves defined by $c_{\frac{1}{2}}(t) = c(\frac{1}{2}t)$ and $c_{-\frac{1}{2}}(t) = c(1 - \frac{1}{2}t)$.

The next two lemmas are lemmas 1 and 3 from [ChGr80] respectively. They will be applied with $r = \mathrm{conj}(M)$.

Lemma 2. *Suppose for some* $r > 0$ *and all* $p \in M$, $\exp_p$ *is nonsingular on the open ball* $B_r(0) \subset T_pM$. *Let* $c \in P_0$ *be in the closure of* $\mathrm{Exp}(\tilde{P}_0) \cap \Omega^{\leq 4r^2}$. *Then either* $c \in Q_0$ *(and thus is not a closed geodesic if nonconstant), or* $c_{\frac{1}{2}}$ *and* $c_{-\frac{1}{2}}$ *are geodesics of length* r *with conjugate endpoints. Furthermore* $C = Q_0 \cap \Omega^{<4r^2}$ *is a connected component of* $\Omega^{<4r^2}$.

Lemma 3. *Assume the hypotheses of lemma 2 and furthermore that any closed geodesic* $c \in \overline{Q}_0 \cap \Omega^{\leq 4r^2}$ *(necessarily of length 2r) has index at least two. Then* $\overline{Q}_0 \cap \Omega^{\leq 4r^2}$ *is the closure of* $Q_0 \cap \Omega^{<4r^2}$ *and is a connected component of* $\Omega^{\leq 4r^2}$

3. Proofs of main results

The main new technical observation in this paper consists of the following lemma.

Lemma 4. *Assume that* M *is an oriented Riemannian manifold with* $\sec \geq 1$ *and* $n = \dim(M)$ *odd. If* $\gamma \subset M$ *is a closed geodesic of length* π, *then the index of* γ *in the free loop space* $\Omega(M)$ *is not one.*

Proof. The proof is by contradiction. Assume that $\gamma \subset M$ is a closed geodesic of length π with $\mathrm{ind}_\Omega(\gamma) = 1$. Let v be a tangent vector to γ of length π and fix the parameterization of γ defined by $\gamma(t) = \exp(tv)$. Let $L(t) = \dot{\gamma}(t)^\perp$ and let $L = L(0)$. For each $t \in \mathbb{R}$, parallel translation along γ defines an orientation preserving isometry $P_t : L \to L(t)$. Let $P = P_1 : L \to L$.

The first step is to show that $P = -\mathrm{Id}$. Consider the $(n-1)$-dimensional space of vector fields along γ defined by

$$X = \{\sin(\pi t)E(t) \,|\, E(t) \text{ is a unit normal parallel field along } \gamma\}.$$

Note that $X \subset T_\gamma\Omega$ and that for each $V \in X$,

$$\begin{aligned} I_\Omega(V,V) &= \int_0^1 \pi^2\cos^2(\pi t) - \pi^2\sin^2(\pi t)\sec(\dot{\gamma},E)(t)\,dt \\ &\leq \int_0^1 \pi^2\cos^2(\pi t) - \pi^2\sin^2(\pi t)\,dt = 0 \end{aligned}$$

since $\sec \geq 1$. As $\mathrm{ind}_\Omega(\gamma) = 1$, lemma 1 implies that

$$\dim(X \cap \mathrm{null}(I_\Omega)) \geq \dim(X) - 1 = n - 2.$$

A vector field $V(t) = \sin(\pi t)E(t) \in X \cap \mathrm{null}(I_\Omega)$ is a periodic Jacobi field. In particular, $V'(0) = V'(1)$, whence $E(0) = -E(1)$. Therefore -1 is an eigenvalue for $P : L \to L$ with multiplicity at least $n-2$. Since $\dim(L) = n-1$ is even and $P : L \to L$ is an orientation preserving isometry, -1 must have multiplicity $n-1$, concluding the proof that $P = -\operatorname{Id}$.

Let $E(t)$ be an arbitrary unit normal parallel field along γ. Consider the two dimensional space of vector fields along γ defined by

$$Y = \{(a\cos(\pi t) + b\sin(\pi t))E(t) \,|\, a, b \in \mathbb{R}\}.$$

As $P = -\operatorname{Id}$ we have that $V(0) = V(1)$ for each vector field $V \in Y$. Therefore, Y is a subspace of $T_\gamma\Omega$. Since $\sec \geq 1$,

$$I(V,V) =$$

$$\pi^2 \int_0^1 (-a\sin(\pi t) + b\cos(\pi t))^2 - (a\cos(\pi t) + b\sin(\pi t))^2 \sec(\dot{\gamma}, E)(t)\, dt$$

$$\leq \pi^2 \int_0^1 (-a\sin(\pi t) + b\cos(\pi t))^2 - (a\cos(\pi t) + b\sin(\pi t))^2\, dt = 0$$

with equality if and only if $\sec(\dot{\gamma}, E)(t) \equiv 1$. As $\operatorname{ind}_\Omega(\gamma) = 1$, lemma 1 implies that

$$\dim(Y \cap \operatorname{null}(I_\Omega)) \geq \dim(Y) - 1 = 1.$$

In particular, there exists a nonzero field $V \in Y$ with $I(V,V) = 0$ so that $\sec(\dot{\gamma}, E) \equiv 1$. It follows easily that the space Y consists of periodic Jacobi fields.

The fact that the unit parallel field $E(t)$ was arbitrary in the last paragraph has two consequences. First, the Poincare map

$$\Theta : L \times L \to L \times L$$

is the identity map. Therefore, $\operatorname{ind}_\Omega(\gamma) = \operatorname{ind}_\Lambda(\gamma)$. Secondly, all normal sectional curvatures of γ are one. The Rauch comparison theorem then implies that $\gamma(t)$ is not conjugated to $\gamma(0)$ for any $0 < t < 1$. By the Morse index theorem $\operatorname{ind}_\Lambda(\gamma) = 0$. In conclusion, $0 = \operatorname{ind}_\Lambda(\gamma) = \operatorname{ind}_\Omega(\gamma) = 1$, a contradiction. □

Next is the proof of proposition 1. The proof is well known given lemma 4 and follows the line of reasoning in [ChGr80] closely.

Proposition 1. *Let M be a simply connected manifold with* $\sec \geq 1$. *If* $\operatorname{conj}(M) \geq \pi/2$, *then* $\operatorname{inj}(M) = \operatorname{conj}(M)$.

Proof. The proof is by contradiction. Assume that $\operatorname{inj}(M) < \operatorname{conj}(M)$. As $\operatorname{inj}(M)$ always equals the smaller of $\operatorname{conj}(M)$ and half the length of a shortest closed geodesic in M, there is a closed geodesic $\tau \subset M$ of length $2\operatorname{inj}(M)$.

A standard argument implies that the geodesic τ has index zero in Ω. To see this, fix $p \in \tau$ and let $p' \in \tau$ be the point at distance inj(M) from p. The geodesic τ consists of two subsegments τ_1 and τ_2 connecting p to p' and of length inj(M). If $\mathrm{ind}_\Omega(\tau) > 0$, then there is a third minimizing geodesic τ_3 joining p to p' and meeting τ orthogonally at the point p' (c.f. the proof of [doCa92, Proposition 3.4, pg. 281]). Since $\mathrm{inj}(M) < \mathrm{conj}(M)$, the points p and p' are not conjugated along either of the segments γ_1 and γ_3. As these segments do not meet smoothly at p' it is possible to find a point $p'' \in \mathrm{Cut}(p)$ nearer to p than p' (c.f. [doCa92, Proposition 2.12, pg. 274]). This contradicts $\mathrm{d}(p, p') = \mathrm{inj}(M)$.

If $\dim(M)$ is even, Synge's trick implies that $\mathrm{ind}_\Omega(\tau) > 0$, a contradiction (c.f. the proof of [doCa92, Propositon 3.4, pg. 274]). From now on, assume that $\dim(M)$ is odd and at least three.

As M is simply connected, τ is null-homotopic. The Abresch-Meyer long homotopy lemma [AbMe97, Lemma 4.1] implies that every null-homotpy of τ passes through a curve of length at least $2\,\mathrm{conj}(M) \geq \pi$. It follows that for $e_0 = 4\,\mathrm{conj}(M)^2$ the space $\Omega^{<e_0}$ is disconnected.

Since $\sec \geq 1$, any closed geodesic of energy greater than e_0 has index at least $\dim(M) - 1 \geq 2$ in Λ and hence in Ω (c.f. [doCa92, Lemma 3.2, pg 276]). A consequence of this fact and the simple connectivity of M is that $\Omega^{\leq e_0}$ is connected (c.f. [ChGr80, Lemma 4, pg. 440]). This will be used to argue that $\Omega^{<e_0}$ is connected as well, the desired contradiction.

The first step is to see that each closed geodesic $\gamma \in \overline{Q}_0 \cap \Omega^{\leq e_0}$ (necessarily of length $2\,\mathrm{conj}(M)$) has index at least two in Ω. If $\mathrm{conj}(M) > \pi/2$, then this follows immediately from the remark at the beginning of the last paragraph. Otherwise, $\mathrm{conj}(M) = \pi/2$ and γ has length π. By lemma 2, $\mathrm{ind}_\Omega(\gamma) \geq 1$ and by lemma 4, $\mathrm{ind}_\Omega(c) \geq 2$, concluding this step.

Now lemma 3 implies that $\overline{Q}_0 \cap \Omega^{\leq e_0}$ is the closure of $Q_0 \cap \Omega^{<e_0}$ and is a connected component of $\Omega^{\leq e_0}$. As $\Omega^{\leq e_0}$ is connected, it follows that $\overline{Q}_0 \cap \Omega^{\leq e_0} = \Omega^{\leq e_0}$. In particular $Q_0 \cap \Omega^{<e_0}$ is dense in $\Omega^{\leq e_0}$ and hence in $\Omega^{<e_0}$. By lemma 2, $Q_0 \cap \Omega^{e_0}$ is a connected component of $\Omega^{<e_0}$. These last two remarks imply that $\Omega^{<e_0}$ is connected, a contradiction.

□

Theorem 1 is an easy consequence of proposition 1 and the next two theorems. The first theorem is due to Grove-Shiohama [GrSh77] and is a generalization of the sphere theorem.

Theorem 4 (Diameter sphere theorem)**.** *If M is a manifold with* $\sec \geq 1$ *and* $\mathrm{diam}(M) > \pi/2$*, then M is homeomorphic to a sphere.*

The second theorem is due to Gromoll-Grove-Wilking [GrGr87, Wi01]. It is a generalization of Berger's rigidity theorem. For a recent alternative proof, see [CaTa07].

Theorem 5 (Diameter rigidity theorem)**.** *If M is a simply connected manifold with* $\sec \geq 1$ *and* $\mathrm{diam}(M) = \pi/2$*, then M is homeomorphic to a sphere or isometric to a compact rank one symmetric space.*

Theorem 1. *Let M be a simply connected manifold with* $\sec \geq 1$*. If* $\mathrm{conj}(M) \geq \pi/2$*, then M is homeomorphic to a sphere or isometric to a compact rank one symmetric space.*

Proof. Proposition 1 implies that $\mathrm{inj}(M) = \mathrm{conj}(M) \geq \pi/2$. Therefore $\mathrm{diam}(M) \geq \mathrm{inj}(M) \geq \pi/2$. The conclusion follows from the diameter sphere and rigidity theorems 4 and 5. □

Theorem 2. *If M is simply connected and satisfies* $\sec \geq 1$ *and* $\mathrm{rad}(M) \geq \pi/2$*, then* $\mathrm{conj}(M) \leq \mathrm{rad}(M)$ *with equality if and only if M is isometric to a compact rank one symmetric space.*

Proof. The proof of the inequality $\mathrm{conj}(M) \leq \mathrm{rad}(M)$ is by contradiction. If $\mathrm{conj}(M) > \mathrm{rad}(M) \geq \pi/2$, then $\mathrm{inj}(M) = \mathrm{conj}(M) > \mathrm{rad}(M)$ by proposition 1. This is a contradiction since $\mathrm{inj}(M) \leq \mathrm{rad}(M)$ always holds.

Now consider the equality case $\mathrm{conj}(M) = \mathrm{rad}(M) \geq \pi/2$. Theorem 1 implies that M is isometric to a compact rank one symmetric space or homeomorphic to a sphere. Moreover, proposition 1 implies that $\mathrm{inj}(M) = \mathrm{conj}(M) = \mathrm{rad}(M)$. Therefore, the conclusion follows from the following lemma.

Lemma. *Assume that M is homeomorphic to a sphere and that* $\mathrm{inj}(M) = \mathrm{rad}(M)$*. Then M is isometric to a constant curvature sphere.*

By the resolution of the Blaschke conjecture for spheres (c.f. [Ber78], [Kaz78], [Wei74], and [Ya80]), a Riemannian metric on the sphere with $\mathrm{inj}(M) = \mathrm{diam}(M)$ is isometric to a constant curvature sphere. It suffices to prove that $\mathrm{diam}(M) \leq \mathrm{inj}(M)$.

Choose a point $p \in M$ with $\mathrm{rad}(p) = \mathrm{rad}(M)$. As $\mathrm{inj}(M) = \mathrm{rad}(M)$, all points $q \in \mathrm{Cut}(p)$ are at distance $\mathrm{inj}(M)$ from p. In the language of [Bes78, Definition 5.22, pg. 132], M is said to have spherical cut locus at the point p. By [Bes78, Proposition 5.44, pg. 138] and [Bes78, Proposition 5.39, pg. 136], it follows that $\mathrm{Cut}(p)$ is a smooth closed submanifold of M. Since $\mathrm{Cut}(p)$ is homotopy equivalent to $\mathbb{R}^n$ via the strong deformation retraction from $M \setminus \{p\}$ to $\mathrm{Cut}(p)$, it follows that $\mathrm{Cut}(p) = \{q\}$ for some $q \in M$.

Choose points $x, y \in M$ such that $d(x,y) = \mathrm{diam}(M)$. Then x and y each lie in a geodesic of length $\mathrm{inj}(M)$ connecting p to q. Denote these geodesics by γ_x and γ_y. If $\gamma_x = \gamma_y$, then clearly $\mathrm{d}(x,y) \leq \mathrm{inj}(M)$. Otherwise x and y lie in the embedded circle $\gamma_x \cup \gamma_y \subset M$ of length $2\,\mathrm{inj}(M)$, whence $\mathrm{d}(x,y) \leq \mathrm{inj}(M)$. In either case, $\mathrm{diam}(M) = \mathrm{d}(x,y) \leq \mathrm{inj}(M)$, concluding the proof.

□

Theorem 3. *Let M be a simply connected manifold with* $\sec \geq 1$. *If for each unit speed geodesic $\gamma : \mathbb{R} \to M$, $\gamma(\pi/2)$ is the first conjugate point to $\gamma(0)$ along γ, then M is isometric to a compact rank one symmetric space.*

Proof. The hypotheses imply that $\mathrm{conj}(M) = \pi/2$ and that $\mathrm{rad}(M) \leq \pi/2$ since geodesics do not minimize beyond their first conjugate point. By proposition 1, $\pi/2 = \mathrm{conj}(M) = \mathrm{inj}(M) \leq \mathrm{rad}(M) \leq \pi/2$ so that $\mathrm{rad}(M) = \pi/2$ as well. The conclusion follows from theorem 2. □

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