Finite time blow-up and condensation for the bosonic Nordheim equation.

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Abstract. The homogeneous bosonic Nordheim equation is a kinetic equation describing the dynamics of the distribution of particles in the space of moments for a homogeneous, weakly interacting, quantum gas of bosons. We show the existence of classical solutions of the homogeneous bosonic Nordheim equation that blow up in finite time. We also prove finite time condensation for a class of weak solutions of the kinetic equation.
Key words. Bosons, Nordheim Boltzmann equation, bounded solution, blow up, Bose-Einstein condensation.

1 Introduction

The dynamics of the distribution of particles in the space of moments $F(t,p)$ for a homogeneous, weakly interacting, quantum gas of bosons can be described by the Nordheim equation:

$$\partial_t F_1 = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} q(F)\,\mathcal{M} d^3p_2 d^3p_3 d^3p_4\,, \quad p_1 \in \mathbb{R}^3\,, \quad t>0, \tag{1.1}$$

$$F_1(0,p) = F_0(p)\,, \quad p_1 \in \mathbb{R}^3, \tag{1.2}$$

$$q(F) = q_3(F) + q_2(F)\,, \qquad \epsilon = \frac{|p|^2}{2}, \tag{1.3}$$

$$\mathcal{M} = \mathcal{M}(p_1,p_2;p_3,p_4) = \delta(p_1+p_2-p_3-p_4)\,\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)\,, \tag{1.4}$$

$$q_3(F) = F_3F_4(F_1+F_2) - F_1F_2(F_3+F_4)\,, \tag{1.5}$$

$$q_2(F) = F_3F_4 - F_1F_2, \tag{1.6}$$

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where we use the notation $F_j = F(t, p_j)$, $j \in \mathbb{R}^3$. The system of units has been chosen in order to have particles with mass equal to one. This system of equations was formulated by Nordheim (cf. [23]). One of the main reasons that explain why this system has been extensively studied in the physical literature is because it has been considered by several authors as a convenient way to approximate the dynamics of formation of Bose-Einstein condensates.

The system (1.1)-(1.6) can be thought as a generalization of the classical Boltzmann equation of gas dynamics. The main difference between both types of equations is the presence of the terms $q_3(F)$ (cf. (1.5)). These terms are cubic in the distribution of particles F in spite of the fact that only binary collisions are taken into account in the derivation of (1.1)-(1.6). The reason for the onset of the cubic terms $q_3(F)$ in (1.1) is that the counting of the number of collisions yielding the evolution of F must be made using Bose statistics, instead of the classical statistics which is used to compute the number of collisions among different types of particles, in the derivation of Boltzmann equation (cf. [14], [23]). From the physical point of view it is possible to acquire an intuitive picture about the quantum statistical effects assuming that the phase space is divided in a family of small macroscopic domains, but large enough to accomodate a large number of quantum cells with a volume of the order of the cube of Planck's constant. The distribution function F can then be characterized by the occupation numbers of each macroscopic cell. In the bosonic case the the wave function describing the whole system must be symmetric with respect to permutations of all the particles. The main consequence of this is that computing the number of collisions between particles Bose-Einstein statistics must be used instead of classical statistical distributions. If we choose the units of length and momentum to have as F the total number of occupied states in a given cell with respect to the total number of quantum states admissible in such a cell, it turns out the the rate of change of the number of occupied states at a point p_1, due to interactions with particles in $[p_2, p_2 + d^3p_2]$ and yielding particles with moments $[p_3, p_3 + d^3p_3]$, $[p_4, p_4 + d^3p_4]$ respectively, is given by:

$$\mathcal{M}\left[F_3F_4(1 + F_1 + F_2) - F_1F_2(1 + F_3 + F_4)\right] d^3p_2 d^3p_3 d^3p,$$

with $\mathcal{M}$ as in (1.4), and where we have also assumed that the unit of time has been choosen in order to obtain relevant changes of F in time scales of order one.

The presence of the terms $q_3(F)$ in (1.1) produces important differences between the behaviour of the solutions of the Nordheim equation and Boltzmann equation. One of the main differences between the dynamical behaviour of the solutions of both equations is that, as we will prove in this paper, there exist solutions of the Nordheim equation which are initially bounded and decay sufficiently fast at infinity to have finite energy but become unbounded in finite time. This behaviour is very different from the behaviour of the solutions of Boltzmann equation. Indeed, T. Carleman proved already, (cf. [3], [4]), that the solutions of the homogeneous Boltzmann equation, with the cross section associated to hard spheres interactions, and initial data bounded by $\left(1 + |p|^2\right)^{-\gamma}$ with $\gamma > 3$ are globally defined and bounded in time (see [32] for a more detailed discussion on that subject). In this paper we will show the existence of a large class of solutions of (1.1)-(1.6) which satisfy similar boundedness conditions for large values of $|p|$ but become unbounded for finite p.

Blow up in finite time for the solutions of (1.1)-(1.6) has been conjectured in the physical literature on the basis of numerical simulations and physical arguments (cf. [15], [16], [17], [18], [27], [28], [31]). These studies are restricted to spherically symmetric distributions F. According to the numerical simulations in those papers, many solutions of (1.1)-(1.6) which are initially bounded develop a singularity in finite time at $p = 0$. The behaviour of this solution at the time of the formation of the singularity can be described by means of a power law $|p|^{-\alpha}$. The time T where such blow up takes place is usually considered to be the time at which a Bose-Einstein condensate is formed. According to the numerical simulations in [16], [27], [28] the number of particles at the value $p = 0$ at the time $t = 0$ is zero, due to the integrability of the singularity there. However, the scenarios for the evolution of the distribution function F suggested in [15], [16], [27], [28] indicate that for times $t > T$ a macroscopic fraction of particles appears at the point $p = 0$. In mathematical terms, the distribution F becomes a measure containing a Dirac mass at $p = 0$ for times $t > T$. Kinetic equations describing the joint evolution of distribution functions in the presence of a condensed part have also been also obtained in several other articles of the physical literature using different types of physical approximations (cf. [1], [6], [10], [13], [25], [30], [33]).

The different behaviour for the solutions of the Boltzmann and Nordheim equations is due to the presence of the cubic terms $q_3(F)$ in (1.1), i.e. the terms not included in the classical Boltzmann equation are a consequence of using the Bose-Einstein statistics. These terms are the dominant ones for large densities, and they are the ones yielding blow-up for F.

The cross section that appears in (1.1)-(1.6) is constant for all the collisions preserving momentum and energy. This is the cross section commonly used in the physical literature and it is usually justified on the basis of the so-called Born approximation. The underlying idea behind this approximation in the simplest case is the following. It is assumed that at the fundamental level the system of quantum particles can be described by a hamiltonian of the form $H = H_0 + H_1$ where H_0 is the hamiltonian describing a noninteracting system with N particles. The term H_0 can include also some confining potential or equivalently some boundary conditions ensuring that the particles remain in a bounded region. The eigenvectors of H_0 can be labelled by means of a set of variables, and the macroscopic states will be defined assuming that a large number of these states are contained in each of them. On the other hand, the term H_1 is typically a pair interaction potential which consists in the sum over all possible pairs of particles of energy interactions induced by a potential V. The kinetic description given by Nordheim equation is expected to be a good approximation of this quantum system if the number of particles N tends to infinity and the interaction potentials and typical particle energies are rescaled in a suitable way. However, no rigorous derivation of Nordheim equation taking as starting point a Hamiltonian system is currently available, although there exist some partial results in this direction (cf. [26]).

Born's approximation allows to obtain the transition probabilities between two given states of the system. The applicability of this approximation requires to have an integrable interaction potential between particles V whose range of interaction is much shorter than the characteristic De Broglie length associated to the particles of the system. However, the constancy of the differential cross section in the center of mass system also takes place in some cases in which the Born approximation is not strictly valid, for instance for hard spheres with a radius much smaller than the characteristic De Broglie length of the system (cf. for instance [12]).

It is clarifying to rewrite (1.1) in the center of mass reference system. This allows to obtain a precise geometrical interpretation of the meaning of Born's approximation. Notice that we assume that the mass of particles is one (see the formula for ϵ in (1.3)). In order to perform some of the integrations in (1.1) we use the change of variables $(p_2, p_3, p_4) \to (p_2, P, Q)$ where:

$$P = \frac{p_3 + p_4}{2} \quad , \quad Q = \frac{p_3 - p_4}{2} = \frac{(p_3 - P) - (p_4 - P)}{2}.$$

Notice that since the mass of the particles has been normalized to one, P is the velocity of the center of mass of the system. On the other hand Q is a vector along the direction connecting the vectors p_3 and p_4. This vector is invariant under change of reference system. It will provide a measure of the deviation of the vector connecting the directions in the center of mass system.

Notice that:

$$\delta\left(p_1 + p_2 - p_3 - p_4\right) \cdot \delta\left(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4\right) = \delta\left(p_1 + p_2 - 2P\right) \delta\left(\frac{\left(p_1 - p_2\right)^2}{4} - Q^2\right),$$

$$d^3p_3 d^3p_4 = 2^3 d^3P d^3Q.$$

We also write Q in spherical coordinates and write p_3, p_4 in terms of $P,\ Q$:

$$Q = |Q|\,\omega \quad \text{with} \quad \omega \in S^2 \quad , \quad p_3 = P + Q \quad , \quad p_4 = P - Q.$$

Then, after some computations we can rewrite (1.1) as:

$$\partial_t F_1 = 8 \int_{\mathbb{R}^3} \int_{S^2} q\left(F\right)\Big|_{P = \frac{p_1 + p_2}{2}, |Q| = \frac{|p_1 - p_2|}{2}} \ |Q|\, d^3p_2 d^2\omega.$$

It becomes apparent from this formula that for any incoming direction p_2 colliding with p_1 the conservation of energy in the center of mass system implies $|Q| = \frac{|p_1 - p_2|}{2}$ and the direction of the vector Q is chosen as a homogeneous probability measure in the unit sphere S^2. This choice of the direction of the outgoing particles by means of a uniformly distributed measure in the space of all directions is the main distinguished feature of Born's approximation.

It is relevant to mention that from the physical point of view, the stationary solutions of (1.1), (1.3)-(1.6) might be expected to be the Bose-Einstein distributions:

$$F_{BE}(p) = m_0 \delta(p - p_0) + \frac{1}{\exp\left(\frac{\beta |p-p_0|^2}{2} + \alpha\right) - 1} \tag{1.7}$$

where $m_0 \geq 0$, $\beta \in (0, \infty]$, $0 \leq \alpha < \infty$ and $\alpha \cdot m_0 = 0$, $p_0 \in \mathbb{R}^3$. Notice, however that the precise sense in which the measures F_{BE} are stationary solutions of (1.1), (1.3)-(1.6) is not clear, because the right-hand side of (1.1) is not well defined for measures containing Dirac masses. There are several possible ways of justifying that the measures F_{BE} in (1.7) are, at least in some sense, steady state distributions of (1.1), (1.3)-(1.6). First notice that $q(F_{BE}) = 0$, with $q(\cdot)$ as in (1.5), (1.6). Nevertheless this argument must be taken with some care, because the in general is not possible to define the right-hand side of (1.1) for more general measures containing Dirac masses. Another way to see that the distributions F_{BE} must play the role of the stationary solutions for (1.1), (1.3)-(1.6) is to notice that these distributions are maximizers of the entropy associated to the system for a given value of the number of particles and energy. More precisely, the entropy for unit of volume of a homogeneous system of bosons with distribution function F in the space p is given by:

$$S = \int \left[(1+F)\log(1+F) - F\log(F)\right] d^3p. \tag{1.8}$$

This functional is increasing along the solutions of (1.1), (1.3)-(1.6). On the other hand it can be seen that the maximizer of the functional (1.8) with the constraints $\int F(p)\, d^3p = M$, $\int F(p)\, p d^3p = p_0$, $\int F(p) \frac{|p|^2}{2} d^3p = E$ is given by the distribution F_{BE} (cf. [11]).

We will restrict our analysis in the following to isotropic distributions. Therefore:

$$F(t,p) = F(t, \mathcal{R}p), \quad \mathcal{R} \in SO(3), \quad p \in \mathbb{R}^3 \quad , \quad t \geq 0.$$

It then follows that there exists a function $f = f(\epsilon, t)$ where ϵ is as in (1.3) such that:

$$f(t,\epsilon) = F(t,p).$$

Given a spherically symmetric solution of (1.1)-(1.6) it is possible to write an equation for $f(t,\epsilon)$. (see [28] for details). We first use the formula

$$\delta(p_1 + p_2 - p_3 - p_4) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ik\cdot(p_1+p_2-p_3-p_4)} d^3k,$$

which is valid in the sense of distributions. We next write $p_2,\ p_3,\ p_4$ as well as k using spherical coordinates. Integrating the angular variables, (1.1) becomes:

$$\partial_t f_1 = 32\pi \int_0^\infty \int_0^\infty \int_0^\infty Dq(f)\, \delta(\epsilon_1 + \epsilon_2 - \epsilon_3 - \epsilon_4)\, d\epsilon_2 d\epsilon_3 d\epsilon_4, \tag{1.9}$$

where:

$$D = \frac{1}{|p_1|}\int_0^\infty \left[\prod_{j=1}^{4} \sin\left(|p_j|\,\lambda\right)\right]\frac{d\lambda}{\lambda^2}.$$

This integral can be explicitly computed by means of elementary arguments. Using the fact that $|p_1|^2 + |p_2|^2 = |p_3|^2 + |p_4|^2$ which is due to the presence of the Dirac mass in (1.9) it follows that (cf. [28]):

$$D = \frac{\pi}{4}\min\left\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\right\}$$

whence, integrating the Dirac mass with respect to the variable ϵ_2 we obtain:

$$\partial_t f_1 = \frac{8\pi^2}{\sqrt{2}}\int_0^\infty\int_0^\infty q(f)\,W d\epsilon_3 d\epsilon_4, \tag{1.10}$$

where:

$$W = \frac{\min\left\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\right\}}{\sqrt{\epsilon_1}}, \quad \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1, \tag{1.11}$$

and $q(\cdot)$ is as in (1.3) with $\epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$.

It is worth to notice that $f(t,\epsilon)$ is not a particle density in the energy space. The density of particles in the energy space is actually given by:

$$g(t,\epsilon) = 4\pi\sqrt{2\epsilon} f(t,\epsilon). \tag{1.12}$$

We can rewrite (1.10), (1.11) using the density g :

$$\partial_t g_1 = 32\pi^3\int_0^\infty\int_0^\infty q\left(\frac{g}{4\pi\sqrt{2\epsilon}}\right)\Phi d\epsilon_3 d\epsilon_4, \tag{1.13}$$

$$\Phi = \min\left\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{\epsilon_4}\right\} \quad , \quad \epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1. \tag{1.14}$$

In spite of the fact that f is not the density of particles in the space of energy, the formulation of the problem (1.10), (1.11) is suitable in order to prove local well-posedness results for classical solutions. On the other hand, in order to understand phenomena like Bose-Einstein condensation, or in general any phenomenon characterized by the presence of a positive amount of particles near the origin, it is often more convenient to use the density of particles g. Indeed, in such a case the formation of condensates is characterized by the onset of a Dirac mass at $\epsilon = 0$. In this paper we will switch between both functions f, g depending on the question under consideration.

Equation (1.13) shows that Dirac mass distributions of g placed at the origin are too singular to yield a well defined right-hand side. Dirac masses at $\epsilon = \epsilon_0 > 0$ are not a serious mathematical difficulty. However, it must be noticed that such measures, in the original set of variables p correspond to measures supported on the sphere $|p| = \sqrt{2\epsilon_0} > 0$. Many of the mathematical difficulties arising in the study of the equations (1.13), (1.14) have their root in the singular character of Dirac masses concentrated at the origin for these equations.

Equation (1.10), (1.11) has been extensively studied in the physical literature, usually by means of numerical simulations (cf. [15], [16], [17], [18], [27], [28], [31]). However, there are not many rigorous mathematical results for this equation. A theory of weak solutions has been developed by X. Lu in [19, 20, 21]. It has been proved in these papers that it is possible to define a concept of measure valued solutions for (1.10), (1.11) and that at least one of such solutions exists globally in time for a large class of initial data. This solution converges in the weak topology of measures to the stationary state F_{BE} having the same mass as the initial data F_0 as $t \to \infty$ for long times (cf [20]).

Another class of solutions of (1.10), (1.11) has been constructed in [7], [8]. The solutions constructed in these papers are locally defined in time. They satisfy the equation in a classical sense for $\epsilon > 0$ and have the singular behaviour $f(t, \epsilon) \sim a(t)\, \epsilon^{-\frac{7}{6}}$ as $\epsilon \to 0$ for some function $a(t)$. The most relevant feature of these solutions is that they yield a nonzero flux of particles towards the origin. Therefore, the number of particles in the region $\{\epsilon > 0\}$ is a decreasing function. It is not clear at the moment if the solutions in [7], [8] can be extended to the set $\{\epsilon \geq 0\}$ in order to obtain some kind of mass preserving weak solution of (1.10), (1.11), but it has been proved in [21] that they are not a weak solution of (1.10), (1.11) in the sense of the definition given in [19].

A review of the currently available mathematical results for the solutions of (1.10), (1.11) can be found in [29]. This paper discuss also the known mathematical properties and the expected behaviour for the function $f(t, \epsilon)$ in the presence of condensate.

In this paper we first prove the existence of a large class of initial data for (1.10), (1.11) for which there exist classical solutions defined for a finite time interval $0 < t < T$ but becoming unbounded as $t \to T$. This blow-up result supports the scenario for the dynamical formation of Bose-Einstein condensates suggested in [15], [16], [27], [28], [31]. The scenario suggested in those papers is the following: there exists a classical, solution f of the Nordheim equation in a time interval, which blows up in finite time T_{max}. This solution can be extended as a weak solution containing a Dirac mass at the origin for all $t > T_{max}$.

The results of this paper support this scenario in the following sense. Theorem 2.3 shows the existence of local in time bounded solutions (mild solutions) that blow up in finite time T_{max} for a non empty set of initial data. In Theorem 2.10 we prove the existence of solutions, which are global weak solutions, that are bounded mild solutions for some finite time, and that, after some time, contain a Dirac mass at the origin. Notice that we do not prove that the onset of the Dirac mass formation takes place precisely at T_{max}, but only at some time $T_{cond} \geq T_{max}$.

We now describe some of the main ideas used in the proof of this blow-up result. As a first step we need to prove local existence of classical solutions for the Cauchy problem associated to (1.10), (1.11) with initial data decreasing like a power law for large ϵ. This problem can be solved using the methods introduced by T. Carleman in his seminal work about the well posedness of the spatially homogeneous Boltzmann equation (cf. [3], [4]). Actually the main difficulty proving local existence for (1.10), (1.11) has more to do with the quadratic terms in (1.6) than with the cubic terms (1.5). The reason for this is that the main difficulty that must be solved both for Boltzmann and Nordheim equations in order to prove local well-posedness is to find a class of functions whose behaviour for large ϵ is preserved in some suitable iterative scheme. For large values of ϵ the dominant terms in the equation are the quadratic ones while the cubic terms can be treated as some kind of perturbation. It was found by T. Carleman that a suitable class of functions that allow to prove well-posedness by means of an iterative argument are the functions bounded as $C(1+\epsilon)^{-\gamma}$ with $\gamma > 3$. Actually this class of functions allows to prove well posedness even for nonspherically symmetric distributions. In the Boltzmann case, due to the conservation of the energy and the number of particles it is possible to prove global existence of solutions (cf. [3], [4]).

The derivation of optimal decay estimates for the solutions of the homogeneous Boltzmann equation which allow to prove conservation of the energy of the system has been extensively studied (cf. [22], [24], as well as the review [32] and the references therein). However, since this is an issue more related to the classical Boltzmann equation than to the specific effects induced by the cubic terms in (1.10), (1.11) we have preferred to use the methods of T. Carleman to prove local well-posedness. The reason being that, although Carleman's method requires to impose decay estimates for the solutions more restrictive than some of the more recent approaches, it uses simpler arguments. In spite of this this approach will be enough to obtain a large class of initial data yielding blow-up for the solutions of (1.10), (1.11) in finite time.

In order to prove the blow-up for the solutions some additional methods are needed. We will use first a crucial monotonicity property that has been obtained in [21]. This property allows to obtain an estimate for the net number of collisions taking place between particles with small energy. Roughly speaking this number measures the rate of change of the number of small particles associated to g. It turns out that, for a given amount of mass in the region $\{\epsilon \in [0, R]\}$, the net number of collisions between particles with small energy yields big changes in the mass of the system, except if the distribution g is very close, in the weak topology, to a Dirac mass supported at some point $\epsilon = \epsilon_0 > 0$. Given that the total mass of the system is bounded such large changes of the mass are not admissible. Therefore, if the initial number of particles with small energy is sufficiently large and the solutions do not become unbounded in finite time, only one alternative is left, namely, the distribution g close, in the weak topology, to a Dirac mass at a particular value of the energy $\epsilon_0 > 0$.

The rigorous proof of this concentration estimate will made use of a key Measure Theory result that describes in a rather precise way the degree of concentration of arbitrary measures defined in intervals $[0, R]$ with R small.

Finally we will prove that the concentration of g at a positive value of the energy cannot take place for a set of times too large if $f_0(\epsilon) \geq \nu > 0$ for small ϵ. This will be seen using two arguments. First, we will prove, using again the monotonicity property mentioned above, that the condition $f_0(\epsilon) \geq \nu > 0$ implies that the number of particles with energy in the interval $[0, R]$ with R small, can be bounded from below for times of order one. Using this, we can see that there would be a fast transfer of particles with energy $\epsilon = \epsilon_0$ towards lowest energy values. This would contradict also the conservation of the total number of particles and then, the only alternative left is blow-up in finite time for f.

This will be seen deriving an estimate for the transfer of particles from the peak at $\epsilon = \epsilon_0$ to the region of smaller particles. This estimate will show that for solutions of (1.10), (1.11) with $f_0(\epsilon) \geq \nu > 0$ the transfer of particles from the peak at $\epsilon = \epsilon_0$ towards smaller particles is very large.

The previous arguments will be made precise deriving detailed estimates of the measure of the set of times in which the distribution g behaves in each specific form.

On the other hand, the reason for the onset of the alternative which states that the net rate of collisions can take place only if g concentrates near a Dirac mass is because for small values of ϵ the dominant terms of the equation are the cubic ones in (1.3). The equation verified by g if only the terms in $q_3(\cdot)$ are kept is (cf. (1.13)):

$$\partial_t g_1 = 32\pi^3 \int_0^\infty \int_0^\infty q_3\left(\frac{g}{4\pi\sqrt{2\epsilon}}\right) \Phi d\epsilon_3 d\epsilon_4 \tag{1.15}$$

and it can be readily seen that every distribution of the form $g = M_0 \delta_{\epsilon_0}$ with $M_0 \geq 0,\ \epsilon_0 > 0$ is a stationary solution of (1.15).

Similar arguments are applied to weak solutions and yield also condensation in finite time.

The plan of the paper is the following. In Section 2 we define the concept of solution which will be used and state the blow-up result which we prove in this paper. In Section 3 we prove a local well posedness result for the equation (1.10), (1.11) for suitable initial data. Section 4 describes a basic monotonicity property of the solutions of (1.10), (1.11) which will be used in Section 5 to derive a first estimate for the number of collisions which change the energy of the particles. Section 6 contains a crucial measure theory result which states in a precise quantitative way that an arbitrary measure in $[0, 1]$, either is concentrated in a small set, or it has its mass spread among two measurable sets which are "sufficiently separated". This measure theory result is used in Section 7 to transform the estimate obtained in Section 5 to another estimate which states that the solutions of (1.10), (1.11), must have their mass concentrated in a narrow peak if they are defined during sufficiently long times. Section 8 contains some estimates which prove that the portion of particle distributions concentrated in narrow peaks with small energy would be transferred very fast towards even smaller energy values. This is used in Section 9 to conclude the Proof of the blow-up result obtained in this paper. Section 10 contains several results which prove that some suitable weak solutions of (1.10), (1.11) yield finite time condensation.

In all the paper C will be a generic numerical constant which can change from line to line. We will denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}$ as $|A|$.

2 Main results of the paper.

2.1 Definition of mild solution.

In order to formulate the main blow-up result of this paper we need to introduce some functional spaces and to precise the concept of solution of (1.10), (1.11).

Given $\gamma \in \mathbb{R}$ we will denote as $L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)$ the space of functions such that:

$$\|f\|_{L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)}=\sup_{\epsilon\geq 0}\left\{(1+\epsilon)^{\gamma} f(\epsilon)\right\}<\infty.$$

Notice that $L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)$ is a Banach space with the norm $\|\cdot\|_{L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)}$.

We will use also the spaces $X_{T_1,T_2}=L^{\infty}_{loc}\left([T_1,T_2);L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)\right)$ of the functions f satisfying:

$$\sup_{t\in K}\|f(t,\cdot)\|_{L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)}<\infty,$$

for any compact $K\subset[T_1,T_2)$. Notice that the spaces X_{T_1,T_2} are not Banach spaces. We will use the space $L^{\infty}\left([T_1,T_2];L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)\right)$ which is the Banach space of functions such that:

$$\|f\|_{L^{\infty}\left([T_1,T_2];L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)\right)}=\sup_{t\in[T_1,T_2]}\|f(t,\cdot)\|_{L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)}<\infty.$$

Definition 2.1 *Suppose that $\gamma>3$ and $0\leq T_1<T_2<+\infty$. We will say that a function $f\in L^{\infty}_{loc}\left([T_1,T_2);L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)\right)$ is a mild solution of (1.10), (1.11) on $(0,T)$ with initial data f_0 if it satisfies:*

$$f(t,\epsilon_1)=f_0(\epsilon_1)\Psi(t,\epsilon_1)+\frac{8\pi^2}{\sqrt{2}}\int_{T_1}^{t}\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}\int_0^{\infty}\int_0^{\infty}f_3f_4(1+f_1+f_2)\,W d\epsilon_3 d\epsilon_4 ds \tag{2.1}$$

a.e. $t\in[T_1,T_2)$, where:

$$\begin{aligned}a(t,\epsilon_1)&=\frac{8\pi^2}{\sqrt{2}}\int_0^{\infty}\int_0^{\infty}f_2(1+f_3+f_4)\,W d\epsilon_3 d\epsilon_4\quad,\quad\Psi(t,\epsilon_1)\\&=\exp\left(-\int_{T_1}^{t}a(s,\epsilon_1)\,ds\right).\end{aligned}\tag{2.2}$$

Remark 2.2 *Since $\gamma>3$, for any $t\in[T_1,T_2)$ the integral term*

$$\int_0^{\infty}\int_0^{\infty}f_3f_4(1+f_1+f_2)\,W d\epsilon_3 d\epsilon_4$$

can be estimated by a constant $C \|f\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)} \left(1 + \|f\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right)$ *where* C *is a numerical constant. The term* $a(t, \epsilon_1)$ *can be estimated by*

$$C\left(1 + \|f\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right) \int_0^\infty \int_0^\infty f_2 W d\epsilon_3 d\epsilon_4.$$

By the definition of W*, and using* ϵ_2 *as one of the integration variables, we estimate* $\int_0^\infty \int_0^\infty f_2 W d\epsilon_3 d\epsilon_4$ *as* $\int_0^\infty f_2 \left(\sqrt{\epsilon_2} + \sqrt{\epsilon_1}\right) W d\epsilon_2 \leq C \|f\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}$*. Therefore, if* $f \in L^\infty_{loc}([T_1, T_2); L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma))$ *all the terms in (2.1) are well defined for* $T_1 \leq t < T_2$*.*

2.2 Blow-up Result.

The main blow-up result of this paper is the following:

Theorem 2.3 *There exist* $\theta_* > 0$ *such that, for all* $M > 0,\ E > 0,\ \nu > 0,\ \gamma > 3$*, there exists* $\rho_0 = \rho_0(M, E, \nu) > 0,\ K^* = K^*(M, E, \nu) > 0,\ T_0 = T_0(M, E)$ *satisfying the following property. For any* $f_0 \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$ *such that*

$$4\pi\sqrt{2} \int_{\mathbb{R}^+} f_0(\epsilon) \sqrt{\epsilon} d\epsilon = M \ , \quad 4\pi\sqrt{2} \int_{\mathbb{R}^+} f_0(\epsilon) \sqrt{\epsilon^3} d\epsilon = E, \tag{2.3}$$

and

$$\sup_{0\leq\rho\leq\rho_0} \left[\min\left\{\inf_{0\leq R\leq\rho} \frac{1}{\nu R^{3/2}} \int_0^R f_0(\epsilon)\sqrt{\epsilon}d\epsilon, \frac{1}{K^* \rho^{\theta_*}} \int_0^\rho f_0(\epsilon)\sqrt{\epsilon}d\epsilon\right\}\right] \geq 1, \tag{2.4}$$

there exists a unique mild solution of (1.10), (1.11) in the sense of Definition 2.1 $f \in L^\infty_{loc}([0, T_{\max}); L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma))$*, with initial data* f_0*, defined for a maximal existence time* $T_{\max} < T_0$ *and that satisfies:*

$$\limsup_{t\to T_{\max}^-} \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^+)} = \infty.$$

The above Theorem means that initial data $f_0 \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$, with a sufficiently large density around $\epsilon = 0$, blows up in finite time. More precisely, the condition (2.4) means that there exists $\rho \in (0, \rho_0)$ satisfying:

$$\int_0^R f_0(\epsilon) \sqrt{\epsilon} d\epsilon \geq \nu R^{\frac{3}{2}} \ \text{ for } 0 < R \leq \rho \ , \quad \int_0^\rho f_0(\epsilon) \sqrt{\epsilon} d\epsilon \geq K^*(\rho)^{\theta_*}, \tag{2.5}$$

The second condition in (2.5) holds if the distribution f_0 has a mass sufficiently large in a ball with radius ρ for some ρ sufficiently small. The first condition is satisfied if $f_0(\epsilon) \geq 3\nu/2$ for all ε sufficiently small. Since θ_* might be small, the first condition in (2.5) does not implies the second.

Our results do not provide an explicit functional relation for the functions $\rho_0(M, E, \nu)$, $K^*(M, E, \nu)$ and $T_0(M, E)$ in terms of their arguments. Therefore, for a given initial data $f_0 \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$ it is not easy to check if condition (2.5) is satisfied. However, it is simple to verify that the class of functions $f_0 \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$ satisfying such condition is not empty. Indeed, we have:

Proposition 2.4 *Given $M > 0, E > 0, \nu > 0, \gamma > 0$ there exists a family of function $f_0 \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$ satisfying (2.3) and for which (2.4) holds with θ_*, K^* and ρ_0 as in Theorem 2.3.*

Proof. Consider $\zeta \in C_0([0,\infty))$ such that:

$$0 \le \zeta \le 2, \quad \int_0^\infty \zeta(s)\sqrt{s}ds = 1, \quad \zeta(s) \ge \frac{1}{4}, \forall s \in [0,1], \quad \text{supp}(\zeta) = [0,2]. \tag{2.6}$$

We also consider functions functions $\overline{f}_1 \ge 0$ and $\overline{f}_2 \ge 0$, $\overline{f}_1 \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$, $\overline{f}_2 \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$ such that

$$\int_0^\infty \overline{f}_1(\epsilon)\sqrt{\epsilon}d\epsilon = \int_0^\infty \overline{f}_2(\epsilon)\sqrt{\epsilon}d\epsilon = \frac{M}{2}, \tag{2.7}$$

$$\int_0^\infty \overline{f}_1(\epsilon)\epsilon^{3/2}d\epsilon = \frac{E}{4}, \quad \int_0^\infty \overline{f}_2(\epsilon)\epsilon^{3/2}d\epsilon = \frac{3E}{4}. \tag{2.8}$$

For example, fix a function $\varphi \ge 0$, $\varphi \in C_0(0,\infty)$ such that

$$\int_0^\infty \varphi(\epsilon)\sqrt{\epsilon}d\epsilon = \frac{M}{2}$$

and let be

$$C_1 = \int_0^\infty \varphi(\epsilon)\epsilon^{3/2}d\epsilon.$$

Then, we may take:

$$\overline{f}_k(\epsilon) = \mu_k^{3/2}\varphi(\mu_k\epsilon), \ \ k = 1, 2, \tag{2.9}$$

where $\mu_1 = \frac{4C_1}{E}$ and $\mu_2 = \frac{4C_1}{3E}$.

We define now

$$f_0(\epsilon) = \frac{1}{\rho^{\beta+\frac{1}{2}}}\zeta\left(\frac{\epsilon}{\rho}\right) + \kappa_1\overline{f}_1(\epsilon) + \kappa_2\overline{f}_2(\epsilon) \tag{2.10}$$

where $0 < \beta < 1$, $\kappa_1 > 0$, $\kappa_2 > 0$ to be precised.

Given $\rho > 0$, we choose κ_1 and κ_2 satisfying:

$$\kappa_1 + \kappa_2 = 1 - \frac{2\rho^{1-\beta}}{M} \tag{2.11}$$

$$\kappa_1 + 3\kappa_2 = 4 - \frac{4C_0\rho^{2-\beta}}{E} \tag{2.12}$$

where

$$C_0 = \int_0^\infty s^{3/2}\zeta(s)ds \in (0,2).$$

With such a choice, the function f_0 satisfies (2.3). The solutions of system (2.11), (2.12) are such that:

$$\kappa_1 = \frac{1}{3} + \lambda_1(\rho, M, E), \quad \lim_{\rho\to 0} \lambda_1(\rho, M, E) = 0, \tag{2.13}$$

$$\kappa_2 = \frac{2}{3} + \lambda_2(\rho, M, E), \quad \lim_{\rho\to 0} \lambda_2(\rho, M, E) = 0 \tag{2.14}$$

Therefore, there exists $\overline{\rho}(M,E) > 0$, such that if $\rho < \overline{\rho}(M,E)$, we have $\kappa_1 > 0$, $\kappa_2 > 0$ and then $f_0 \geq 0$.

We now choose β and ρ in order for f_0 to satisfy (2.5). To this end we first observe that:

$$\int_0^R f_0(\epsilon)\sqrt{\epsilon}d\epsilon \geq \frac{1}{\rho^{\beta+1/2}} \int_0^R \zeta\left(\frac{\epsilon}{\rho}\right)\sqrt{\epsilon}d\epsilon$$

Using (2.6) we obtain:

$$\forall R \leq \rho, \quad \frac{1}{\rho^{\beta+1/2}} \int_0^R \zeta\left(\frac{\epsilon}{\rho}\right)\sqrt{\epsilon}d\epsilon \geq \frac{1}{4\rho^{\beta+1/2}} \int_0^R \sqrt{\epsilon}d\epsilon = \frac{R^{3/2}}{6\rho^{\beta+1/2}} \tag{2.15}$$

Then, the first condition in (2.5) holds if:

$$\rho < \left(\frac{1}{6\nu}\right)^{\frac{1}{\beta+1/2}}. \tag{2.16}$$

Using (2.15) with $R = \rho$ as well as (2.6), we obtain

$$\frac{1}{\rho^{\beta+1/2}} \int_0^R \zeta\left(\frac{\epsilon}{\rho}\right)\sqrt{\epsilon}d\epsilon \geq \frac{1}{6}\rho^{1-\beta}$$

We now chose β such that

$$1 - \theta_* < \beta < 1. \tag{2.17}$$

Then, if

$$\rho < \left(\frac{1}{6K^*}\right)^{\frac{1}{\beta-1+\theta_*}} \tag{2.18}$$

the second condition in (2.5) is satisfied. Choosing then

$$0 < \rho < \min\left(\rho_0(M,E,\nu), \overline{\rho}(M,E), \left(\frac{1}{6K^*(M,E)}\right)^{\frac{1}{\beta-1+\theta_*}}, \left(\frac{1}{6\nu}\right)^{\frac{1}{\beta+1/2}}\right) \tag{2.19}$$

and β as in (2.17), all the conditions in (2.3) (2.5) are then satisfied.

An example of functions ζ, $\overline{f}_1, \overline{f}_2$, and constants μ_1, μ_2 satisfying all the requirements are the following: $\zeta = \frac{3}{2}\chi_{[0,1]}$, $\varphi = \frac{3M}{4}\chi_{[0,1]}$, $\mu_1 = \frac{6M}{5E}$, $\mu_2 = \frac{2M}{5E}$ (cf. Figure 2.1 below). ■

It is apparent from the above arguments that the function f_0 may be chosen as regular as desired. This shows that the onset of blow up is not related with the regularity of the solution.

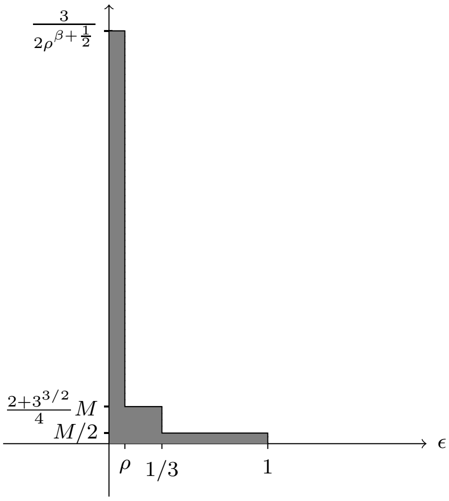

Figure 2.1: An example of initial datum (2.10) for $M > 0, \frac{2M}{5E} = 1, 0 < \rho << 1$.

Remark 2.5 *The result is local for given values of E and M in the sense that the main requirements for blow-up are to have a lower estimate for f_0 for small energies as well as a sufficiently large amount of particles in the interval $\epsilon \in [0, \rho]$.*

2.3 Weak solutions.

The theory of weak solutions of (1.10), (1.11) has been developed by X. Lu in [19, 20, 21]. It allows to deal with measured valued solutions and suits very well to the purpose of considering the finite time formation of Dirac mass in the solutions of (1.10), (1.11).

Since we are interested in the condensation phenomena, it is convenient to use the equation for the mass density g, instead of f instead of (1.10), (1.11). We will denote as $\mathcal{M}_+ (\mathbb{R}^+; 1+\epsilon)$ the set of Radon measures g in $\mathbb{R}^+$ satisfying:

$$\int (1+\epsilon)\, g(\epsilon)\, d\epsilon < \infty$$

We will use the notation $g(\epsilon)$ in spite of the fact that g is a measure.

Definition 2.6 *We will say that $g \in C([0,T); \mathcal{M}_+(\mathbb{R}^+; (1+\epsilon)))$ is a weak solution of (1.13), (1.14) on $(0,T)$, with initial datum $g_0 \in \mathcal{M}_+(\mathbb{R}^+; 1+\epsilon)$, if, for any*

$\varphi \in C_0^2([0,T),[0,\infty))$, the following identity holds:

$$-\int_{\mathbb{R}^+} g_0(\epsilon)\varphi(0,\epsilon)\,d\epsilon = \int_0^T\int_{\mathbb{R}^+} g\partial_t\varphi d\epsilon dt +$$
$$+\frac{1}{2^{\frac{5}{2}}}\int_0^T\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{g_1g_2g_3\Phi}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}}Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 dt +$$
$$+\frac{\pi}{2}\int_0^T\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 dt \tag{2.20}$$

where Φ is as in (1.14) and:

$$Q_\varphi = \varphi(\epsilon_3) + \varphi(\epsilon_1+\epsilon_2-\epsilon_3) - 2\varphi(\epsilon_1) \tag{2.21}$$

Definition 2.7 *We say that f is a weak solution of (1.10), (1.11) in $(0,T)$ with initial datum f_0, if $g_0 = 4\pi\sqrt{2\epsilon}f_0(\epsilon) \in \mathcal{M}_+(\mathbb{R}^+;1+\epsilon)$, and $g = 4\pi\sqrt{2\epsilon}f(t,\epsilon) \in C([0,T);\mathcal{M}_+(\mathbb{R}^+;(1+\epsilon)))$ is a weak solution of (1.13), (1.14) on $(0,T)$, with initial datum g_0 in the sense of Definition 2.6.*

Remark 2.8 *We prove in Lemma 3.13 that every mild solution f in the sense of Definition 2.1 is a weak solution in the sense of Definition 2.7.*

It has been proved by X. Lu in Theorem 2 of [19] that for all $g_0 \in \mathcal{M}_+(\mathbb{R}^+;1+\epsilon)$, there exists a global weak solution of (1.13), (1.14) in the sense of the Definition 2.6. This solution satisfies (2.20), (2.21) for all $\varphi \in C_b^2([0,\infty))$ and with $T=\infty$. Moreover the two quantities $\int_{[0,\infty)} g(t,\epsilon)d\epsilon$ and $\int_{[0,\infty)} \epsilon g(t,\epsilon)d\epsilon$ are constant in time, for all $t \geq 0$.

2.4 Finite time condensation results.

We may now state our two results on finite time condensation.

Theorem 2.9 *There exist $\theta_* > 0$ with the following property. For all $M>0$, $E>0$, $\nu>0$, there exists $\rho_0 = \rho_0(M,E,\nu) > 0$, $K^* = K^*(M,E,\nu) > 0$, $T_0 = T_0(M,E)$ such that for any weak solution of (1.13), (1.14) on $(0,T_0)$ in the sense of Definition 2.6 with $g_0 \in \mathcal{M}_+(\mathbb{R}^+;(1+\epsilon))$ satisfying*

$$4\pi\sqrt{2}\int_{\mathbb{R}^+} g_0(\epsilon)\,d\epsilon = M \ , \quad 4\pi\sqrt{2}\int_{\mathbb{R}^+} g_0(\epsilon)\,\epsilon d\epsilon = E \tag{2.22}$$

$$\sup_{0\leq\rho\leq\rho_0}\left[\min\left\{\inf_{0\leq R\leq\rho}\frac{1}{\nu R^{3/2}}\int_0^R g_0(\epsilon)d\epsilon, \frac{1}{K^*\rho^{\theta_*}}\int_0^\rho g_0(\epsilon)d\epsilon\right\}\right] \geq 1, \tag{2.23}$$

we have:

$$\sup_{0<t\leq T_0}\int_{\{0\}} g(t,\epsilon)\,d\epsilon > 0. \tag{2.24}$$

Notice that the construction of the weak solutions of (1.13), (1.14) does not rule out the possibility of having "instantaneous condensation", i.e.:

$$\sup_{0<t\leq T^*}\int_{\{0\}} g\left(t,\epsilon\right)d\epsilon>0$$

for any $T^*>0$. However, it is possible to construct weak solutions of (1.13), (1.14) such that $\sup_{0<t\leq T_*}\int_{\{0\}} g\left(t,\epsilon\right)d\epsilon=0$ for some $0<T^*<T_0$, but satisfying (2.24). For such solutions we would have then condensation in a finite, but positive time, as it has been suggested in the physical literature (cf. [15], [16], [27], [28]). More precisely, we have:

Theorem 2.10 *There exist* $\theta_*>0$ *with the following property. For all* $M>0,\ E>0,\ \nu>0,\ \gamma>3$, *there exists* $\rho_0=\rho_0\left(M,E,\nu\right)>0,\ K^*=K^*\left(M,E,\nu\right)>0,\ T_0=T_0\left(M,E\right)$ *such that for any* $f_0\in L^\infty\left(\mathbb{R}^+;\left(1+\epsilon\right)^\gamma\right)$ *satisfying*

$$4\pi\sqrt{2}\int_{\mathbb{R}^+} f_0\left(\epsilon\right)\sqrt{\epsilon}d\epsilon=M\ \ ,\ \ 4\pi\sqrt{2}\int_{\mathbb{R}^+} f_0\left(\epsilon\right)\sqrt{\epsilon^3}d\epsilon=E \tag{2.25}$$

$$\sup_{0\leq\rho\leq\rho_0}\left[\min\left\{\inf_{0\leq R\leq\rho}\frac{1}{\nu R^{3/2}}\int_0^R f_0(\epsilon)\sqrt{\epsilon}d\epsilon,\frac{1}{K^*\rho^{\theta_*}}\int_0^\rho f_0(\epsilon)\sqrt{\epsilon}d\epsilon\right\}\right]\geq 1, \tag{2.26}$$

there exists a weak solution g *of (1.13), (1.14), with* $g_0\left(\epsilon\right)=4\pi\sqrt{2\epsilon}f_0\left(\epsilon\right)$, *and there exists* $T_*>0$ *such that the following holds:*

$$\sup_{0\leq t\leq T_*}\left\|f\left(t,\cdot\right)\right\|_{L^\infty\left(\mathbb{R}^+\right)}<\infty\ \ ,\ \ \sup_{T_*<t\leq T_0}\int_{\{0\}} g\left(t,\epsilon\right)d\epsilon>0 \tag{2.27}$$

where $g=4\pi\sqrt{2\epsilon}f$.

3 Local well-posedness Theorem.

As a first step we prove the local well-posedness of (1.10), (1.11) for initial data in $L^\infty\left(\mathbb{R}^+;\left(1+\epsilon\right)^\gamma\right)$ with $\gamma>3$. We first summarize some topological results which will be used in the arguments.

3.1 Topological preliminaries.

We introduce a concept of weak topology in the spaces $L^\infty\left(\mathbb{R}^+;\left(1+\epsilon\right)^\gamma\right)$ by means of the functionals:

$$L_\varphi\left[f\right]=\int_{\mathbb{R}^+} f\varphi d\epsilon\ \ ,\ \ \varphi\in C_0\left(\left[0,\infty\right)\right). \tag{3.1}$$

We will denote as Θ the topology induced by the functionals φ in $L^\infty\left(\mathbb{R}^+;\left(1+\epsilon\right)^\gamma\right)$. We define also the set:

$$\mathcal{K}_R^\gamma=\left\{f\geq 0,\ f\in L^\infty\left(\mathbb{R}^+;\left(1+\epsilon\right)^\gamma\right):\left\|f\right\|_{L^\infty\left(\mathbb{R}^+;\left(1+\epsilon\right)^\gamma\right)}\leq R\right\}\ \ ,\ \ 0<T<\infty.$$

Lemma 3.1 *For any $R > 0,\ \gamma > 1$, the topological space $(\mathcal{K}_R^\gamma, \Theta)$ is metrizable and compact.*

Proof. In order to check that the topology Θ is metrizable in $\mathcal{K}_R^\gamma$ we select a countable set of functions $\bar{\varphi}_n \in C_0([0,\infty))$ which is dense in $C_0([0,\infty))$ in the sense of the uniform topology in compact sets. We then define a metric in $\mathcal{K}_R^\gamma$ by means of:

$$\text{dist}(f,g) = \sum_n \frac{2^{-n}}{\|\bar{\varphi}_n\|_{L^1([0,\infty))}} \left| \int_{[0,\infty)} (f-g)\,\bar{\varphi}_n d\epsilon \right|. \tag{3.2}$$

It is now standard to check that every set in Θ contains a ball with the form $\left\{g \in \mathcal{K}_{T,R}^\gamma : \text{dist}(f_0, g) < \delta\right\}$ for some $f_0 \in \mathcal{K}_{T,R}^\gamma$ and $\delta > 0$ and also that such a ball contains a set of Θ.

In order to prove the compactness of $\mathcal{K}_R^\gamma$ with this topology it is enough to prove, due to the metrizability of the space, that every sequence has a convergent subsequence. Given a sequence $\{f_n\} \subset \mathcal{K}_R^\gamma$ we have, since $\gamma > 1$, that $\int_{\mathbb{R}^+} f_n d\epsilon \leq CR$ for some C independent on n. Then, there exists a subsequence of $\{f_n\}$ which will be denoted by the same indexes, as well as a Radon measure $\mu \in \mathcal{M}_+(\mathbb{R}^+)$ such that $f_n \rightharpoonup \mu$ as $n \to \infty$ in $\mathcal{M}_+(\mathbb{R}^+)$.

The definition of $L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$ and $\mathcal{K}_R^\gamma$ imply:

$$\int_{\mathbb{R}^+} f_n (1+\epsilon)^\gamma \varphi d\epsilon \leq R \int_{\mathbb{R}^+} \varphi d\epsilon = R\,\|\varphi\|_{L^1([0,\infty))},$$

for any $\varphi \in C_0([0,\infty))$. Taking subsequences and passing to the limit in this formula we obtain:

$$\int_{\mathbb{R}^+} \mu (1+\epsilon)^\gamma \varphi d\epsilon \leq R \int_{\mathbb{R}^+} \varphi d\epsilon = R\,\|\varphi\|_{L^1([0,\infty))},$$

and this implies that $\mu(1+\epsilon)^\gamma \in \left(L^1([0,\infty))\right)^* = L^\infty([0,\infty))$. (cf. [2], [5]). A similar argument yields $\mu \geq 0$. Then $\mu \in \mathcal{K}_R^\gamma$ and the result follows. ■

We now define the space of functions in which we will prove well posedness. We will denote as $\mathcal{X}_{R,T}^\gamma$ the space of functions $C([0,T]; (\mathcal{K}_R^\gamma, \Theta))$ endowed with the metric:

$$\underset{\mathcal{X}_{R,T}^\gamma}{\text{dist}}(f_1, f_2) = \sup_{t\in[0,T]} d_{\mathcal{K}_R^\gamma}(f_1(t), f_2(t)), \tag{3.3}$$

where $d_{\mathcal{K}_R^\gamma}$ is the distance associated to the topological space $(\mathcal{K}_R^\gamma, \Theta)$ (cf. Lemma 3.1). The space $\mathcal{X}_{R,T}^\gamma$ is a complete metric space for any $R > 0,\ 0 < T < \infty,\ \gamma > 1$. We recall in the next Proposition the characterization of the compact sets of $\mathcal{X}_{R,T}^\gamma$.

Proposition 3.2 *Let $R > 0,\ 0 < T < \infty,\ \gamma > 1$. Suppose that $\mathcal{F} \subset \mathcal{X}_{R,T}^\gamma$ is an equicontinuous family of functions, i.e. for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $f \in \mathcal{F}$ and any $t_1, t_2 \in [0,T]$ satisfying $|t_1 - t_2| < \delta$ we have $d_{\mathcal{K}_R^\gamma}(f(t_1), f(t_2)) < \varepsilon$. Then, the family $\mathcal{F}$ is compact in the topology of $\mathcal{X}_{R,T}^\gamma$.*

Proof. It is just a consequence of Arzelà-Ascoli Theorem for continuous functions with values in general compact metric spaces (cf. [5], [9]). ■

It will be convenient to reformulate Proposition 3.2 in a form that is more convenient to use in terms of test functions.

Proposition 3.3 *Suppose that $\mathcal{F} \subset \mathcal{X}_{R,T}^{\gamma}$ is a family of functions satisfying that, for any $\varphi \in C_0([0,\infty))$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $f \in \mathcal{F}$ and any $t_1, t_2 \in [0,T]$ satisfying $|t_1 - t_2| < \delta$ the functions $\psi_{\varphi}(t; f) = \int_{\mathbb{R}^+} f(t,\epsilon)\varphi(\epsilon)\, d\epsilon$ satisfy:*

$$|\psi_{\varphi}(t_1; f) - \psi_{\varphi}(t_2; f)| < \varepsilon,$$

for any $f \in \mathcal{F}$. Then, the family $\mathcal{F}$ is compact in $\mathcal{X}_{R,T}^{\gamma}$.

Proof. We use the metric defined in (3.2). Given $\varepsilon > 0$ as well as the definition of $\mathcal{X}_{R,T}^{\gamma}$ it follows that there exists N large enough such that

$$\sum_{n \geq N} \frac{2^{-n}}{\|\bar{\varphi}_n\|_{L^1([0,\infty))}} \left| \int_{[0,\infty)} (f(t_1) - f(t_2))\, \bar{\varphi}_n d\epsilon \right| < \frac{\varepsilon}{2},$$

for any $t_1, t_2 \in [0,T]$. On the other hand, using the property satisfied by the family $\mathcal{F}$ it follows that, there exists $\delta > 0$ such that, if $|t_1 - t_2| < \delta$ we have:

$$\sum_{n < N} \frac{2^{-n}}{\|\bar{\varphi}_n\|_{L^1([0,\infty))}} \left| \int_{[0,\infty)} (f(t_1) - f(t_2))\, \bar{\varphi}_n d\epsilon \right| < \frac{\varepsilon}{2}.$$

Therefore $d_{\mathcal{K}_R^{\gamma}}(f(t_1), f(t_2)) < \varepsilon$ and applying Proposition 3.2 the result follows. ■

3.2 Statement of the Local Well-Posedness Theorem.

Theorem 3.4 *Suppose that $f_0 \in L^{\infty}(\mathbb{R}^+; (1+\epsilon)^{\gamma})$ with $\gamma > 3$. There exists $T > 0$, depending only on $\|f_0(\cdot)\|_{L^{\infty}(\mathbb{R}^+;(1+\epsilon)^{\gamma})}$, and there exists a unique mild solution of (1.10), (1.11), $f \in L_{loc}^{\infty}([0,T); L^{\infty}(\mathbb{R}^+; (1+\epsilon)^{\gamma}))$ in the sense of Definition 2.1.*

The obtained solution f satisfies:

$$4\pi\sqrt{2}\int_0^{\infty} f_0(\epsilon)\, \epsilon^w d\epsilon = 4\pi\sqrt{2}\int_0^{\infty} f(t,\epsilon)\, \epsilon^w d\epsilon\,, \quad t \in (0,T)\;, \quad w \in \left\{\frac{1}{2}, \frac{3}{2}\right\}. \tag{3.4}$$

The function f is in the space $W^{1,\infty}((0,T); L^{\infty}(\mathbb{R}^+))$ and it satisfies (1.10) a.e. $\epsilon \in \mathbb{R}^+$ for any $t \in (0, T_{\max})$. Moreover, f can be extended as a mild solution of (1.10), (1.11) to a maximal time interval $(0, T_{\max})$ with $0 < T_{\max} \leq \infty$. If $T_{\max} < \infty$ we have:

$$\limsup_{t \to T_{\max}^-} \|f(t,\cdot)\|_{L^{\infty}(\mathbb{R}^+)} = \infty.$$

We will split the Proof of Theorem 3.4. We first prove the existence of one mild solution using Schauder's fixed point Theorem. Suppose that $T>0$. We define for each $\gamma>3$ the following auxiliary operator $\mathcal{T}:\mathcal{X}_T^{\gamma}\to\mathcal{X}_T^{\gamma}$:

$$\begin{aligned}\mathcal{T}(f)(t,\epsilon_1)&=f_0(\epsilon_1)\Psi(t,\epsilon_1)+\\&+\frac{8\pi^2}{\sqrt{2}}\int_0^t\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}\int_0^{\infty}\int_0^{\infty}f_3f_4(1+f_1+f_2)W\,d\epsilon_3d\epsilon_4ds,\end{aligned}\tag{3.5}$$

where Ψ is as in (2.2). Given $\gamma>3$ we define the following functional ψ_{γ} : $L^{\infty}(\mathbb{R}^+;(1+\epsilon)^{\gamma})\to\mathbb{R}^+$.

$$\psi_{\gamma}[f]=\|f\|_{L^{\infty}(\mathbb{R}^+;(1+\epsilon)^{\gamma})}+\int_0^{\infty}\epsilon^{\frac{3}{2}}f(\epsilon)\,d\epsilon.\tag{3.6}$$

The following estimate will play a crucial role in all this Section.

Proposition 3.5 *Let $\gamma>3$. Suppose that the operator:*

$$\mathcal{T}:L^{\infty}_{loc}\left([0,T);L^{\infty}\left(\mathbb{R}^+;(1+\epsilon)^{\gamma}\right)\right)\to L^{\infty}_{loc}\left([0,T);L^{\infty}\left(\mathbb{R}^+;(1+\epsilon)^{\gamma}\right)\right)$$

is defined as in (3.5). There exists $0<\theta<1$ and $C>0$ both of them depending only on γ such that for any $0\le t\le T$:

$$\begin{aligned}&\psi_{\gamma}[f(t,\cdot)]\le\psi_{\gamma}[f_0]+Ct\left(1+\sup_{0\le s\le t}\|f(t,\cdot)\|_{L^{\infty}(\mathbb{R}^+)}\right)\left(\sup_{0\le s\le t}\int f(s,\epsilon)\,d\epsilon\right)^2+\\&+Ct\sup_{0\le s\le t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^+;(1+\epsilon)^{\gamma})}\times\\&\times\left[\left(1+2\sup_{0\le s\le t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^+)}\right)\sup_{0\le s\le t}\int_0^{\infty}\left(\sqrt{\epsilon}+(\epsilon)^{\frac{3}{2}}\right)f(s,\epsilon)\,d\epsilon+\right.\\&\qquad\left.+\left(\sup_{0\le s\le t}\int\left(1+\epsilon^{\frac{3}{2}}\right)f(s,\epsilon)\,d\epsilon\right)^2\right]+\theta\sup_{0\le s\le t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^+;(1+\epsilon)^{\gamma})}\,.\end{aligned}\tag{3.7}$$

In order to prove Proposition 3.5 we need two auxiliary Lemmas.

Lemma 3.6 *Let $\gamma>3$ and $\mathcal{T}$ be as in Proposition (3.5). There exists $C>0$ depending only on γ such that:*

$$\begin{aligned}\int_0^{\infty}\epsilon^{\frac{3}{2}}\mathcal{T}(f)(t,\epsilon)\,d\epsilon&\le\int_0^{\infty}(\epsilon)^{\frac{3}{2}}f_0(\epsilon)\,d\epsilon+Ct\left(1+2\sup_{0\le s\le t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^+)}\right)\times\\&\times\left(\sup_{0\le s\le t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^+;(1+\epsilon)^{\gamma})}\right)\left(\sup_{0\le s\le t}\int_0^{\infty}\sqrt{\epsilon}f(s,\epsilon)\,d\epsilon\right),\end{aligned}\tag{3.8}$$

for $0\le t\le T$.

Proof. Notice that, using the symmetry of the integral under the exchange $\epsilon_3 \leftrightarrow \epsilon_4$ as well as the fact that $W \leq \sqrt{\frac{\epsilon_4}{\epsilon_1}}$ if $\epsilon_3 \geq \epsilon_4$ and that $\gamma > 3$.:

$$\begin{aligned}
&\left\| \int \int W f_3 f_4 \left(1 + f_1 + f_2\right) d\epsilon_3 d\epsilon_4 \right\|_{L^1\left(\mathbb{R}^+; \epsilon^{\frac{3}{2}} d\epsilon\right)} \\
&\leq C \left(1 + 2\|f\|_{L^\infty(\mathbb{R}^+)}\right) \|f\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)} \int_0^\infty \frac{d\epsilon_1}{(1+\epsilon_1)^{\gamma-2}} \int_0^\infty f_4 \sqrt{\epsilon_4} d\epsilon_4 \qquad (3.9) \\
&\leq C \left(1 + 2\|f\|_{L^\infty(\mathbb{R}^+)}\right) \|f\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)} \int_0^\infty f_4 \sqrt{\epsilon_4} d\epsilon_4.
\end{aligned}$$

Using the fact that $f \geq 0$ we obtain $0 \leq \Psi(\epsilon_1, t) \leq 1$, $0 \leq \frac{\Psi(\epsilon_1,t)}{\Psi(\epsilon_1,s)} \leq 1$, whence (3.8) follows. ■

We need to derive detailed estimates of the function $a(t, \epsilon_1)$ in (2.2).

Lemma 3.7 *Suppose that $f \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$ with $\gamma > 3$. The function $a(t, \epsilon_1)$ defined in (2.2) can be written as:*

$$a(t, \epsilon_1) = \frac{8\pi^2 \sqrt{\epsilon_1}}{\sqrt{2}} \int_0^\infty f(t, \epsilon) \sqrt{\epsilon} d\epsilon + S[f](t, \epsilon_1), \qquad (3.10)$$

where

$$S[f](t, \epsilon_1) = S_1[f](t, \epsilon_1) + S_2[f](t, \epsilon_1), \qquad (3.11)$$

with:

$$S_1[f] = \frac{8\pi^2 \epsilon_1}{\sqrt{2}} \int_0^\infty f_2 \omega\left(\frac{\epsilon_2}{\epsilon_1}\right) d\epsilon_2 \quad , \quad S_2[f] = \frac{8\pi^2}{\sqrt{2}} \int_0^\infty \int_0^\infty f_2 \left(f_3 + f_4\right) W d\epsilon_3 d\epsilon_4, \qquad (3.12)$$

and:

$$\omega(x) = \frac{x^{\frac{3}{2}}}{3} \quad , \quad x \leq 1 \quad \text{and} \quad \omega(x) = \left(x - \sqrt{x} + \frac{1}{3}\right) \quad , \quad x \geq 1. \qquad (3.13)$$

If $f \geq 0$ we have $S_1[f] \geq 0$ and $S_2[f] \geq 0$.

Proof. Using the change of variables $\epsilon_2 = \epsilon_3 + \epsilon_4 - \epsilon_1$, $\xi = \epsilon_3 - \epsilon_4$ we obtain:

$$\begin{aligned}
a(t, \epsilon_1) = {} & \frac{4\pi^2}{\sqrt{2}} \int_0^\infty f_2 d\epsilon_2 \int_{-(\epsilon_2+\epsilon_1)}^{(\epsilon_2+\epsilon_1)} d\xi W\left(\epsilon_1, \epsilon_2, \frac{\epsilon_2 + \epsilon_1 + \xi}{2}, \frac{\epsilon_2 + \epsilon_1 - \xi}{2}\right) + \\
& + S_2[f](t, \epsilon_1),
\end{aligned}$$

where $S_2[f]$ is as in (3.12).

Using the symmetry of the function $W\left(\epsilon_1, \epsilon_2, \frac{\epsilon_2+\epsilon_1+\xi}{2}, \frac{\epsilon_2+\epsilon_1-\xi}{2}\right)$ with respect to the transformation $\xi \to (-\xi)$, as well as the definition of W, we obtain:

$$a(t, \epsilon_1) = \frac{8\pi^2 \epsilon_1}{\sqrt{2}} \int_0^\infty f_2 \Omega\left(\frac{\epsilon_2}{\epsilon_1}\right) d\epsilon_2 + S_2[f](t, \epsilon_1),$$

with:

$$\Omega(x)=\int_0^{(x+1)} d\xi W\left(1,x,\frac{x+1-\xi}{2}\right),\quad x\geq 0.$$

We can compute $\Omega(x)$ treating separately the cases $x\leq 1$ and $x>1$. Using the definition of W we obtain:

$$\Omega(x)=\sqrt{x}+\frac{(x)^{\frac{3}{2}}}{3}\ \text{ if } x\leq 1\quad,\quad \Omega(x)=\left(x+\frac{1}{3}\right)\ \text{ if } x>1.$$

We can then write $\Omega(x)=\sqrt{x}+\omega(x)$ with $\omega(\cdot)$ as in (3.13). This gives (3.10). Using the fact that $\omega(x)\geq 0$ we conclude the Proof of Lemma 3.7. ■

The following Lemma is important to control the behaviour of $\mathcal{T}(f)(t,\epsilon)$ for large values of ϵ. Its proof uses in a crucial way the structure of the quadratic terms of the equation (1.10).

Lemma 3.8 *Let $\gamma>3$ and $\mathcal{T}$ be as in Proposition (3.5). There exists $\theta\in(0,1)$ and $C>0$ both of them depending only on γ and such that, for any $t\in[0,T]$ the following estimate holds:*

$$\begin{aligned}
\|\mathcal{T}(f)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}(t) &\leq \|f_0\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}+\\
&+t\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\left(\sup_{0\leq s\leq t}\int\left(1+\epsilon^{\frac{3}{2}}\right)f(s,\epsilon)\,d\epsilon\right)^2+\\
&+Ct\left(1+\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right)\left(\sup_{0\leq s\leq t}\int_0^\infty f(s,\epsilon)\,d\epsilon\right)^2+\\
&+Ct\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\left(\sup_{0\leq s\leq t}\int_0^\infty\left(1+(\epsilon)^{\frac{3}{2}}\right)f(s,\epsilon)\,d\epsilon\right)+\\
&+\theta\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\,.
\end{aligned}\tag{3.14}$$

Proof. We estimate the operator $\mathcal{T}(f)$ in the norm $L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)$. Notice that (3.5) implies:

$$\|\mathcal{T}(f)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\leq\|f_0\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}+J_1+J_2+J_3,\tag{3.15}$$

with:

$$J_1=\frac{8\pi^2}{\sqrt{2}}\left\|\int_0^t\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}\int_0^\infty\int_0^\infty f_3f_4W\,d\epsilon_3d\epsilon_4ds\right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)},$$

$$J_2=\left\|\int_0^t\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}f_1\int_0^\infty\int_0^\infty f_3f_4W\,d\epsilon_3d\epsilon_4ds\right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)},$$

$$J_3=\left\|\int_0^t\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}\int_0^\infty\int_0^\infty f_3f_4f_2W\,d\epsilon_3d\epsilon_4ds\right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}.$$

The terms J_2 can be readily estimated:

$$J_2 \leq t \sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)} \left(\sup_{0\leq s\leq t} \int \left(1+\epsilon^{\frac{3}{2}}\right) f(s,\epsilon)\, d\epsilon \right)^2 . \tag{3.16}$$

In order to estimate J_3 we use the symmetry in the variables $\epsilon_3,\ \epsilon_4$ to obtain:

$$J_3 \leq 2\int_0^t \left\| \int_0^\infty f_2 d\epsilon_2 \int_{(0,\infty)} \chi_{\{\epsilon_3\geq\epsilon_4\}} f_3 f_4 d\epsilon_4 \right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)} ds. \tag{3.17}$$

Using the fact that in the region $\{\epsilon_3 \geq \epsilon_4\},\ \epsilon_2 \geq 0$ we have $\epsilon_3 \geq \frac{\epsilon_1}{2}$ as well as the definition of the norm $\|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}$ we arrive at:

$$\begin{aligned} J_3 &\leq C\int_0^t \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)} \left\| \frac{1}{(2+\epsilon_1)^\gamma} \int_0^\infty f_2 d\epsilon_2 \int_{\frac{\epsilon_1}{2}}^\infty f_4 d\epsilon_4 \right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)} ds \\ &\leq Ct \sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)} \left(\sup_{0\leq s\leq t} \int_0^\infty f(s,\epsilon)\, d\epsilon \right)^2 . \end{aligned} \tag{3.18}$$

The term J_1 must be estimated more carefully. We use at this point ideas closely related to those in [3],[4]. Suppose that $\frac{1}{2} < \mu < 1$ and let us write $M_0(t) = \int g(t,\epsilon)\, d\epsilon$. This number is the total mass of the particles and for solutions of (1.10), (1.11) can be expected to be constant. However, this has not yet been proved and we must therefore keep $M_0(t)$ as a function of t. Using (1.12), (2.2) and Lemma 3.7 we obtain $\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)} \leq \exp\left(-\pi\sqrt{\epsilon_1}\int_s^t M_0(\xi)\, d\xi\right)$. Then, splitting the domain of integration in J_1 in the two subdomains indicated in Figure 3.1 we obtain:

$$\begin{aligned} J_1 &\leq J_{1,1} + J_{1,2}, \\ J_{1,1} &= \frac{8\pi^2}{\sqrt{2}} \left\| \int_0^t \exp\left(-\pi\sqrt{\epsilon_1}\int_s^t M_0(\xi)\, d\xi\right) \left(\int_{(1-\mu)\epsilon_1}^\infty f_3 d\epsilon_3 \right)^2 ds \right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}, \\ J_{1,2} &= \frac{16\pi^2}{\sqrt{2}} \left\| \int_0^t \exp\left(-\pi\sqrt{\epsilon_1}\int_s^t M_0(\xi)\, d\xi\right) \times \right. \\ &\qquad \left. \times \left(\int_{\mu\epsilon_1}^\infty f_3 d\epsilon_3 \right) \left(\int_0^{(1-\mu)\epsilon_1} \sqrt{\frac{\epsilon_4}{\epsilon_1}} f_4 d\epsilon_4 \right) ds \right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}. \end{aligned}$$

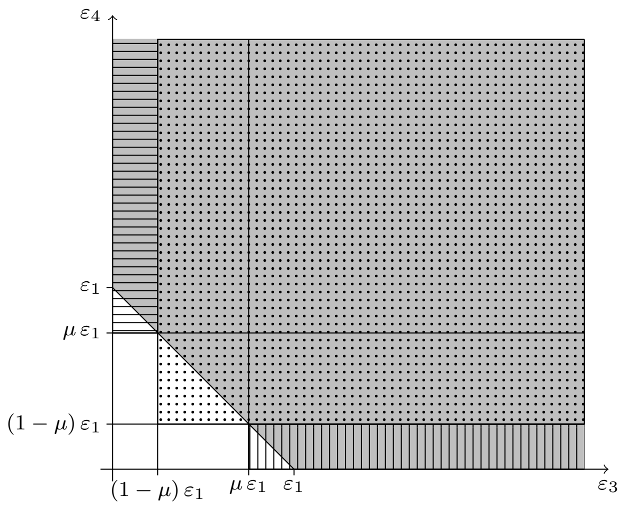

Figure 3.2: ($\varepsilon = 3$, $\mu = 3/4$) The domain of integration of the integral J_1, in grey, is covered by the union of the domain with points, the domain with vertical lines and that with horizontal lines. In the part of the grey domain covered by points the function W is bounded by one. In the part of vertical and horizontal lines the grey domain covered by vertical lines it is less than or equal to $\sqrt{\varepsilon_4}/\sqrt{\varepsilon_3}$.

We first estimate $J_{1,1}$:

$$J_{1,1} \leq \frac{8\pi^2}{\sqrt{2}} \left\| \int_0^t \exp\left(-\pi\sqrt{\epsilon_1}\int_s^t M_0(\xi)\,d\xi\right)\left(\int_{(1-\mu)\epsilon_1}^{\infty} f_3 d\epsilon_3\right)^2 ds\right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}$$

$$\leq C\left\|\frac{1}{1+(\epsilon_1)^{\frac{3}{2}}}\int_0^t \left(\int_{(1-\mu)\epsilon_1}^{\infty} f_3 d\epsilon_3\right)\left(\int_{(1-\mu)\epsilon_1}^{\infty}\left(1+(\epsilon_4)^{\frac{3}{2}}\right) f_4 d\epsilon_4\right) ds\right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}$$

$$\leq C\left\|\int_0^t \frac{\|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}\left(\int_0^{\infty}\left(1+(\epsilon_4)^{\frac{3}{2}}\right) f_4 d\epsilon_4\right) ds\right\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}$$

$$\leq C\int_0^t \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\left(\int_0^{\infty}\left(1+(\epsilon_4)^{\frac{3}{2}}\right) f_4 d\epsilon_4\right) ds,$$

where the constant C depends on μ, γ.

We now estimate $J_{1,2}$ which is the most delicate term. We fix $L > 0$ and treat separately the cases $\epsilon_1 \leq L$ and $\epsilon_1 > L$. In the region where $\epsilon_1 \leq L$ we have the pointwise estimate:

$$J_{1,2} \leq \frac{16\pi^2}{\sqrt{2}}\int_0^t \exp\left(-\pi\sqrt{\epsilon_1}\int_s^t M_0(\xi)\,d\xi\right)\left(\int_{\mu\epsilon_1}^{\infty} f_3 d\epsilon_3\right)\left(\int_0^{(1-\mu)\epsilon_1}\sqrt{\frac{\epsilon_4}{\epsilon_1}} f_4 d\epsilon_4\right) ds$$

$$\leq \frac{16\pi^2}{\sqrt{2}}\sqrt{(1-\mu)}\int_0^t \left(\int_0^{\infty} f_3 d\epsilon_3\right)^2 ds \quad , \quad \epsilon_1 \leq L.$$

On the other hand, if $\epsilon_1 > L$ we obtain:

$$J_{1,2} \leq 2\pi \int_0^t \frac{\exp\left(-\pi\sqrt{\epsilon_1}\int_s^t M_0(\xi)\,d\xi\right)}{\sqrt{\epsilon_1}} \left(\int_{\mu\epsilon_1}^{\infty} f_3 d\epsilon_3\right) M_0(s)\,ds$$

$$\leq \frac{2\pi}{(\gamma-1)} \left(\sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right) \int_0^t \frac{\exp\left(-\pi\sqrt{\epsilon_1}\int_s^t M_0(\xi)\,d\xi\right)}{\sqrt{\epsilon_1}} \frac{M_0(s)\,ds}{(1+\mu\epsilon_1)^{\gamma-1}}$$

$$\leq \frac{2}{(\gamma-1)} \left(\sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right) \frac{1}{\epsilon_1 (1+\mu\epsilon_1)^{\gamma-1}},$$

where we have used the definition of $M_0(s)$ and (1.12). Using now the inequality $\frac{1}{\epsilon_1} \leq \frac{L+1}{L}\frac{1}{1+\mu\epsilon_1}$ that holds for $\epsilon_1 \geq L$ we obtain:

$$J_{1,2} \leq \frac{2}{(\gamma-1)} \left(\sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right) \frac{L+1}{L}\frac{1}{(1+\mu\epsilon_1)^{\gamma}},\ \varepsilon_1 \geq L.$$

We now use that $\gamma > 3$. Choosing then L large and μ sufficiently close to one (both depending on γ we obtain:

$$\frac{2}{(\gamma-1)}\frac{L+1}{L}\frac{1}{(1+\mu\epsilon_1)^{\gamma}} \leq \frac{\theta}{(1+\epsilon_1)^{\gamma}}$$

with $\theta < 1$ independent on ϵ_1. Therefore, we obtain, adding the contributions from the regions where $\epsilon_1 \leq L$ and $\epsilon_1 > L$:

$$J_{1,2} \leq \frac{16\pi^2}{\sqrt{2}}\sqrt{(1-\mu)}\,(1+L)^{\gamma} \int_0^t \left(\int_0^{\infty} f_3 d\epsilon_3\right)^2 ds + \theta \sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\,. \tag{3.19}$$

We combine now (3.16), (3.18), (3.19) to obtain (3.14) whence the Lemma follows. ■

Proof of Proposition 3.5. It is just a consequence of (3.6), Lemma 3.6 and Lemma 3.8. ■

We now prove the following result.

Lemma 3.9 *Assume $\gamma > 3$. There exists $B_0 > 0$ depending on $\|f_0\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}$, such that, for any $R > B_0$ there exists $T_*(R)$ such that, for any $T < T_*(R)$ the operator $\mathcal{T}$, defined by means of (3.5), transforms $\mathcal{X}_{R,T}^{\gamma}$ into itself.*

Proof. Since $\gamma > 3$ we can estimate $\int \left(1+\epsilon^{\frac{3}{2}}\right) f(\epsilon)\,d\epsilon$ by of $C\,\|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}$. Using Lemma 3.8 we obtain:

$$\|\mathcal{T}(f)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}(t) \leq \|f_0\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)} + Ct\left(\sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right)^3 +$$
$$+ Ct\left(1+\sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right)\left(\sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right)^2 +$$
$$+ Ct\left(\sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}\right)^2 + \theta \sup_{0\leq s\leq t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)},$$

if $0 \leq t \leq T$. Therefore, since $\|f\|_{L^{\infty}([0,T];L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma}))} \leq R$ we obtain:

$$\sup_{0\leq t\leq T} \|\mathcal{T}(f)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}(t) \leq \|f_0\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})} + CT\left(R^2+R^3\right) + \theta R. \quad (3.20)$$

Since $\theta < 1$ it then follows that, choosing R sufficiently large and then T small (depending on B) we would obtain that the right-hand side of (3.20) is smaller than R, whence the result follows. ■

We also have the following:

Lemma 3.10 *Suppose that $R \geq B_0$ and $T < T_*(R)$ are as in Lemma 3.9. Then, the operator $\mathcal{T}$ defined by means of (3.5) is continuous in the metric space $\mathcal{X}_{R,T}^{\gamma}$.*

Proof. Since $\mathcal{X}_{R,T}^{\gamma}$ is a metric space, it is sufficient to check the result for sequences. Suppose then that we have a sequence $\{f_n\}_{n\geq 0} \in \mathcal{X}_{R,T}^{\gamma}$ and $\mu \in \mathcal{X}_{R,T}^{\gamma}$ such that $f_n \to \mu$ in the topology of $\mathcal{X}_{R,T}^{\gamma}$. Let $\varphi \in C_0([0,\infty))$ be a test function. We need to show that we can pass to the limit in:

$$\int_{\mathbb{R}^{+}} \mathcal{T}(f_n)(t,\epsilon_1)\varphi(\epsilon_1)\,d\epsilon_1$$

uniformly in $t \in [0,T]$. We notice first that, due to the boundedness of the functions in $\mathcal{X}_{R,T}^{\gamma}$, weak convergence of a sequence $\{f_n\}$ in the topology Θ to $\mu \in \mathcal{X}_{R,T}^{\gamma}$ implies the convergence of integrals like $\int_{[R_1,R_2]} f_n(t,\epsilon)\,d\epsilon$ to $\int_{[R_1,R_2]} \mu d\epsilon$ for any $0 \leq R_1 \leq R_2 < \infty$. Indeed, this can be seen approximating the characteristic function of the interval $[R_1,R_2]$ by a set of continuous functions $\varphi_m \in C_0([0,T]\times[0,\infty))$ in the topology of $L^1([0,T]\times[0,\infty))$. The L^{∞} estimates for the functions $f_n \in \mathcal{K}_T$ imply that, for any given $\varepsilon > 0$ choosing m large enough we have $\left|\int_{[R_1,R_2]} f_n d\epsilon - \int f_n\varphi_m d\epsilon\right| < \frac{\varepsilon}{2}$ uniformly in $t \in [0,T]$. Choosing then n large enough we obtain $\left|\int_{[R_1,R_2]} \mu\varphi_m d\epsilon - \int f_n\varphi_m d\epsilon\right| < \frac{\varepsilon}{2}$, whence the desired convergence follows.

We consider first the term $f_0(\epsilon_1)\Psi(t,\epsilon_1)$. The function $f_0(\epsilon_1)$ is fixed, independent on n. We need to compute the pointwise limit as $n \to \infty$ of

$$\begin{aligned} a_n(t,\epsilon_1) &= \frac{8\pi^2}{\sqrt{2}} \int_0^{\infty}\int_0^{\infty} f_{n,2}(1+f_{n,3}+f_{n,4})\,W d\epsilon_3 d\epsilon_4 \\ \Psi_n(t,\epsilon_1) &= \exp\left(-\int_0^t a_n(s,\epsilon_1)\,ds\right). \end{aligned}$$

Since φ is compactly supported we need to compute this limit only in bounded regions. Suppose that $\epsilon_1 \leq L_1$. We then rewrite $a_n(t,\epsilon_1)$ as:

$$\begin{aligned} a_n(t,\epsilon_1) = &\frac{8\pi^2}{\sqrt{2}} \int\int_{[0,L_2]^2} f_{n,2}(1+f_{n,3}+f_{n,4})\,W d\epsilon_3 d\epsilon_4 + \\ &+ \frac{8\pi^2}{\sqrt{2}} \int\int_{\mathbb{R}_+^2\setminus[0,L_2]^2} f_{n,2}(1+f_{n,3}+f_{n,4})\,W d\epsilon_3 d\epsilon_4, \end{aligned} \quad (3.21)$$

where $L_2 > 2L_1$. In order to pass to the limit in the first integral on the right-hand side we change variables to have as integration variables (ϵ_2, ϵ_3) for the integral containing the term $f_{n,2}f_{n,3}$ and the integration variables (ϵ_2, ϵ_4) for the integral containing the term $f_{n,2}f_{n,4}$. In the case of the term containing only $f_{n,2}$ either choice of variables of integration is valid. We then obtain integrals in sets with test functions chosen as characteristic functions multiplying a continuous function which contains the dependence on W. Due to the estimates for the functions $f_n \in \mathcal{K}_T$ we can pass to the limit in these integrals as $n \to \infty$. The resulting limit is, after returning to the original integration variables:

$$\frac{8\pi^2}{\sqrt{2}} \int \int_{[0,L_2]^2} \mu_2 (1 + \mu_3 + \mu_4) W d\epsilon_3 d\epsilon_4.$$

The last integral in (3.21) can be estimated using ϵ_2 as one of the integration variables and the fact that $L_2 > 2L_1$:

$$\frac{8\pi^2}{\sqrt{2}} \int \int_{\mathbb{R}_+^2 \setminus [0,L_2]^2} f_{n,2} (1 + f_{n,3} + f_{n,4}) W d\epsilon_3 d\epsilon_4 \leq C \int_{\frac{L_2}{2}}^{\infty} f_n(\epsilon) \epsilon d\epsilon.$$

Using then the fact that $\int_0^\infty f_n(t,\epsilon) \epsilon^{\frac{3}{2}} d\epsilon \leq 2 \|f_0\|_{\mathcal{Y}}$ we obtain:

$$\frac{8\pi^2}{\sqrt{2}} \int \int_{\mathbb{R}_+^2 \setminus [0,L_2]^2} f_{n,2} (1 + f_{n,3} + f_{n,4}) W d\epsilon_3 d\epsilon_4 \leq \frac{C}{\sqrt{L_2}},$$

where C is independent on n. Taking then the limit $L_2 \to \infty$ and then $n \to \infty$ we obtain that:

$$a_n(t,\epsilon_1) \to a(t,\epsilon_1) = \frac{8\pi^2}{\sqrt{2}} \int \int \mu_2 (1 + \mu_3 + \mu_4) W d\epsilon_3 d\epsilon_4, \tag{3.22}$$

as $n \to \infty$ for each $\epsilon_1 \geq 0$ uniformly in $t \in [0,T]$. Since $a_n(t,\epsilon_1) \geq 0$ we have $\Psi_n(t,\epsilon_1) \leq 1$. We can use then Lebesgue's Theorem to obtain:

$$\int_{\mathbb{R}+} f_0(\epsilon_1) \Psi_n(t,\epsilon_1) \varphi(t,\epsilon_1) d\epsilon_1 dt \to \int_{\mathbb{R}+} f_0(\epsilon_1) \Psi(t,\epsilon_1) \varphi(t,\epsilon_1) d\epsilon_1 dt, \tag{3.23}$$

as $n \to \infty$, uniformly in $t \in [0,T]$ with:

$$\Psi(t,\epsilon_1) = \exp\left(-\int_0^t a(s,\epsilon_1) ds\right),$$

and $a(t,\epsilon_1)$ as in (3.22).

We now pass to the limit in the term resulting from the last term in (3.5). To this end we compute the limits of the following integrals:

$$I_{1,n} = \frac{8\pi^2}{\sqrt{2}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty \frac{\Psi_n(t,\epsilon_1)}{\Psi_n(s,\epsilon_1)} f_{n,3} f_{n,4} W \varphi(t,\epsilon_1) d\epsilon_1 d\epsilon_3 d\epsilon_4 ds,$$
$$I_{2,n} = \frac{8\pi^2}{\sqrt{2}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty \frac{\Psi_n(t,\epsilon_1)}{\Psi_n(s,\epsilon_1)} f_{n,1} f_{n,3} f_{n,4} W \varphi(t,\epsilon_1) d\epsilon_1 d\epsilon_3 d\epsilon_4 ds,$$
$$I_{3,n} = \frac{8\pi^2}{\sqrt{2}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty \frac{\Psi_n(t,\epsilon_1)}{\Psi_n(s,\epsilon_1)} f_{n,2} f_{n,3} f_{n,4} W \varphi(t,\epsilon_1) d\epsilon_1 d\epsilon_3 d\epsilon_4 ds.$$

The limit of the first integral, $\lim_{n\to\infty} I_{1,n}$, is obtained in the same way as (3.23). We split the integral $I_{1,n}$ in the variables (ϵ_3, ϵ_4) in the regions $[0,L]^2$ and $\mathbb{R}_+^2 \setminus [0,L_2]^2$ with L_2 large to be determined. On the other hand, the integration in ϵ_1 takes place in a compact set due to the fact that φ is compactly supported. The contribution to the integral due to the set $(\epsilon_3,\epsilon_4) \in \mathbb{R}_+^2 \setminus [0,L_2]^2$ is uniformly small if L_2 is large (independently on n) arguing as before. We can take the limit of $\frac{\Psi_n(t,\epsilon_1)}{\Psi_n(s,\epsilon_1)}$ as $n\to\infty$ using Lebesgue's Theorem, whence:

$$I_{1,n} \to \frac{8\pi^2}{\sqrt{2}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty \frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)} \mu_3\mu_4 W\varphi(t,\epsilon_1)\, d\epsilon_1 d\epsilon_3 d\epsilon_4 ds\,, \quad n\to\infty.$$

In order to take the limit of $I_{2,n}$ we first use the fact that the integration takes place in a bounded interval $[0,L_1]$ due to the choice of φ. We use Egorov's Theorem to approximate uniformly, for each ε_0 small, $\frac{\Psi_n(t,\epsilon_1)}{\Psi_n(s,\epsilon_1)}$ as $\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}$ in a set $A \subset [0,L_1]$ such that $|[0,L_1]\setminus A| \le \varepsilon_0$. Notice that the set A depends on t and s. We then have

$$\left| \int_0^{L_1} \frac{\Psi_n(t,\epsilon_1)}{\Psi_n(s,\epsilon_1)} f_{n,1} W d\epsilon_1 - \int_A \frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)} f_{n,1} W d\epsilon_1 \right| \le C \|f\|_{\mathcal{X}_{R,T}^\gamma} \varepsilon_0,$$

uniformly in bounded sets of ϵ_3,ϵ_4 in bounded sets if n is chosen sufficiently large. It then follows that:

$$I_{2,n} \to \frac{8\pi^2}{\sqrt{2}} \int_0^t \int_0^\infty \int_0^\infty \int_0^\infty \frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)} \mu_1\mu_3\mu_4 W\varphi(t,\epsilon_1)\, d\epsilon_1 d\epsilon_3 d\epsilon_4 ds\,, \quad n\to\infty.$$

We now consider the limit of the term $I_{3,n}$. We can again restrict the domain of integration to a large cube $[0,L]^3$ because the contribution of the tails can be estimated uniformly in n as $L\to\infty$. On the other hand we can apply again Egorov's Theorem to show that for any $\varepsilon_0 > 0$ there exists a set $A\subset[0,L]$, depending on s and t, such that $|[0,L]\setminus A| \le \varepsilon_0$ with the property that $\frac{\Psi_n(t,\epsilon_1)}{\Psi_n(s,\epsilon_1)}$ converges uniformly to $\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}$ uniformly on A. We now change the integration variables by means of:

$$(\epsilon_1,\epsilon_3,\epsilon_4) \to (\epsilon_2,\epsilon_3,\epsilon_4) = (\epsilon_3+\epsilon_4-\epsilon_1, \epsilon_3, \epsilon_4)\,.$$

This transformation brings the set $([0,L]\setminus A)\times[0,L]^2$ to a new set B whose intersection with the domain of integration Ω_L has a measure of order $C_L\varepsilon_0$, with C_L depending on L, but not on ε_0, n. We then need to take the limit as $n\to\infty$ in:

$$\begin{aligned} &\frac{8\pi^2}{\sqrt{2}} \int_0^t \int_0^L \int_0^L \int_0^L \frac{\Psi_n(t,\epsilon_3+\epsilon_4-\epsilon_2)}{\Psi_n(s,\epsilon_3+\epsilon_4-\epsilon_2)} f_{n,2} f_{n,3} f_{n,4} W\varphi(t,\epsilon_1)\, d\epsilon_2 d\epsilon_3 d\epsilon_4 ds \\ &= \frac{8\pi^2}{\sqrt{2}} \int_0^t \int\int\int_{\Omega_L\setminus B} [\cdots] + \frac{8\pi^2}{\sqrt{2}} \int_0^t \int\int\int_B [\cdots]\,. \end{aligned}$$

The last integral can be estimated as:

$$\frac{8\pi^2}{\sqrt{2}} \int_0^t \int\int\int_B [\cdots] \le C_L\varepsilon_0.$$

On the other hand, in the integral $\frac{8\pi^2}{\sqrt{2}}\int_0^t\int\int\int_{\Omega_L\setminus B}[\cdots]$ we have uniform convergence of $\frac{\Psi_n(t,\cdot)}{\Psi_n(s,\cdot)}$ to $\frac{\Psi(t,\cdot)}{\Psi(s,\cdot)}$ as $n\to\infty$. We can then complete the integral to the domain Ω_L, adding a term which gives an error of order $C_L\varepsilon_0$. Making them ε_0 small and then L large, and returning to the original set of variables we obtain:

$$I_{3,n}\to\frac{8\pi^2}{\sqrt{2}}\int_0^t\int_0^\infty\int_0^\infty\int_0^\infty\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}\mu_2\mu_3\mu_4W\varphi(t,\epsilon_1)\,d\epsilon_1d\epsilon_3d\epsilon_4ds\,,\quad n\to\infty.$$

This concludes the proof of the continuity of $\mathcal{T}$ in the topology Θ. ■

Lemma 3.11 *Suppose that* $R\geq B_0$ *and* $T<T_*(R)$ *are as in Lemma 3.9. The operator* $\mathcal{T}$*, defined by means of (3.5) and restricted to the metric space* $\mathcal{X}_{R,T}^{\gamma}$ *is compact.*

Proof. By Proposition 3.3 it is enough to show that the functions $\psi_\varphi(t;\mathcal{T}(f))=\int_{\mathbb{R}^+}f(t,\epsilon)\varphi(\epsilon)\,d\epsilon$ are uniformly continuous in $t\in[0,T]$ for $f\in\mathcal{X}_{R,T}^{\gamma}$. Therefore, we need to prove that the functions

$$\Psi_\varphi^{(1)}(t;f)=\int_{\mathbb{R}^+}f_0(\epsilon_1)\Psi(t,\epsilon_1)\varphi(\epsilon_1)\,d\epsilon_1,$$

$$\Psi_\varphi^{(2)}(t;f)=\frac{8\pi^2}{\sqrt{2}}\int_0^t\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}\int_0^\infty\int_0^\infty\int_0^\infty f_3f_4(1+f_1+f_2)\varphi(\epsilon_1)Wd\epsilon_1d\epsilon_3d\epsilon_4ds,$$

are uniformly continuous for $f\in\mathcal{X}_{R,T}^{\gamma},\ t\in[0,T]$. We prove first that the family of functions $\left\{\Psi_\varphi^{(1)}(\cdot;f):f\in\mathcal{X}_{R,T}^{\gamma}\right\}$ is uniformly Lischitz in $t\in[0,T]$. To this end, we just differentiate these functions with respect to t. This can be made *a.e.* $t\in[0,T]$ due to the uniform boundedness of $a(t,\epsilon_1)$ in compact sets of $\epsilon_1\in[0,\infty)$. Then:

$$\begin{aligned}\frac{\partial\Psi_\varphi^{(1)}(t;f)}{\partial t}&=-\int_{\mathbb{R}^+}f_0(\epsilon_1)a(t,\epsilon_1)\Psi(t,\epsilon_1)\varphi(\epsilon_1)\,d\epsilon_1\\ \frac{\partial\Psi_\varphi^{(2)}(t;f)}{\partial t}&=\frac{8\pi^2}{\sqrt{2}}\int_0^\infty\int_0^\infty\int_0^\infty f_3f_4(1+f_1+f_2)\varphi(\epsilon_1)Wd\epsilon_1d\epsilon_3d\epsilon_4-\\ &-\frac{8\pi^2}{\sqrt{2}}\int_0^t\frac{\Psi(t,\epsilon_1)a(t,\epsilon_1)}{\Psi(s,\epsilon_1)}\int_0^\infty\int_0^\infty\int_0^\infty f_3f_4(1+f_1+f_2)\varphi(\epsilon_1)Wd\epsilon_1d\epsilon_3d\epsilon_4ds\end{aligned}$$

a.e. $t\in[0,T]$. The right-hand side of these formulas can be bounded easily using that $f\in\mathcal{X}_{R,T}^{\gamma}$. It then follows that the family of functions $\mathcal{T}\left(\mathcal{X}_{R,T}^{\gamma}\right)$ is equicontinuous and then Proposition 3.2 implies that it is compact. ■

Proposition 3.12 *Let* $f_0\in L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)$ *with* $\gamma>3$*. There exists* $T>0$ *depending only on* $\|f_0(\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}$ *and at least one mild solution of (1.10), (1.11) in the sense of Definition 2.1.*

Proof. Given $\|f_0(\cdot)\|_{L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma)}$ we choose $R \geq B_0$ and $T < T_*(R)$ are as in Lemma 3.9. Then the operator $\mathcal{T}$ transforms the space $\mathcal{X}_{R,T}^\gamma$ into itself due to Lemma 3.9. Moreover, this operator is continuous due to Lemma 3.10 if we endow $\mathcal{X}_{R,T}^\gamma$ with the topology Θ and due to Lemmas 3.1, 3.3 and 3.11 the operator $\mathcal{T} : \mathcal{X}_{R,T}^\gamma \to \mathcal{X}_{R,T}^\gamma$ is compact. Then Schauder-Tikhonov Theorem (cf. [5], p. 456) implies the existence of a fixed point of $\mathcal{T}$ in $f \in \mathcal{X}_{R,T}^\gamma$. Due to the definition of $\mathcal{T}$ it follows that f satisfies (2.1). ∎

Proposition 3.12 yields a solution of (1.10), (1.11) in the sense of Definition 2.1. In order to check that the total number of particles and the total energy are constant in time it is convenient to prove that the derived solution is a weak solution in some suitable sense.

Lemma 3.13 *Suppose that $\gamma > 3$ and $f \in L_{loc}^\infty([0,T);L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma))$ is a mild solution of (1.10), (1.11) in the sense of Definition 2.1. Let g be as in (1.12). Suppose that $\varphi \in C_0^2([0,T)\times[0,\infty))$ Then, the function $\psi_\varphi(t) = \int_{\mathbb{R}^+} g(t,\epsilon)\varphi(t,\epsilon)d\epsilon$ is Lipschitz continuous in $t \in [0,T]$ and the following identity holds:*

$$\partial_t\left(\int_{\mathbb{R}^+} g\varphi d\epsilon\right) = \int_{\mathbb{R}^+} g\partial_t\varphi d\epsilon + \frac{1}{2^{\frac{5}{2}}}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{g_1g_2g_3\Phi}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}}Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 + \\ + \frac{\pi}{2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3\,, \quad a.e.\ t \in [0,T]\,, \tag{3.24}$$

where:

$$\Phi = \min\left\{\sqrt{\epsilon_1}, \sqrt{\epsilon_2}, \sqrt{\epsilon_3}, \sqrt{(\epsilon_1+\epsilon_2-\epsilon_3)_+}\right\} \tag{3.25}$$

$$Q_\varphi = \varphi(\epsilon_3) + \varphi(\epsilon_1+\epsilon_2-\epsilon_3) - 2\varphi(\epsilon_1)\,. \tag{3.26}$$

Proof. Using (2.1) we obtain that g satisfies:

$$g(t,\epsilon_1) = g_0(\epsilon_1)\Psi(t,\epsilon_1) + \\ + \pi\int_0^t \frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}\int_0^\infty\int_0^\infty \frac{g_3g_4}{\sqrt{\epsilon_3\epsilon_4}}\left(1 + \frac{g_1}{4\pi\sqrt{2\epsilon_1}} + \frac{g_2}{4\pi\sqrt{2\epsilon_2}}\right)\Phi d\epsilon_3 d\epsilon_4 ds$$

a.e. $t \in [0,T)$. Multiplying by a test function $\varphi \in C_0([0,T)\times[0,\infty))$ and integrating in $\epsilon \in [0,\infty)$ we obtain:

$$\int_0^\infty g(t,\epsilon_1)\varphi(t,\epsilon_1)d\epsilon_1 = \int_0^\infty g_0(\epsilon_1)\varphi(t,\epsilon_1)\Psi(t,\epsilon_1)d\epsilon_1 + \\ + \pi\int_0^t\int_0^\infty\int_0^\infty\int_0^\infty \frac{g_3g_4\Phi}{\sqrt{\epsilon_3\epsilon_4}}\left(1 + \frac{g_1}{4\pi\sqrt{2\epsilon_1}} + \frac{g_2}{4\pi\sqrt{2\epsilon_2}}\right)\varphi(t,\epsilon_1)\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)}d\epsilon_1 d\epsilon_3 d\epsilon_4 ds.$$

Notice that the integral $\psi_\varphi(t) = \int_0^\infty g(t,\epsilon_1)\varphi(t,\epsilon_1)d\epsilon_1$ is Lipschitz continuous with respect to t in $t \in [0,T)$, since $a(t,\epsilon_1)$ in (2.2) is uniformly bounded in compact sets of $[0,T)\times[0,\infty)$. Therefore $\psi_\varphi(t)$ is differentiable $a.e.$ $t \in [0,T)$ and its derivative is given by:

$$\partial_t \left(\int_0^\infty g(t,\epsilon_1)\varphi(t,\epsilon_1)\,d\epsilon_1 \right) = \int_0^\infty g(t,\epsilon_1)\,\partial_t\varphi(t,\epsilon_1)\,d\epsilon_1 + $$
$$+\pi \int_0^\infty\int_0^\infty\int_0^\infty \frac{g_3 g_4 \Phi}{\sqrt{\epsilon_3\epsilon_4}} \left(1+\frac{g_1}{4\pi\sqrt{2\epsilon_1}}+\frac{g_2}{4\pi\sqrt{2\epsilon_2}}\right)\varphi(t,\epsilon_1)\,d\epsilon_1 d\epsilon_3 d\epsilon_4 -$$
$$- \int_0^\infty \varphi(t,\epsilon_1)\,a(t,\epsilon_1)\,g(t,\epsilon_1)\,d\epsilon_1.$$

Using (2.2) we obtain:

$$\partial_t \left(\int_0^\infty g(t,\epsilon_1)\varphi(t,\epsilon_1)\,d\epsilon_1 \right) = \int_0^\infty g(t,\epsilon_1)\,\partial_t\varphi(t,\epsilon_1)\,d\epsilon_1 + $$
$$+\pi \int_0^\infty\int_0^\infty\int_0^\infty \frac{g_3 g_4 \Phi}{\sqrt{\epsilon_3\epsilon_4}} \left(1+\frac{g_1}{4\pi\sqrt{2\epsilon_1}}+\frac{g_2}{4\pi\sqrt{2\epsilon_2}}\right)\varphi(t,\epsilon_1)\,d\epsilon_1 d\epsilon_3 d\epsilon_4 -$$
$$-\pi \int_0^\infty\int_0^\infty\int_0^\infty \frac{g_1 g_2 \Phi}{\sqrt{\epsilon_1\epsilon_2}} \left(1+\frac{g_3}{4\pi\sqrt{2\epsilon_3}}+\frac{g_4}{4\pi\sqrt{2\epsilon_4}}\right)\varphi(t,\epsilon_1)\,d\epsilon_1.$$

In order to obtain (3.24) we perform the two following simple operations. We relabel the integration variables in the cubic terms in order to write them as integrals with respect to the variables $(\epsilon_1,\epsilon_2,\epsilon_3)$. In the quadratic terms we symmetrize the integrals with respect to variables that appear in the functions g. ■

Remark 3.14 *Lemma 3.13 shows that any function* $f \in L^\infty_{loc}([0,T);L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma))$ *with* $\gamma>3$ *that is a mild solution of (1.10), (1.11) in the sense of Definition 2.1 is also a weak solution of (1.10), (1.11) on* $(0,T)$ *in the sense of definition 2.7.*

We can now prove that mass and energy are preserved for the obtained solution.

Lemma 3.15 *Suppose that* $\gamma>3$ *and* $f \in L^\infty_{loc}([0,T);L^\infty(\mathbb{R}^+;(1+\epsilon)^\gamma))$ *is a mild solution of (1.10), (1.11) in the sense of Definition 2.1. Let* g *be as in (1.12). Then:*

$$\partial_t\left(\int_{\mathbb{R}^+} g\epsilon d\epsilon\right) = \partial_t\left(\int_{\mathbb{R}^+} g d\epsilon\right) = 0\,,\quad a.e.\ t\in[0,T)\,.$$

Proof. We apply (3.24) with the test functions:

$$\varphi_n = \zeta_n(\epsilon)\,\epsilon$$

where ζ_n is a cutoff function satisfying $\zeta_n(\epsilon)=1$ if $\epsilon\leq n,\ \zeta_n(\epsilon)=0$ if $\epsilon\geq n+1,\ \zeta_n\geq 0,\ \zeta_n'\leq 0,\ \zeta_n\in C^\infty(\mathbb{R})$. Notice that, since $\gamma>3$, it is possible to pass to the limit in the cubic term:

$$\lim_{n\to\infty}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{g_1g_2g_3\Phi}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}}Q_{\varphi_n}d\epsilon_1d\epsilon_2d\epsilon_3 = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{g_1g_2g_3\Phi}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}}Q_{\varphi_\infty}d\epsilon_1d\epsilon_2d\epsilon_3,$$

where $\varphi_\infty(\epsilon)=\epsilon$. In order to pass to the limit in the quadratic term a more careful argument is needed. We first estimate the integral:

$$\left|\int_{\mathbb{R}^+}\Phi Q_{\varphi_n}d\epsilon_3\right| \leq C\min\{\sqrt{\epsilon_1},\sqrt{\epsilon_2}\}(1+\epsilon_1+\epsilon_2)^2$$
$$\leq C\min\{\sqrt{\epsilon_1},\sqrt{\epsilon_2}\}\left(1+(\epsilon_1)^2+(\epsilon_2)^2\right).$$

Then, the quadratic integral can then estimated uniformly in n as:

$$C\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{g_1g_2}{\sqrt{\epsilon_1\epsilon_2}}\min\{\sqrt{\epsilon_1},\sqrt{\epsilon_2}\}\left(1+(\epsilon_1)^2+(\epsilon_2)^2\right)d\epsilon_1d\epsilon_2 \leq$$
$$\leq C\int_{\mathbb{R}^+}d\epsilon_1\int_0^{\epsilon_1}d\epsilon_2\frac{g_1g_2}{\sqrt{\epsilon_1}}\left(1+(\epsilon_1)^2\right)\leq C\int_{\mathbb{R}^+}f_1(\epsilon_1)^2d\epsilon_1,$$

and since $\gamma>3$ this integral is convergent and we can take the limit $n\to\infty$. Then:

$$\partial_t\left(\int_{\mathbb{R}^+}g\varphi d\epsilon\right)=\int_{\mathbb{R}^+}g\partial_t\varphi d\epsilon+\frac{1}{2^{\frac{5}{2}}}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{g_1g_2g_3\Phi}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}}Q_\varphi d\epsilon_1d\epsilon_2d\epsilon_3+$$
$$+\frac{\pi}{2}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}Q_\varphi d\epsilon_1d\epsilon_2d\epsilon_3\,,\quad a.e.\ t\in[0,T)\,, \tag{3.27}$$

with $\varphi(\epsilon)=\epsilon$. Since $Q_\varphi=0$ we obtain $\partial_t\left(\int_{\mathbb{R}^+}g\epsilon d\epsilon\right)=0,\ \ a.e.\ t\in[0,T)$. A similar argument using the sequence of test functions $\varphi_n=\zeta_n(\epsilon)$ yields $\partial_t\left(\int_{\mathbb{R}^+}gd\epsilon\right)=0$, $a.e.\ t\in[0,T)$. ■

As a next step we prove uniqueness of the mild solutions of (1.10), (1.11). Our goal is to prove the following:

Proposition 3.16 *Suppose that $\gamma>3$ and $f,\ \tilde{f}\in L^\infty_{loc}\left([0,T);L^\infty\left(\mathbb{R}^+;(1+\epsilon)^\gamma\right)\right)$ are mild solutions of (1.10), (1.11) in the sense of Definition 2.1 with initial data $f(0,\cdot)=\tilde{f}(0,\cdot)=f_0(\cdot)\in L^\infty\left(\mathbb{R}^+;(1+\epsilon)^\gamma\right)$. Then $f=\tilde{f}$.*

In order to prove Proposition 3.16 we begin with a preliminary computation.

Lemma 3.17 *Suppose that $f,\ \tilde{f}$ are as in Proposition 3.16. Then, the following identity holds:*

$$\left(f(t,\epsilon_1)-\tilde{f}(t,\epsilon_1)\right)=f_0(\epsilon_1)\,\Omega(t,\epsilon_1)\left[\exp\left(-\int_0^t S[f](s,\epsilon_1)\,ds\right)-\right.$$
$$\left.-\exp\left(-\int_0^t S\left[\tilde{f}\right](s,\epsilon_1)\,ds\right)\right]+$$
$$+\frac{8\pi^2}{\sqrt{2}}\int_0^t\Omega(t-s,\epsilon_1)\times$$
$$\times\left[\exp\left(-\int_s^t S[f](s,\epsilon_1)\,ds\right)-\exp\left(-\int_s^t S\left[\tilde{f}\right](s,\epsilon_1)\,ds\right)\right]J[f](s,\epsilon_1)\,ds+$$
$$+\frac{8\pi^2}{\sqrt{2}}\int_0^t\Omega(t-s,\epsilon_1)\exp\left(-\int_s^t S\left[\tilde{f}\right](s,\epsilon_1)\,ds\right)\left[J[f](s,\epsilon_1)-J\left[\tilde{f}\right](s,\epsilon_1)\right]ds, \tag{3.28}$$

where $M_0 = 4\pi \int_0^\infty f_0(\epsilon)\sqrt{2\epsilon}d\epsilon$, $S[\cdot]$ *is defined as in (3.11), (3.12) and:*

$$J[f] = \int_0^\infty \int_0^\infty f_3 f_4 (1 + f_1 + f_2) W d\epsilon_3 d\epsilon_4, \tag{3.29}$$

$$\Omega(t, \epsilon_1) = \exp\left(-\pi M_0 t \sqrt{\epsilon_1}\right).$$

Proof. By Lemma 3.15 we have:

$$4\pi \int_0^\infty f(t,\epsilon)\sqrt{2\epsilon}d\epsilon = 4\pi \int_0^\infty \tilde{f}(t,\epsilon)\sqrt{2\epsilon}d\epsilon = M_0 \quad , \quad a.e. \;\; t \in [0, T). \tag{3.30}$$

Let $\Psi(t,\epsilon_1)$ be as in (2.2) and let us denote as $\tilde{\Psi}(t,\epsilon_1)$ the corresponding function associated to $\tilde{f}$. Due to (3.30) we have:

$$\Psi(t,\epsilon_1) = \Omega(t,\epsilon_1)\exp\left(-\int_0^t S[f](s,\epsilon_1)\,ds\right) \tag{3.31}$$

$$\tilde{\Psi}(t,\epsilon_1) = \Omega(t,\epsilon_1)\exp\left(-\int_0^t S\left[\tilde{f}\right](s,\epsilon_1)\,ds\right). \tag{3.32}$$

By definition f and $\tilde{f}$ satisfy (2.1). Taking the difference of these equations we obtain:

$$\begin{aligned}\left(f(t,\epsilon_1) - \tilde{f}(t,\epsilon_1)\right) = {} & f_0(\epsilon_1)\left(\Psi(t,\epsilon_1) - \tilde{\Psi}(t,\epsilon_1)\right) + \\ & + \frac{8\pi^2}{\sqrt{2}}\int_0^t \left[\frac{\Psi(t,\epsilon_1)}{\Psi(s,\epsilon_1)} - \frac{\tilde{\Psi}(t,\epsilon_1)}{\tilde{\Psi}(s,\epsilon_1)}\right] J[f](s,\epsilon_1)\,ds + \\ & + \frac{8\pi^2}{\sqrt{2}}\int_0^t \frac{\tilde{\Psi}(t,\epsilon_1)}{\tilde{\Psi}(s,\epsilon_1)}\left[J[f](s,\epsilon_1) - J\left[\tilde{f}\right](s,\epsilon_1)\right] ds.\end{aligned}$$

Using (3.31) and (3.32)we deduce (3.28) ■

In the next Lemmas we estimate the differences of the terms containing $S[f]$, $S\left[\tilde{f}\right]$ and $J[f]$, $J\left[\tilde{f}\right]$ respectively.

Lemma 3.18 *Suppose that f and $\tilde{f}$ are as in Proposition 3.16. Then, we have the following estimates:*

$$\begin{aligned}&\left|\exp\left(-\int_s^t S[f](\xi,\epsilon_1)\,d\xi\right) - \exp\left(-\int_s^t S\left[\tilde{f}\right](\xi,\epsilon_1)\,d\xi\right)\right| \\ &\leq \int_s^t \left|S[f](\xi,\epsilon_1) - S\left[\tilde{f}\right](\xi,\epsilon_1)\right| d\xi \quad , \quad 0 \leq s \leq t \leq T\,, \;\; \epsilon_1 > 0\,, \;\; 0 \leq s \leq t \leq T,\end{aligned} \tag{3.33}$$

$$\left|S[f](t,\epsilon_1) - S\left[\tilde{f}\right](t,\epsilon_1)\right| \leq \frac{C}{1+\sqrt{\epsilon_1}}\left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}, \epsilon_1 > 0\,, \;\; 0 \leq t \leq T, \tag{3.34}$$

for some suitable constant $C > 0$ depending only on γ.

Proof. The estimate (3.33) is a consequence of the inequality $\left|e^{-A}-e^{-B}\right| \leq |A-B|,\ A \geq 0,$
$B \geq 0$. In order to prove (3.34) we first notice that:

$$\left|S\left[f\right](s,\epsilon_1)-S\left[\tilde{f}\right](s,\epsilon_1)\right| \leq \left|S_1\left[f\right](s,\epsilon_1)-S_1\left[\tilde{f}\right](s,\epsilon_1)\right| + \\ + \left|S_2\left[f\right](s,\epsilon_1)-S_2\left[\tilde{f}\right](s,\epsilon_1)\right|$$

The definition of $S_1[\cdot]$ in (3.12) yields:

$$\left|S_1\left[f\right](s,\epsilon_1)-S_1\left[\tilde{f}\right](s,\epsilon_1)\right| \leq \frac{C}{\left(1+\sqrt{\epsilon_1}\right)}\int_0^\infty \left|f_2-\tilde{f}_2\right|(\epsilon_2)^{\frac{3}{2}}\,d\epsilon_2 + \\ + C\int_{\epsilon_1}^\infty \epsilon_2\left|f_2-\tilde{f}_2\right|d\epsilon_2.$$

Then, using that $\gamma > 3$:

$$\left|S_1\left[f\right](s,\epsilon_1)-S_1\left[\tilde{f}\right](s,\epsilon_1)\right| \leq \frac{C}{1+\sqrt{\epsilon_1}}\left\|f_2-\tilde{f}_2\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)} + \\ + \frac{C}{\left(1+\epsilon_1\right)^{\gamma-2}}\left\|f_2-\tilde{f}_2\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)} \\ \leq \frac{C}{1+\sqrt{\epsilon_1}}\left\|f_2-\tilde{f}_2\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}. \tag{3.35}$$

On the other hand, using (3.12):

$$\left|S_2\left[f\right](s,\epsilon_1)-S_2\left[\tilde{f}\right](s,\epsilon_1)\right| \leq C\int_0^\infty\int_0^\infty \left|f_2\left(f_3+f_4\right)-\tilde{f}_2\left(\tilde{f}_3+\tilde{f}_4\right)\right| W d\epsilon_3 d\epsilon_4.$$

Then, using also the symmetry we obtain the following estimate for $\epsilon_1 \geq 1$:

$$\left|S_2\left[f\right](s,\epsilon_1)-S_2\left[\tilde{f}\right](s,\epsilon_1)\right| \leq C\int_0^\infty\int_0^\infty \left|f_2-\tilde{f}_2\right| f_3 W d\epsilon_3 d\epsilon_4 + \\ + C\int_0^\infty\int_0^\infty \tilde{f}_2\left|f_3-\tilde{f}_3\right| W d\epsilon_3 d\epsilon_4 \\ \leq \frac{C}{\sqrt{\epsilon_1}}\int_0^\infty \left|f_2-\tilde{f}_2\right| d\epsilon_2 \int_0^\infty f_3\sqrt{\epsilon_3}d\epsilon_3 + \\ + \frac{C}{\sqrt{\epsilon_1}}\int_0^\infty \tilde{f}_2 d\epsilon_2\int_0^\infty \left|f_3-\tilde{f}_3\right|\sqrt{\epsilon_3}d\epsilon_3,$$

whence

$$\left|S_2\left[f\right](s,\epsilon_1)-S_2\left[\tilde{f}\right](s,\epsilon_1)\right| \leq \frac{C}{\sqrt{\epsilon_1}}\int_0^\infty \left|f-\tilde{f}\right|\left(1+\sqrt{\epsilon}\right)d\epsilon.$$

We can obtain also estimates for $\epsilon_1 \leq 1$ using $W \leq 1$. Then, combining the estimates for $\epsilon_1 \leq 1$ and $\epsilon_1 \geq 1$

$$\left|S_2[f](s,\epsilon_1) - S_2\left[\tilde{f}\right](s,\epsilon_1)\right| \leq \frac{C}{\left(1+\sqrt{\epsilon_1}\right)} \int_0^\infty \left|f - \tilde{f}\right| \left(1+\sqrt{\epsilon}\right) d\epsilon$$
$$\leq \frac{C}{\left(1+\sqrt{\epsilon_1}\right)} \left\|f_2 - \tilde{f}_2\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}. \quad (3.36)$$

Using (3.35) and (3.36) we obtain (3.34). ■

We now estimate the difference $\left[J[f](s,\epsilon_1) - J\left[\tilde{f}\right](s,\epsilon_1)\right]$ in (3.28) as well as the functions $J[f](s,\epsilon_1)\,,\ J\left[\tilde{f}\right](s,\epsilon_1)$. The following result requires to use the same type of detailed estimates for $a(\epsilon,t)$ used in the Proof of Lemma 3.8.

Lemma 3.19 *There exists a constant $C > 0$, depending only on γ, such that, for any $f,\ \tilde{f}$ as in Proposition 3.16:*

$$0 \leq \max\left\{J[f](t,\epsilon_1), J\left[\tilde{f}\right](t,\epsilon_1)\right\} \leq \frac{C}{(1+\epsilon_1)^{\gamma-\frac{1}{2}}} \quad , \quad \epsilon_1 > 0 \quad , \quad 0 \leq t \leq T, \quad (3.37)$$

$$\left|J[f](t,\epsilon_1) - J\left[\tilde{f}\right](t,\epsilon_1)\right| \leq \frac{\sqrt{2}}{8\pi^2}\frac{M_0\theta\sqrt{\epsilon_1}}{(1+\epsilon_1)^\gamma}\left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)} +$$
$$+ \frac{C\sqrt{\epsilon_1}}{(1+\epsilon_1)^\gamma}\int_0^\infty \left|f - \tilde{f}\right| \sqrt{\epsilon} d\epsilon + \frac{C\left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}, \quad (3.38)$$

for $\epsilon_1 > 0 \quad , \quad 0 \leq t \leq T$.

Proof. Using (3.29) and the fact that $W \leq \min\left\{\frac{\sqrt{\epsilon_3}}{\sqrt{\epsilon_1}}, \frac{\sqrt{\epsilon_4}}{\sqrt{\epsilon_1}}\right\}$ and $W = 0$ if $\epsilon_3+\epsilon_4 \leq \epsilon_1$ we obtain (3.37).

In order to estimate $\left[J[f](s,\epsilon_1) - J\left[\tilde{f}\right](s,\epsilon_1)\right]$ we first use (3.29) to obtain:

$$\left|J[f](s,\epsilon_1) - J\left[\tilde{f}\right](s,\epsilon_1)\right| \leq K_1 + K_2 + K_3,$$

$$K_1 = \int_0^\infty \int_0^\infty \left|f_3 f_4 - \tilde{f}_3\tilde{f}_4\right| W d\epsilon_3 d\epsilon_4\,, \quad K_2 = \int_0^\infty \int_0^\infty \left|f_3 f_4 f_1 - \tilde{f}_3\tilde{f}_4\tilde{f}_1\right| W d\epsilon_3 d\epsilon_4,$$
$$K_3 = \int_0^\infty \int_0^\infty \left|f_3 f_4 f_2 - \tilde{f}_3\tilde{f}_4\tilde{f}_2\right| W d\epsilon_3 d\epsilon_4.$$

The term K_1 must be carefully estimated, with the methods used in the Proof of Lemma 3.8. Using the symmetry with respect to the variables $\epsilon_3,\ \epsilon_4$ we obtain:

$$K_1 \leq \int_0^\infty \int_0^\infty f_3 \left|f_4 - \tilde{f}_4\right| W d\epsilon_3 d\epsilon_4 + \int_0^\infty \int_0^\infty \tilde{f}_4 \left|f_3 - \tilde{f}_3\right| W d\epsilon_3 d\epsilon_4$$
$$= \int_0^\infty \int_0^\infty \left(f_3 + \tilde{f}_3\right) \left|f_4 - \tilde{f}_4\right| W d\epsilon_3 d\epsilon_4.$$

We now introduce numbers $L > 0$ and $0 < \mu < 1$ as in the Proof of Lemma 3.8. We then estimate K_1 for $\epsilon_1 < L$ as:

$$\begin{aligned} K_1 &\leq \frac{C}{\left(1+\sqrt{\epsilon_1}\right)} \int_0^\infty \left(f_3 + \tilde{f}_3\right) d\epsilon_3 \int_0^\infty \left|f_4 - \tilde{f}_4\right| \sqrt{\epsilon_4} d\epsilon_4 \\ &\leq \frac{C}{\left(1+\sqrt{\epsilon_1}\right)} \int_0^\infty \left|f_4 - \tilde{f}_4\right| \sqrt{\epsilon_4} d\epsilon_4 \quad , \quad \epsilon_1 < L. \end{aligned} \tag{3.39}$$

On the other hand, in order to estimate K_1 for $\epsilon_1 \geq L$ we introduce an auxiliary parameter $\mu < 1$ as in Lemma 3.8 and whose precise value will be determined later. We then have, using also (1.11):

$$\begin{aligned} K_1 \leq & \frac{1}{\sqrt{\epsilon_1}} \left(\int_{(1-\mu)\epsilon_1}^\infty \left(f_3 + \tilde{f}_3\right) d\epsilon_3\right) \left(\int_{(1-\mu)\epsilon_1}^\infty \left|f_4 - \tilde{f}_4\right| \sqrt{\epsilon_4} d\epsilon_4\right) + \\ & + \frac{1}{\sqrt{\epsilon_1}} \left(\int_{\mu\epsilon_1}^\infty \left(f_3 + \tilde{f}_3\right) d\epsilon_3\right) \left(\int_0^\infty \left|f_4 - \tilde{f}_4\right| \sqrt{\epsilon_4} d\epsilon_4\right) + \\ & + \frac{1}{\sqrt{\epsilon_1}} \left(\int_0^\infty \left(f_3 + \tilde{f}_3\right) \sqrt{\epsilon_3} d\epsilon_3\right) \left(\int_{\mu\epsilon_1}^\infty \left|f_4 - \tilde{f}_4\right| d\epsilon_4\right), \end{aligned}$$

if $\epsilon_1 \geq L$. Taking into account the definitions of $\|\cdot\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}$ and M_0 we obtain:

$$\begin{aligned} K_1 \leq & \frac{C_\mu}{\sqrt{\epsilon_1}} \frac{1}{(\epsilon_1)^{2\gamma-2}} \left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)} + \frac{C_\mu}{(\epsilon_1)^{\gamma-\frac{1}{2}}} \int_0^\infty \left|f_4 - \tilde{f}_4\right| \sqrt{\epsilon_4} d\epsilon_4 + \\ & + \frac{\sqrt{2}}{8\pi^2} \frac{2M_0}{(\gamma-1)} \frac{\left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(\mu\epsilon_1)^{\gamma-\frac{1}{2}}}, \end{aligned}$$

with $C_\mu > 0$ depending on μ. Choosing μ sufficiently close to one, assuming that L is large, and using the fact that $\gamma > 3$ we obtain:

$$\frac{2}{(\gamma-1)} \frac{1}{(\mu\epsilon_1)^\gamma} + \frac{C}{(\epsilon_1)^{2\gamma-1}} \leq \frac{\theta}{(1+\epsilon_1)^\gamma} \quad , \quad \epsilon_1 \geq L$$

for some $\theta < 1$. Then:

$$K_1 \leq \frac{\sqrt{2}}{8\pi^2} \frac{M_0\theta\sqrt{\epsilon_1}}{(1+\epsilon_1)^\gamma} \left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)} + \frac{C\sqrt{\epsilon_1}}{(1+\epsilon_1)^\gamma} \int_0^\infty \left|f_4 - \tilde{f}_4\right| \sqrt{\epsilon_4} d\epsilon_4 \quad , \quad \epsilon_1 \geq L.$$

Combining this estimate with (4.7) in order to include also the contribution of the region $\{\epsilon_1 < L\}$ we obtain:

$$K_1 \leq \frac{\sqrt{2}}{8\pi^2} \frac{M_0\theta\sqrt{\epsilon_1}}{(1+\epsilon_1)^\gamma} \left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)} + \frac{C\sqrt{\epsilon_1}}{(1+\epsilon_1)^\gamma} \int_0^\infty \left|f - \tilde{f}\right| \sqrt{\epsilon} d\epsilon \quad , \quad \epsilon_1 \geq 0. \tag{3.40}$$

Notice that C depends only in γ and θ. We now estimate K_2:

$$K_2 \leq \int_0^\infty \int_0^\infty \left[\left|f_3 - \tilde{f}_3\right| f_4 f_1 + \tilde{f}_3 \left|f_4 - \tilde{f}_4\right| f_1 + \tilde{f}_3 \tilde{f}_4 \left|f_1 - \tilde{f}_1\right|\right] W d\epsilon_3 d\epsilon_4.$$

The first two terms on the right-hand side can be easily estimated using the boundedness of f :

$$\int_0^\infty \int_0^\infty \left[\left|f_3 - \tilde{f}_3\right| f_4 f_1 + \tilde{f}_3 \left|f_4 - \tilde{f}_4\right| f_1\right] W d\epsilon_3 d\epsilon_4 \leq$$
$$\leq \frac{C}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}} \int_0^\infty \left|f_3 - \tilde{f}_3\right| \sqrt{\epsilon_3} d\epsilon_3.$$

On the other hand:

$$\int_0^\infty \int_0^\infty \tilde{f}_3 \tilde{f}_4 \left|f_1 - \tilde{f}_1\right| W d\epsilon_3 d\epsilon_4 \leq \frac{C \left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}.$$

Then:

$$\begin{aligned} K_2 &\leq \frac{C}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}} \int_0^\infty \left|f_3 - \tilde{f}_3\right| \sqrt{\epsilon_3} d\epsilon_3 + \frac{C \left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}} \\ &\leq \frac{C \left\|f - \tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}. \end{aligned} \tag{3.41}$$

We now estimate K_3. Notice that:

$$\begin{aligned} K_3 &\leq \int_0^\infty \int_0^\infty \left|f_3 f_4 - \tilde{f}_3 \tilde{f}_4\right| f_2 W d\epsilon_3 d\epsilon_4 + \int_0^\infty \int_0^\infty \tilde{f}_3 \tilde{f}_4 \left|f_2 - \tilde{f}_2\right| W d\epsilon_3 d\epsilon_4 \\ &= K_{3,1} + K_{3,2}. \end{aligned}$$

If $\epsilon_1 \leq 1$ both terms on the right-hand side can be estimated by $C \int_0^\infty \left|f_4 - \tilde{f}_4\right| d\epsilon_4$. If $\epsilon_1 > 1$ we use the fact that at least one of the integration variables ϵ_3 or ϵ_4 is larger than $\frac{\epsilon_1}{2}$. Then:

$$K_{3,2} \leq \frac{C}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}} \int_0^\infty \int_0^\infty \tilde{f}_4 \left|f_2 - \tilde{f}_2\right| \sqrt{\epsilon_4} d\epsilon_3 d\epsilon_4 \leq \frac{C}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}} \int_0^\infty \left|f - \tilde{f}\right| d\epsilon.$$

Using also the fact that at least one of the integration variables is larger than $\frac{\epsilon_1}{2}$ we obtain:

$$\begin{aligned} K_{3,1} \leq &\int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_3 \int_0^\infty d\epsilon_4 f_3 \left|f_4 - \tilde{f}_4\right| f_2 W + \int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_3 \int_0^\infty d\epsilon_4 \left|f_3 - \tilde{f}_3\right| f_4 f_2 W + \\ &+ \int_0^\infty d\epsilon_3 \int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_4 f_3 \left|f_4 - \tilde{f}_4\right| f_2 W + \int_0^\infty d\epsilon_3 \int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_4 \left|f_3 - \tilde{f}_3\right| f_4 f_2 W. \end{aligned}$$

Therefore, using the definition of the norms $\|\cdot\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}$:

$$\begin{aligned}K_{3,1} \leq & \frac{C}{(1+\epsilon_1)^\gamma}\int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_3 \int_0^\infty d\epsilon_4 \left|f_4-\tilde{f}_4\right| f_2 W + \\ & + \frac{C\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^\gamma}\int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_3 \int_0^\infty d\epsilon_4 f_4 f_2 W + \\ & + \frac{C\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^\gamma}\int_0^\infty d\epsilon_3 \int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_4 f_3 f_2 W + \\ & + \frac{C}{(1+\epsilon_1)^\gamma}\int_0^\infty d\epsilon_3 \int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_4 \left|f_3-\tilde{f}_3\right| f_2 W\end{aligned}$$

and, relabelling variables:

$$\begin{aligned}K_{3,1} \leq & \frac{C}{(1+\epsilon_1)^\gamma}\int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_3 \int_0^\infty d\epsilon_4 \left|f_4-\tilde{f}_4\right| f_2 W + \\ & + \frac{C\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^\gamma}\int_{\frac{\epsilon_1}{2}}^\infty d\epsilon_3 \int_0^\infty d\epsilon_4 f_4 f_2 W,\end{aligned}$$

whence:

$$\begin{aligned}K_{3,1} \leq & \frac{C}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}\int_0^\infty d\epsilon_4 \left|f_4-\tilde{f}_4\right| \sqrt{\epsilon_4} + \frac{C\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}} \\ \leq & \frac{C\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}.\end{aligned}$$

Then:

$$\begin{aligned}K_3 \leq & \frac{C}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}\int_0^\infty \left|f-\tilde{f}\right| d\epsilon + \frac{C\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}} \\ \leq & \frac{C\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}.\end{aligned} \tag{3.42}$$

Combining (3.39), (3.40), (3.41), (3.42) we obtain (3.38). ■

Proof of Proposition 3.16. We estimate the differences $\left(f(t,\epsilon_1)-\tilde{f}(t,\epsilon_1)\right)$. We have (cf. (3.28)):

$$\left|f(t,\epsilon_1)-\tilde{f}(t,\epsilon_1)\right| \leq f_0(\epsilon_1)\,\Omega(t,\epsilon_1)\int_0^t ds\left|S[f](s,\epsilon_1)-S\left[\tilde{f}\right](s,\epsilon_1)\right|+$$
$$+\frac{8\pi^2}{\sqrt{2}}\int_0^t \Omega(t-s,\epsilon_1)\int_s^t d\xi\left|S[f](\xi,\epsilon_1)-S\left[\tilde{f}\right](\xi,\epsilon_1)\right| J[f](s,\epsilon_1)\,ds+$$
$$+\frac{8\pi^2}{\sqrt{2}}\int_0^t \Omega(t-s,\epsilon_1)\exp\left(-\int_s^t S\left[\tilde{f}\right](\xi,\epsilon_1)\,d\xi\right)\left|J[f](s,\epsilon_1)-J\left[\tilde{f}\right](s,\epsilon_1)\right| ds.$$

Then, using the estimates for $f_0 \in L^\infty\left(\mathbb{R}^+;(1+\epsilon)^\gamma\right)$ as well as Lemma 3.18 and (3.37):

$$\left|f(t,\epsilon_1)-\tilde{f}(t,\epsilon_1)\right| \leq \frac{C}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}\int_0^t ds\left\|f(s)-\tilde{f}(s)\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}+$$
$$+\frac{C}{(1+\epsilon_1)^{\gamma}}\int_0^t ds\int_s^t d\xi\left\|f(\xi)-\tilde{f}(\xi)\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}+$$
$$+\frac{8\pi^2}{\sqrt{2}}\int_0^t \Omega(t-s,\epsilon_1)\exp\left(-\int_s^t S\left[\tilde{f}\right](\xi,\epsilon_1)\,d\xi\right)\left|J[f](s,\epsilon_1)-J\left[\tilde{f}\right](s,\epsilon_1)\right| ds.$$

By Lemma 3.19:

$$\left|f(t,\epsilon_1)-\tilde{f}(t,\epsilon_1)\right| \leq \frac{C}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}\int_0^t ds\left\|f(s)-\tilde{f}(s)\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}+$$
$$+\frac{C}{(1+\epsilon_1)^{\gamma}}\int_0^t ds\int_s^t d\xi\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}(\xi)+$$
$$+\int_0^t \Omega(t-s,\epsilon_1)\frac{M_0\theta\sqrt{\epsilon_1}}{(1+\epsilon_1)^\gamma}\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}+$$
$$+\frac{C}{(1+\epsilon_1)^{\gamma}}\int_0^t ds\Omega(t-s,\epsilon_1)\sqrt{\epsilon_1}\int_0^\infty\left|f-\tilde{f}\right|\sqrt{\epsilon}d\epsilon+$$
$$+\frac{C}{(1+\epsilon_1)^{\gamma+\frac{1}{2}}}\int_0^t\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}(s)\,ds. \tag{3.43}$$

Then, using $\int_0^t \Omega(t-s,\epsilon_1)\,M_0\sqrt{\epsilon_1}\leq 1$:

$$\sup_{0\leq s\leq t}\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}(s) \leq CT\sup_{0\leq s\leq t}\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}(s)+$$
$$+\theta\sup_{0\leq s\leq t}\left\|f-\tilde{f}\right\|_{L^\infty(\mathbb{R}^+,(1+\epsilon)^\gamma)}(s)+$$
$$+C\sup_{0\leq s\leq t}\left(\int_0^\infty\left|f-\tilde{f}\right|\left(1+\sqrt{\epsilon}\right)d\epsilon\right)\quad \text{if } 0\leq t\leq T.$$

Since $\theta < 1$ we have:

$$\sup_{0\leq s\leq t}\left\|f-\tilde{f}\right\|_{L^{\infty}(\mathbb{R}^{+},(1+\epsilon)^{\gamma})}(s) \leq CT\sup_{0\leq s\leq t}\left\|f-\tilde{f}\right\|_{L^{\infty}(\mathbb{R}^{+},(1+\epsilon)^{\gamma})}(s)+ \\ +C\sup_{0\leq s\leq t}\left(\int_{0}^{\infty}\left|f-\tilde{f}\right|\left(1+\sqrt{\epsilon}\right)d\epsilon\right).$$

On the other hand, multiplying (3.43) by $\left(1+\sqrt{\epsilon_1}\right)$ and integrating we obtain:

$$\sup_{0\leq s\leq t}\left(\int_{0}^{\infty}\left|f-\tilde{f}\right|\left(1+\sqrt{\epsilon}\right)d\epsilon\right) \leq CT\sup_{0\leq s\leq t}\left\|f-\tilde{f}\right\|_{L^{\infty}(\mathbb{R}^{+},(1+\epsilon)^{\gamma})}(s)+ \\ +CT\sup_{0\leq s\leq t}\left(\int_{0}^{\infty}\left|f-\tilde{f}\right|\left(1+\sqrt{\epsilon}\right)d\epsilon\right).$$

Then, assuming that T is small we obtain:

$$\sup_{0\leq s\leq t}\left(\int_{0}^{\infty}\left|f-\tilde{f}\right|\left(1+\sqrt{\epsilon}\right)d\epsilon\right) \leq CT\sup_{0\leq s\leq t}\left\|f-\tilde{f}\right\|_{L^{\infty}(\mathbb{R}^{+},(1+\epsilon)^{\gamma})}(s).$$

Then:

$$\sup_{0\leq s\leq t}\left\|f-\tilde{f}\right\|_{L^{\infty}(\mathbb{R}^{+},(1+\epsilon)^{\gamma})}(s) \leq CT\sup_{0\leq s\leq t}\left\|f-\tilde{f}\right\|_{L^{\infty}(\mathbb{R}^{+},(1+\epsilon)^{\gamma})}(s)\ ,\quad 0\leq t\leq T,$$

and choosing T small we obtain $f = \tilde{f}$ for $0 \leq t \leq T$. This gives the uniqueness of solutions for short times. Uniqueness for arbitrarily long times can be obtained with a similar argument using the fact that a solution defined in an interval $[0,T]$, with $T > 0$, is also a mild solution in any interval $[T^*,T]$ with $0 < T^* < T$ and initial datum $f(\cdot,T^*)$ at time $t = T^*$. ■

In order to conclude the Proof of Theorem 3.4 it only remains to show that the solutions can be extended as long as $\sup_{0\leq t\leq T}\|f(t,\cdot)\|_{L^{\infty}(\mathbb{R}^{+})}$ remains bounded. To this end we prove the following:

Lemma 3.20 *Suppose that* $\gamma > 3$ *and* $f \in L^{\infty}_{loc}\left([0,T);L^{\infty}\left(\mathbb{R}^{+};(1+\epsilon)^{\gamma}\right)\right)$ *with* $T > 0$ *is a mild solutions of (1.10), (1.11) in the sense of Definition 2.1. Suppose that* $\sup_{0\leq t\leq T}\|f(t,\cdot)\|_{L^{\infty}(\mathbb{R}^{+})} < \infty$. *Then, it is possible to extend the solution to a larger time interval* $[0,T+\delta)$ *for some* $\delta > 0$.

Proof. We recall that $f=\mathcal{T}(f)$ for $t\in(0,T)$. Using Lemma 3.8 we then obtain the estimate:

$$\begin{aligned}\|f\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}(t)\leq\|f_0\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}+t\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}\times\\ \times\left(\sup_{0\leq s\leq t}\int\left(1+\epsilon^{\frac{3}{2}}\right)f(s,\epsilon)\,d\epsilon\right)^{2}+\\ +Ct\left(1+\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}\right)\left(\sup_{0\leq s\leq t}\int_0^{\infty}f(s,\epsilon)\,d\epsilon\right)^{2}+\\ +Ct\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}\left(\sup_{0\leq s\leq t}\int_0^{\infty}(\epsilon)^{\frac{3}{2}}f(s,\epsilon)\,d\epsilon\right)+\\ +\theta\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}.\end{aligned}$$

We now use that the energy $\int_0^{\infty}\epsilon^{\frac{3}{2}}f(t,\epsilon)\,d\epsilon$ remains constant in time for mild solutions (cf. Lemma 3.15). Then, splitting the domain of integration in the regions $\{\epsilon\geq 1\}$ and $\{\epsilon<1\}$ we derive the estimate:

$$\sup_{0\leq s\leq t}\int_0^{\infty}f(s,\epsilon)\,d\epsilon\leq\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+})}+\sup_{0\leq s\leq t}\int_0^{\infty}(\epsilon)^{\frac{3}{2}}f(s,\epsilon)\,d\epsilon.$$

By assumption $\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+})}\leq C$ for $0\leq t\leq T$. Then:

$$\sup_{0\leq s\leq t}\int_0^{\infty}f(s,\epsilon)\,d\epsilon\leq,C$$

whence:

$$\begin{aligned}\|f\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}(t)\leq\|f_0\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}+Ct\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}+\\ +Ct\left(1+\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}\right)+\\ +Ct\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}+\theta\sup_{0\leq s\leq t}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}.\end{aligned}$$

Since $\theta<1$ it then follows that there exists $t^{*}>0$ such that:

$$\sup_{0\leq s\leq t^{*}}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}\leq(1+a)\|f_0\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}+Ct^{*},$$

for some $a>0$. This estimate can be iterated starting at $t=t^{*}$. It then follows, after a number of iterations that:

$$\sup_{0\leq s\leq T}\|f(s,\cdot)\|_{L^{\infty}(\mathbb{R}^{+};(1+\epsilon)^{\gamma})}\leq C.$$

Then, applying Proposition 3.12 we obtain that it is possible to extend the solution to a larger time interval $[0,T+\delta)$. Indeed, suppose that f is defined as

$f = f^{(1)}$ for $0 \leq t \leq t^*$ and $f = f^{(2)}$ for $t^* \leq t \leq t^{**}$ where $f^{(1)}$ is a mild solution of (1.10), (1.11) with initial data f_0 in the interval $0 \leq t \leq t^*$ and f_2 is a mild solution of (1.10), (1.11) defined for $t^* \leq t \leq t^{**}$ such that $f^{(2)} = f^{(1)}$ for $t = t^*$. It follows from (2.1) that f is a mild solution of (1.10), (1.11) in the interval $0 \leq t \leq t^{**}$. ■

Proof of Theorem 3.4. It is a consequence of Proposition 3.12, Lemma 3.15, Proposition 3.16 and Lemma 3.20. ■

4 Monotonicity properties of the kernel $Q_3[f]$.

In this Section we recall a crucial monotonicity property of the kernel $Q_3[f]$ that captures in a precise way the fact that the cubic terms in (1.10) have some tendency to yield concentration of $f(\epsilon, t)$ to concentrate towards regions with smaller values of ϵ. This property has been obtained in [21].

Proposition 4.1 *Let $q_3(\cdot)$ as in (1.5). Let us denote as $\mathcal{S}^3$ the group of permutations of the three elements $\{1, 2, 3\}$. Suppose that $\varphi \in C(\mathbb{R}^+)$ is a test function. The following identity holds for any for any f such that $h = \sqrt{\epsilon} f(\epsilon) \in \mathcal{M}_+(\mathbb{R}^+)$:*

$$\int_{(\mathbb{R}^+)^3} d\epsilon_1 d\epsilon_3 d\epsilon_4 \Phi \; q_3(f)(\epsilon_1) \sqrt{\epsilon_1} \varphi(\epsilon_1) = \int_{(\mathbb{R}^+)^3} d\epsilon_1 d\epsilon_2 d\epsilon_3 \, f_1 \, f_2 \, f_3 \mathcal{G}_\varphi(\epsilon_1 \, \epsilon_2 \, \epsilon_3), \quad (4.1)$$

where:

$$\mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3) = \frac{1}{6} \sum_{\sigma \in \mathcal{S}^3} H_\varphi\left(\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}, \epsilon_{\sigma(3)}\right) \Phi\left(\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}; \epsilon_{\sigma(3)}\right), \quad (4.2)$$

$$H_\varphi(x, y, z) = \varphi(z) + \varphi(x + y - z) - \varphi(x) - \varphi(y), \quad (4.3)$$

with Φ as in (1.14) and:

$$\mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3) = \mathcal{G}_\varphi\left(\epsilon_{\sigma(1)}, \epsilon_{\sigma(2)}, \epsilon_{\sigma(3)}\right) \quad \text{for any } \sigma \in \mathcal{S}^3. \quad (4.4)$$

Moreover, if the function φ is convex we have $\mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3) \geq 0$ and if φ is concave we have $\mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3) \leq 0$. For any test function φ the function $\mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3)$ vanishes along the diagonal $\left\{(\epsilon_1, \epsilon_2, \epsilon_3) \in (\mathbb{R}^+)^3 : \epsilon_1 = \epsilon_2 = \epsilon_3\right\}$.

Remark 4.2 *The interpretation of this Theorem is simple if we think the process in terms of particles whose dynamics is driven by means of the collision terms $Q_3[f]$. Notice that such dynamics can be thought as a classical dynamics in which given three particles two of them are selected as incoming particles and the last one is one of the outgoing particles. The energy is the fourth one is then determined by means of the conservation of energy. Given three particles with energies $\epsilon_1, \epsilon_2, \epsilon_3$ we consider all the processes in which they can be involved, either as initial or final particles. The probabilities of these processes depend on the specific choice made of incoming and outgoing particles. We then compute the average change of $\Delta = \sum_{k=1}^3 \varphi(\epsilon_k)$ in* *these processes. If φ is concave the change of Δ is nonnegative, and if φ is convex, such a change is nonpositive. If we take, for instance the convex function $\varphi(\epsilon) = \epsilon^{-r}$ with $r > 0$, the monotonicity property states that particles tend to move on average towards smallest values of ϵ.*

Remark 4.3 *Notice that $\mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3) = 0$ if $\varphi = 1$ or $\varphi = \epsilon$. This could be expected due to the fact that the kinetic equation (or the particle interpretation of this process) formally conserves the number of particles and the energy. Moreover, we have $\mathcal{G}_\varphi(\epsilon_1, \epsilon_1, \epsilon_1) = 0$. The meaning of this identity is that the distribution of particles is not modified by the cubic terms of the equation if there is only one type of them. Notice that this implies also that Dirac masses $g(\epsilon) = \delta_{\epsilon=\epsilon^*}$ with $\epsilon^* > 0$ are stationary solutions of the kinetic equation containing only cubic terms (cf. (1.15)). This stationarity is the source of many of the technical difficulties in the forthcoming analysis.*

Definition 4.4 *We will use repeatedly the auxiliary functions ϵ_+, ϵ_0, ϵ_- defined from $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$ as follows:*

$$\begin{aligned}
\epsilon_+(\epsilon_1, \epsilon_2, \epsilon_3) &= \max\{\epsilon_1, \epsilon_2, \epsilon_3\}, \\
\epsilon_-(\epsilon_1, \epsilon_2, \epsilon_3) &= \min\{\epsilon_1, \epsilon_2, \epsilon_3\}, \\
\epsilon_0(\epsilon_1, \epsilon_2, \epsilon_3) &= \epsilon_k \in \{\epsilon_1, \epsilon_2, \epsilon_3\} \text{ such that } \epsilon_-(\epsilon_1, \epsilon_2, \epsilon_3) \le \epsilon_k \le \epsilon_+(\epsilon_1, \epsilon_2, \epsilon_3),
\end{aligned}$$

with $k \in \{1, 2, 3\}$.

Proof of Proposition 4.1. Notice that (4.1)-(4.3) are just a consequence of the identity

$$\int_{\mathbb{R}^+} Q_3[f](\epsilon_1)\sqrt{\epsilon_1}\varphi_1 d\epsilon_1 = \frac{1}{2}\int_{(\mathbb{R}^+)^3} d\epsilon_1 d\epsilon_2 d\epsilon_3 \Phi f_1 f_2 f_3 (\varphi_3 + \varphi_4 - \varphi_1 - \varphi_2), \quad (4.5)$$

combined with a symmetrization argument. In order to prove that $\mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3)$ has the indicated signs for convex or concave functions φ, we use the fact that the symmetry of $\mathcal{G}_\varphi$ under perturbations yields:

$$\mathcal{G}_\varphi(\epsilon_1, \epsilon_2, \epsilon_3) = \mathcal{G}_\varphi(\epsilon_+, \epsilon_-, \epsilon_0). \quad (4.6)$$

Using (1.14) and Definition 4.4 we obtain:

$$\Phi(\epsilon_+, \epsilon_-; \epsilon_0) = \Phi(\epsilon_0, \epsilon_+; \epsilon_-) = \sqrt{\epsilon_-} \quad , \quad \Phi(\epsilon_0, \epsilon_-; \epsilon_+) = \sqrt{(\epsilon_0 + \epsilon_- - \epsilon_+)_+}. \quad (4.7)$$

We have also the symmetry property $\Phi(\epsilon_j, \epsilon_\ell; \epsilon_k) = \Phi(\epsilon_\ell, \epsilon_j; \epsilon_k)$, $j, \ell, k \in \{1, 2, 3\}$. Then, using (4.2) we obtain:

$$\begin{aligned}
\mathcal{G}_\varphi(\epsilon_+, \epsilon_-, \epsilon_0) = \frac{1}{3}\Big[&H_\varphi(\epsilon_+, \epsilon_-; \epsilon_0)\sqrt{\epsilon_-} + H_\varphi(\epsilon_0, \epsilon_+; \epsilon_-)\sqrt{\epsilon_-} + \\
&+ H_\varphi(\epsilon_0, \epsilon_-; \epsilon_+)\sqrt{(\epsilon_0 + \epsilon_- - \epsilon_+)_+}\Big], \quad (4.8)
\end{aligned}$$

and using (4.3):

$$\mathcal{G}_{\varphi}(\epsilon_+,\epsilon_-,\epsilon_0) = \frac{1}{3}\Big[\sqrt{\epsilon_-}\left[\varphi(\epsilon_+ + \epsilon_- - \epsilon_0) + \varphi(\epsilon_+ + \epsilon_0 - \epsilon_-) - 2\varphi(\epsilon_+)\right] + \\ + \sqrt{(\epsilon_0 + \epsilon_- - \epsilon_+)_+}\left[\varphi(\epsilon_+) + \varphi(\epsilon_0 + \epsilon_- - \epsilon_+) - \varphi(\epsilon_0) - \varphi(\epsilon_-)\right]\Big]. \tag{4.9}$$

Suppose now that $\varphi = \varphi(\epsilon)$ is a convex function for $\epsilon > 0$. Then:

$$\frac{1}{2}\left[\varphi(\epsilon + z) + \varphi(\epsilon - z)\right] \geq \varphi(\epsilon) \quad , \quad \epsilon > 0, \quad z \geq 0, \quad \epsilon - z > 0. \tag{4.10}$$

On the other hand we can prove the following property for convex functions. Suppose that ψ is a convex function in $\epsilon > 0$. Then for any $0 < \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \epsilon_4$ satisfying $\epsilon_1 + \epsilon_4 = \epsilon_2 + \epsilon_3$ we have:

$$\psi(\epsilon_1) + \psi(\epsilon_4) \geq \psi(\epsilon_2) + \psi(\epsilon_3). \tag{4.11}$$

To prove (4.11) we define the function $W(z) = \psi\left(\frac{\epsilon_1+\epsilon_4}{2} + z\right) + \psi\left(\frac{\epsilon_1+\epsilon_4}{2} - z\right)$ for $z \geq 0$, $\frac{\epsilon_1+\epsilon_4}{2} - z > 0$. If $\psi \in C^2$ we would have:

$$W'(0) = \psi'\left(\frac{\epsilon_1+\epsilon_4}{2}\right) - \psi'\left(\frac{\epsilon_1+\epsilon_4}{2}\right) = 0,$$
$$W''(z) = \psi''\left(\frac{\epsilon_1+\epsilon_4}{2} + z\right) + \psi''\left(\frac{\epsilon_1+\epsilon_4}{2} - z\right) \geq 0.$$

It then follows that $W'(z) \geq 0$ if $z \geq 0$, whence:

$$W(z_2) \geq W(z_1) \quad \text{if} \quad 0 \leq z_1 < z_2.$$

Choosing $z_1 = \epsilon_3 - \frac{\epsilon_1+\epsilon_4}{2} = \epsilon_3 - \frac{\epsilon_2+\epsilon_3}{2}$ and $z_2 = \epsilon_4 - \frac{\epsilon_1+\epsilon_4}{2}$ we obtain (4.11). If ψ does not have two derivatives, the result can be proved extending ψ as a linear function for negative values, convolving the resulting function with a mollifier and passing to the limit in the desired identity.

Using (4.11) with $f = \varphi$ and $\epsilon_1 = \epsilon_+ + \epsilon_- - \epsilon_0,\ \epsilon_2 = \epsilon_-,\ \epsilon_3 = \epsilon_0,\ \epsilon_4 = \epsilon_+$ we obtain:

$$\varphi(\epsilon_+) + \varphi(\epsilon_0 + \epsilon_- - \epsilon_+) - \varphi(\epsilon_0) - \varphi(\epsilon_-) \geq 0. \tag{4.12}$$

Plugging (4.10), (4.12) into (4.9) we obtain $\mathcal{G}_{\varphi}(\epsilon_+,\epsilon_-,\epsilon_0) \geq 0$ for any convex function φ. On the other hand, a similar argument shows that $\mathcal{G}_{\varphi}(\epsilon_+,\epsilon_-,\epsilon_0) \leq 0$ for any concave function φ. This concludes the proof. ■

5 Estimating the number of collisions between small particles.

The main goal of this Section is to derive an estimate for the number of vectors $(\epsilon_1,\epsilon_2,\epsilon_3) \in [0,R]^3$, that we call triples, that are sufficiently separated from the diagonal $\{\epsilon_1 = \epsilon_2 = \epsilon_3\}$ for $R \leq \frac{1}{2}$ (cf. (5.22)). The first step is to derive a precise estimate for the number of "triple collisions" taking place in the system.

Proposition 5.1 *Suppose that f is a weak solution of (1.10), (1.11) on $(0,T)$ in the sense of Definition 2.7, with initial data f_0, and let g be as in (1.12). Then, there exists a numerical constant $B>0$, independent on f_0 and T, such that, for any $R\in(0,1)$ we have:*

$$B\int_0^T dt\int_{\left[0,\frac{R}{2}\right]^3}\left[\prod_{m=1}^{3}g_m d\epsilon_m\right]\frac{(\epsilon_0)^{\frac{3}{2}}}{(\epsilon_+)^{\frac{3}{2}}}\left(\frac{\epsilon_0-\epsilon_-}{\epsilon_0}\right)^2\leq 2\pi R^{\frac{3}{2}}\int_0^T dt\left(\int_{[0,R]}g(\epsilon)\,d\epsilon\right)^2+ \\ +MR, \tag{5.1}$$

where M is as in (2.3) and the functions ϵ_-, ϵ_0, ϵ_+ are as in Definition 4.4.

Proof. We use (3.24), (3.25), (3.26) with test function $\varphi(\epsilon)=\psi\left(\frac{\epsilon}{R}\right), R>0, \epsilon>0$, where:

$$\psi(s)=\left\{\begin{array}{ccc} s^\theta & , & 0<s<1 \\ 1 & , & s\geq 1\end{array}\right\},\quad 0<\theta<1.$$

Let $\mathcal{G}_\varphi(\epsilon_1,\epsilon_2,\epsilon_3)$ as in (4.2). Then, since the function φ is concave, Proposition 4.1 implies that $\mathcal{G}_\varphi(\epsilon_1,\epsilon_2,\epsilon_3)\leq 0$. Using (4.1) we then obtain:

$$\begin{aligned}&\int_{\mathbb{R}^+}Q_3[f](\epsilon_1)\sqrt{\epsilon_1}\varphi(\epsilon_1)\,d\epsilon_1\\ &=\int_{\left[0,\frac{R}{2}\right]^3}d\epsilon_1 d\epsilon_2 d\epsilon_3\, f_1\, f_2\, f_3\mathcal{G}_\varphi(\epsilon_1\,\epsilon_2\,\epsilon_3)+\int_{(\mathbb{R}^+)^3\setminus\left[0,\frac{R}{2}\right]^3}d\epsilon_1 d\epsilon_2 d\epsilon_3\, f_1\, f_2\, f_3\mathcal{G}_\varphi(\epsilon_1\,\epsilon_2\,\epsilon_3)\\ &\leq\int_{\left[0,\frac{R}{2}\right]^3}d\epsilon_1 d\epsilon_2 d\epsilon_3\, f_1\, f_2\, f_3\mathcal{G}_\varphi(\epsilon_1\,\epsilon_2\,\epsilon_3).\end{aligned}$$

Using Definition 4.4 and (4.6) we obtain:

$$\int_{\mathbb{R}^+}Q_3[f](\epsilon_1)\sqrt{\epsilon_1}\varphi(\epsilon_1)\,d\epsilon_1\leq\int_{\left[0,\frac{R}{2}\right]^3}d\epsilon_1 d\epsilon_2 d\epsilon_3\, f_1\, f_2\, f_3\mathcal{G}_\varphi(\epsilon_+,\epsilon_-,\epsilon_0). \tag{5.2}$$

Using (4.9) we can compute $\mathcal{G}_\varphi(\epsilon_+,\epsilon_-,\epsilon_0)$ for $(\epsilon_1\,\epsilon_2\,\epsilon_3)\in\left[0,\frac{R}{2}\right]^3$:

$$\begin{aligned}\mathcal{G}_\varphi(\epsilon_+,\epsilon_-,\epsilon_0)=\frac{1}{3}\Bigg[&\sqrt{\epsilon_-}\left[\left(\frac{\epsilon_++\epsilon_--\epsilon_0}{R}\right)^\theta+\left(\frac{\epsilon_++\epsilon_0-\epsilon_-}{R}\right)^\theta-2\left(\frac{\epsilon_+}{R}\right)^\theta\right]+\\ &+\sqrt{(\epsilon_0+\epsilon_--\epsilon_+)_+}\left[\left(\frac{\epsilon_+}{R}\right)^\theta+\left(\frac{\epsilon_0+\epsilon_--\epsilon_+}{R}\right)^\theta-\left(\frac{\epsilon_0}{R}\right)^\theta-\left(\frac{\epsilon_-}{R}\right)^\theta\right]\Bigg].\end{aligned}$$

Integrating (3.24) and using (5.2) as well as the nonnegativity of φ we deduce:

$$\begin{aligned}&\frac{-1}{2^{\frac{5}{2}}}\int_0^T dt\int_{\left[0,\frac{R}{2}\right]^3}d\epsilon_1 d\epsilon_2 d\epsilon_3\, g_1\, g_2\, g_3\frac{\mathcal{G}_\varphi(\epsilon_+,\epsilon_-,\epsilon_0)}{\sqrt{\epsilon_+\epsilon_-\epsilon_0}}\leq\\ &\qquad\leq\frac{\pi}{2}\int_0^T dt\int_{(\mathbb{R}^+)^3}\frac{g_1 g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3+\int_{\mathbb{R}^+}g(\epsilon_1,0)\varphi_1 d\epsilon_1,\end{aligned} \tag{5.3}$$

where $T>0$ is otherwise arbitrary. Notice that, since $\varphi\leq 1$ we have:

$$\int_{\mathbb{R}^+} g(\epsilon_1,0)\varphi_1 d\epsilon_1 \leq M. \tag{5.4}$$

We now estimate the first term on the right-hand side of (5.3) as follows. We split the integral as:

$$\int_{(\mathbb{R}^+)^3}\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3=\int_{(\mathbb{R}^+)^3\setminus[0,R]^3}[\cdots]+\int_{[0,R]^3}[\cdots].$$

If $(\epsilon_1,\epsilon_2,\epsilon_3)\in(\mathbb{R}^+)^3\setminus[0,R]^3$ we have $(\varphi_3+\varphi_4-\varphi_1-\varphi_2)=(1+\varphi_4-1-1)=(\varphi_4-1)\leq 0$, whence $\int_{(\mathbb{R}^+)^3\setminus[0,R]^3}[\cdots]\leq 0$. Therefore, using that $\varphi\leq 1$, as well as the fact that $\int_{[0,R]}\Phi d\epsilon_3\leq 2\sqrt{R}\min\left\{\sqrt{\epsilon_1},\sqrt{\epsilon_2}\right\}\max\left\{\sqrt{\epsilon_1},\sqrt{\epsilon_2}\right\}$ if $(\epsilon_1,\epsilon_2)\in[0,R]^2$:

$$\begin{aligned}\int_{(\mathbb{R}^+)^3}\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3&\leq 2\int_{[0,R]^3}\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}d\epsilon_1 d\epsilon_2 d\epsilon_3\\&\leq 4\sqrt{R}\int_{[0,R]^2}\frac{g_1g_2}{\sqrt{\epsilon_1\epsilon_2}}\min\{\sqrt{\epsilon_1},\sqrt{\epsilon_2}\}\max\{\sqrt{\epsilon_1},\sqrt{\epsilon_2}\}d\epsilon_1 d\epsilon_2,\end{aligned}\tag{5.5}$$

whence:

$$\int_{(\mathbb{R}^+)^3}\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3\leq 4\sqrt{R}\int_{[0,R]^2}g_1g_2 d\epsilon_1 d\epsilon_2=4\sqrt{R}\left(\int_{[0,R]}g d\epsilon\right)^2. \tag{5.6}$$

Plugging (5.4), (5.6) into (5.3) we obtain:

$$-\frac{1}{2^{\frac{5}{2}}}\int_0^T dt\int_{\left[0,\frac{R}{2}\right]^3}g_1\,g_2\,g_3\frac{\mathcal{G}_\varphi(\epsilon_+,\epsilon_-,\epsilon_0)}{\sqrt{\epsilon_+\epsilon_-\epsilon_0}}d\epsilon_1 d\epsilon_2 d\epsilon_3\leq 2\pi\sqrt{R}\left(\int_{[0,R]}g d\epsilon\right)^2+M. \tag{5.7}$$

In order to derive a lower estimate of $\mathcal{G}_\varphi(\epsilon_+,\epsilon_-,\epsilon_0)$ we need some calculus inequalities. To this end we define:

$$\sigma(X_-,X_0,X_+)=(X_++X_--X_0)^\theta+(X_++X_0-X_-)^\theta-2(X_+)^\theta,$$

with $0\leq X_-\leq X_0\leq X_+$. Then, we write

$$\sigma(X_-,X_0,X_+)=(X_+)^\theta\sigma\left(\frac{X_-}{X_+},\frac{X_0}{X_+},1\right)=(X_+)^\theta\sigma\left(0,\frac{X_0-X_-}{X_+},1\right).$$

The function $\sigma(0,Z,1)$ is decreasing on Z if $Z>0$. Suppose that $(X_0-X_-)>\frac{X_+}{2}$. Then $\sigma\left(0,\frac{X_0-X_-}{X_+},1\right)\leq\sigma\left(0,\frac{1}{2},1\right)$. The same convexity argument that was used in the Proof of (4.10) yields $\sigma\left(0,\frac{1}{2},1\right)<0$. Then $\sigma\left(\frac{X_-}{X_+},\frac{X_0}{X_+},1\right)\leq -A_{1,\theta}$ for some $A_{1,\theta}>0$ if $(X_0-X_-)>\frac{X_+}{2}$. Suppose now that $X_-\leq X_0$, $(X_0-X_-)\leq\frac{X_+}{2}$ we can use Taylor's Theorem to obtain $\sigma\left(\frac{X_-}{X_+},\frac{X_0}{X_+},1\right)\leq -A_{2,\theta}\left(\frac{X_0-X_-}{X_+}\right)^2$, with $A_{2,\theta}>0$. Then:

$$\sigma\left(X_-,X_0,X_+\right)\leq -A_\theta\left(X_+\right)^\theta\left(\frac{X_0-X_-}{X_+}\right)^2\ ,\ 0\leq X_-\leq X_0\leq X_+,$$

with $A_\theta>0$. Since $(\epsilon_1,\epsilon_2,\epsilon_3)\in\left[0,\frac{R}{2}\right]^3$ we have that the function $\varphi(s)$ is evaluated by means of $\left(\frac{s}{R}\right)^\theta$ in all the terms we then have:

$$\mathcal{G}_\varphi\left(\epsilon_+,\epsilon_-,\epsilon_0\right)\leq -\frac{A_\theta}{3}\sqrt{\epsilon_-}\left(\frac{\epsilon_+}{R}\right)^\theta\left(\frac{\epsilon_0-\epsilon_-}{\epsilon_+}\right)^2.$$

We can assume by definiteness that $\theta=\frac{1}{2}$. Using (5.7) we obtain:

$$B\int_0^T dt\int_{\left[0,\frac{R}{2}\right]^3}d\epsilon_1 d\epsilon_2 d\epsilon_3\left[\prod_{k=1}^3 g_k\right]\frac{1}{\sqrt{\epsilon_+\epsilon_0}}\left(\frac{\epsilon_+}{R}\right)^\theta\left(\frac{\epsilon_0-\epsilon_-}{\epsilon_+}\right)^2\leq$$
$$\leq 2\pi\sqrt{R}\int_0^T dt\left(\int_{[0,R]}g(\epsilon)\,d\epsilon\right)^2+M,$$

where $B>0$ is independent on R, g_0 and T. After some computations we arrive at:

$$B\int_0^T dt\int_{\left[0,\frac{R}{2}\right]^3}d\epsilon_1 d\epsilon_2 d\epsilon_3\left[\prod_{k=1}^3 g_k\right]\frac{\left(\epsilon_+\right)^{\theta-1}\left(\epsilon_0\right)^{\frac{3}{2}}}{\left(\epsilon_+\right)^{\frac{3}{2}}}\left(\frac{\epsilon_0-\epsilon_-}{\epsilon_0}\right)^2\leq$$
$$\leq 2\pi\sqrt{R}R^\theta\int_0^T dt\left(\int_{[0,R]}g(\epsilon)\,d\epsilon\right)^2+MR^\theta.$$

Using the fact that $\theta<1$ and $\epsilon_+\leq R$ we obtain $\left(\epsilon_+\right)^{\theta-1}\geq(R)^{\theta-1}$ whence (5.1) follows. ■

5.1 An estimate for nonnegative sequences.

We will need the following technical Lemma.

Lemma 5.2 *Let us consider two sequences of nonnegative numbers* $\{I_k\}_{k=0}^\infty$ *and* $\{A_k\}_{k=0}^\infty$ *satisfying the inequalities:*

$$I_M^2\sum_{M<k-1}I_j\leq A_M \tag{5.8}$$

for any $M=0,1,2,\ldots$ *. Then for any* $M_0\geq 0$ *we have:*

$$\sum_{M_0\leq j\leq \ell<k-1}I_k I_\ell I_j\leq\sum_{M_0\leq j\leq\ell}\sqrt{A_j A_\ell}. \tag{5.9}$$

Proof. The estimate (5.8) implies:

$$I_M \left(\sum_{k>M+1} I_k \right)^{\frac{1}{2}} \leq \sqrt{A_M}.$$

Taking the values $M = j$ and $M = \ell$ and multiplying the resulting inequalities we obtain:

$$I_j I_\ell \left(\sum_{k>j+1} I_k \right)^{\frac{1}{2}} \left(\sum_{k>\ell+1} I_k \right)^{\frac{1}{2}} \leq \sqrt{A_j A_\ell},$$

for any $j, \ell = 0, 1, 2, ...$Summing this inequality for $M_0 \leq j \leq \ell$ we obtain

$$\sum_{M_0 \leq j \leq \ell} I_j I_\ell \left(\sum_{k>j+1} I_k \right)^{\frac{1}{2}} \left(\sum_{k>\ell+1} I_k \right)^{\frac{1}{2}} \leq \sum_{M_0 \leq j \leq \ell} \sqrt{A_j A_\ell}.$$

Using that, for $j \leq \ell$ we have $\sum_{k>j+1} I_k \geq \sum_{k>\ell+1} I_k$ we obtain:

$$\sum_{M_0 \leq j \leq \ell} I_j I_\ell \left(\sum_{k>\ell+1} I_k \right) = \sum_{M_0 \leq j \leq \ell < k-1} I_j I_\ell I_k \leq \sum_{M_0 \leq j \leq \ell} \sqrt{A_j A_\ell}.$$

■

5.2 From one estimate for the rate of collisions to an estimate for the number of triples.

5.2.1 Notation and some geometrical results.

As a next step we transform estimate (5.1) into a new one that does not contain the power laws of ϵ and only contains the measures g. The resulting formula is more convenient to derive estimates for the number of particles concentrated near $\epsilon = 0$.

We will need in all the following some suitable notation. Given $a > 0$, we define a sequence of intervals $\{\mathcal{I}_k\}_{k=0}^{\infty}$ contained in the interval $[0, 1]$ by means of:

$$\mathcal{I}_k(b) = b^{-k} \left(\frac{1}{b}, 1 \right] \quad , \quad k = 0, 1, 2, ..., \quad b > 1. \tag{5.10}$$

Notice that $\bigcup_{k=0}^{\infty} \mathcal{I}_k(b) = (0, 1]$, $\mathcal{I}_k(b) \cap \mathcal{I}_j(b) = \varnothing$ if $k \neq j$. Given a measure $g \in \mathcal{M}_+([0, 1])$, we remark that, if $\int_{\{0\}} g(\epsilon) d\epsilon = 0$ we have:

$$\int_{[0,1]} g(\epsilon) d\epsilon = \sum_{k=0}^{\infty} \int_{\mathcal{I}_k(b)} g(\epsilon) d\epsilon. \tag{5.11}$$

We need to define also the "extended" intervals:

$$\mathcal{I}_k^{(E)}(b) = \mathcal{I}_{k-1}(b) \cup \mathcal{I}_k(b) \cup \mathcal{I}_{k+1}(b), \quad k = 0, 1, 2, ... \tag{5.12}$$

where, by convenience, we assume that $\mathcal{I}_{-1}(b) = \varnothing$.

We remark that each $\epsilon \in (0,1]$ belongs to three sets $\mathcal{I}_k^{(E)}(b)$. It also readily follows that:

$$3\int_{[0,1]} g(\epsilon)\, d\epsilon = \sum_{k=0}^{\infty} \int_{\mathcal{I}_k^{(E)}(b)} g(\epsilon)\, d\epsilon.$$

We will write $\mathcal{I}_k = \mathcal{I}_k(b)$, $\mathcal{I}_k^{(E)} = \mathcal{I}_k^{(E)}(b)$ if the dependence of the intervals in b is clear in the argument.

We also define, for further references, a set $\mathcal{P}_b$ of subsets of $[0,1]$:

$$\mathcal{P}_b = \left\{ A \subset [0,1] : A = \bigcup_j \mathcal{I}_{k_j}(b) \text{ for some set of indexes } \{k_j\} \subset \{1,2,...\} \right\}. \tag{5.13}$$

Notice that the elements of $\mathcal{P}_b$ consists of unions of elements of the family $\{\mathcal{I}_k(b)\}$. The set $\{k_j\}$ can contain a finite or infinity number of elements.

Given $A \in \mathcal{P}_b$ we define an extended set $A^{(E)}$ as follows. Suppose that $A = \bigcup_{j=1}^{\infty} \mathcal{I}_{k_j}(b)$. We then define:

$$A^{(E)} = \bigcup_{j=1}^{\infty} \mathcal{I}_{k_j}^{(E)}(b). \tag{5.14}$$

We will also need the following family of rescaled intervals. Given $R \in (0,1]$ and $b > 1$ we define: $\{\mathcal{I}_k(b,R)\}$, $\left\{\mathcal{I}_k^{(E)}(b,R)\right\}$ by means of:

$$\mathcal{I}_k(b,R) = R\mathcal{I}_k(b) \quad , \quad \mathcal{I}_k^{(E)}(b,R) = R\mathcal{I}_k^{(E)}(b) \quad , \quad k = 0,1,2,..., \tag{5.15}$$

with $\{\mathcal{I}_k(b)\}$, $\left\{\mathcal{I}_k^{(E)}(b)\right\}$ as in (5.10), (5.12). We define also a class of sets $\mathcal{P}_b(R)$ as follows:

$$\mathcal{P}_b(R) = \{A \subset [0,R] : A = RB, \quad B \in \mathcal{P}_b\}, \tag{5.16}$$

where $\mathcal{P}_b$ is as in (5.14). We can also define the concept of extended sets. Given $A \in \mathcal{P}_b(R)$, with the form $A = RB$, $B \in \mathcal{P}_b$ we define:

$$A^{(E)} = RB^{(E)}. \tag{5.17}$$

We now define the following family of subsets of the cube $[0,1]^3$:

$$\mathcal{S}_{R,\rho} = \left\{(\epsilon_1,\epsilon_2,\epsilon_3) \in [0,R]^3 : |\epsilon_0 - \epsilon_-| > \rho\epsilon_0\right\}, \quad 0 < R \le 1, \quad 0 < \rho < 1 \ . \tag{5.18}$$

where we will assume in the following that ϵ_-, ϵ_0, ϵ_+ are as in Definition 4.4. We finally define also the sets:

$$\mathcal{I}_{j,\ell,k}(b) = \mathcal{I}_j(b) \times \mathcal{I}_\ell(b) \times \mathcal{I}_k(b) \subset [0,1]^3 \quad , \quad j,k,\ell = 0,1,2,... \tag{5.19}$$

Lemma 5.3 *Suppose that* $0 < R \leq 1, \quad 0 < \rho < 1$. *Then:*

$$\mathcal{S}_{R,\rho} \subset \left(\bigcup_{\sigma \in S^3} \left[\bigcup_{N(R,b) \leq j \leq \ell < k-1} \mathcal{I}_{\sigma(j,\ell,k)}(b) \right] \right) \subset \mathcal{S}_{(bR)\wedge 1, \left(1-\frac{1}{b}\right)}, \tag{5.20}$$

where $bR \wedge 1 = \min\{bR, 1\}$, $N(R,b) = \left[\frac{\log\left(\frac{1}{R}\right)}{\log(b)} \right]$, $b = \frac{1}{1-\rho}$.

Proof. Suppose that $(\epsilon_1, \epsilon_2, \epsilon_3) \in \mathcal{S}_{R,\rho}$. Due to the invariance of the result under permutations of the indexes we can assume without loss of generality that $\epsilon_3 = \epsilon_- < \epsilon_2 = \epsilon_0 \leq \epsilon_1 = \epsilon_+$. The choice of $N(R,b)$ implies that there exist $\ell,\ j$ such that $\epsilon_2 \in \mathcal{I}_\ell,\ \epsilon_1 \in \mathcal{I}_j$ with $j \leq \ell$. Due to the definition of $\mathcal{S}_{R,\rho}$ and $\mathcal{I}_\ell$ we have $\epsilon_3 < (1-\rho)\epsilon_2 \leq (1-\rho) b^{-\ell}$. Since $(1-\rho) = b^{-2}$ we then have $\epsilon_3 < b^{-(\ell+2)}$ whence $\epsilon_3 \in \bigcup_{k > \ell+1} \mathcal{I}_k$. Therefore:

$$(\epsilon_1, \epsilon_2, \epsilon_3) \in \bigcup_{N(R,b) \leq j \leq \ell < k-1} \mathcal{I}_{j,\ell,k}(b).$$

This gives the first inclusion in (5.20). In order to prove the second inclusion, we assume, without loss of generality, that $\epsilon_3 = \epsilon_-,\ \epsilon_2 = \epsilon_0,\ \epsilon_1 = \epsilon_+$. Suppose that $\epsilon_2 \in \mathcal{I}_\ell,\ \epsilon_1 \in \mathcal{I}_j$ with $j \leq \ell$. Then $\epsilon_3 \leq b^{-(\ell+2)}$, $\epsilon_2 > b^{-(\ell+1)}$. Therefore $|\epsilon_0 - \epsilon_-| > \bar{\rho}\epsilon_0$ if $\bar{\rho} \leq \left(1 - \frac{1}{b}\right)$ and the result follows. ■

5.3 Estimating the number of triples not too close to the diagonal.

We now prove the following result which provides a precise estimate for the number of triples $(\epsilon_1, \epsilon_2, \epsilon_3) \in [0,R]^3$ whose distance to the diagonal is comparable to their distance to the origin.

Lemma 5.4 *Suppose that* $g \in L^\infty([0,T]; \mathcal{M}_+([0,1]))$, *satisfies (5.1) for any* $0 \leq R \leq 1$ *and* $T > 0$. *Suppose also that:*

$$\int_{\{0\}} g(\epsilon, t)\, d\epsilon = 0, \quad \textit{for any } t \in [0,T]. \tag{5.21}$$

Let $0 < \rho < 1$ *and* $\mathcal{S}_{R,\rho}$ *as in (5.18). Then, for any* $T > 0$ *we have:*

$$B \int_0^T dt \int_{\mathcal{S}_{R,\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m \right] \leq \frac{2b^{\frac{7}{2}} R}{\rho^2 \left(\sqrt{b} - 1\right)^2} \left[2\pi \int_0^T dt \left(\int_{[0,1]} g(\epsilon)\, d\epsilon \right)^2 + M \right], \tag{5.22}$$

with b *as in (5.20) and* $R \in \left[0, \frac{1}{2}\right]$ *and* B *as in (5.1).*

Remark 5.5 *Notice that Lemma 5.4 provides a general estimate for arbitrary measures g satisfying (5.1) even if they are completely unrelated to the equation (1.10), (1.11).*

Proof. We define:

$$f_R(t) = \frac{1}{R}\int_{\left[0,\frac{R}{2}\right]^3}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\frac{(\epsilon_0)^{\frac{3}{2}}}{(\epsilon_+)^{\frac{3}{2}}}\left(\frac{\epsilon_0-\epsilon_-}{\epsilon_0}\right)^2 \tag{5.23}$$

Notice that (5.1) implies that:

$$B\int_0^T f_R(t)\,dt \le \left[2\pi\int_0^T dt\left(\int_{[0,1]} g(\epsilon)\,d\epsilon\right)^2 + M\right] \quad , \quad 0 < R \le 1. \tag{5.24}$$

A crucial point in the following is the fact that this estimate is uniform in R.

Let us select $b > 1$ as in (5.20). We define sets $\mathcal{I}_{j,\ell,k}(b)$ by means of (5.10), (5.19). Notice that $\mathcal{I}_{(j_1,\ell_1,k_1)}(b) \cap \mathcal{I}_{(j_2,\ell_2,k_2)}(b) = \varnothing$ if $(j_1,\ell_1,k_1) \neq (j_2,\ell_2,k_2)$. Lemma 5.3 as well as the definition of $N(R,b)$ in (5.20) imply:

$$\mathcal{S}_{\frac{R}{2},\rho} \subset \left(\bigcup_{\sigma\in S^3}\left[\bigcup_{N\left(\frac{R}{2},b\right)\le j\le\ell<k-1}\mathcal{I}_{\sigma(j,\ell,k)}(b)\right]\right) \subset \left[0,\frac{bR}{2}\right]^3. \tag{5.25}$$

Notice that for each j,ℓ,k the sets in the family $\left\{\mathcal{I}_{\sigma(j,\ell,k)}(b) : \sigma \in S^3\right\}$ are disjoint, except if $j=\ell$. In such a case the permutation σ that keeps k constant and exchange the indexes j and ℓ implies $\mathcal{I}_{\sigma(j,\ell,k)}(b) = \mathcal{I}_{j,\ell,k}(b)$. Therefore, each point $(\epsilon_1,\epsilon_2,\epsilon_3)$ is contained at most in two of the sets $\mathcal{I}_{\sigma(j,\ell,k)}(b)$ in (5.25) and we then have:

$$\frac{1}{2}\sum_{\sigma\in S^3}\left[\sum_{N\left(\frac{R}{2},b\right)\le j\le\ell<k-1}\int_{\mathcal{I}_{\sigma(j,\ell,k)}(b)}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi(\epsilon_1,\epsilon_2,\epsilon_3)\right] \tag{5.26}$$
$$\le \int_{\mathcal{V}_{R,b}}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi(\epsilon_1,\epsilon_2,\epsilon_3),$$

where:

$$\mathcal{V}_{R,b} = \bigcup_{\sigma\in S^3}\left[\bigcup_{N\left(\frac{R}{2},b\right)\le j\le\ell<k-1}\mathcal{I}_{\sigma(j,\ell,k)}(b)\right] \quad , \quad \Psi(\epsilon_1,\epsilon_2,\epsilon_3) = \frac{(\epsilon_0)^{\frac{3}{2}}}{(\epsilon_+)^{\frac{3}{2}}}\left(\frac{\epsilon_0-\epsilon_-}{\epsilon_0}\right)^2. \tag{5.27}$$

Moreover, since:

$$\int_{\mathcal{I}_{\sigma(j,\ell,k)}(b)}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi(\epsilon_1,\epsilon_2,\epsilon_3) = \int_{\mathcal{I}_{j,\ell,k}(b)}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi(\epsilon_1,\epsilon_2,\epsilon_3) \ , \quad \sigma\in S^3, \tag{5.28}$$

and the cardinal of S^3 is six, (5.26) implies:

$$\left[\sum_{N\left(\frac{R}{2},b\right)\leq j\leq \ell<k-1}\int_{\mathcal{I}_{j,\ell,k}(b)}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi\left(\epsilon_1,\epsilon_2,\epsilon_3\right)\right]\leq$$

$$\leq\frac{1}{3}\int_{\mathcal{V}_{R,b}}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi\left(\epsilon_1,\epsilon_2,\epsilon_3\right). \tag{5.29}$$

On the other hand, (5.25) and the definition of $f_R(t)$ in (5.23) imply:

$$\int_{\mathcal{V}_{R,b}}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi\left(\epsilon_1,\epsilon_2,\epsilon_3\right)\leq bRf_{bR}(t). \tag{5.30}$$

The nonnegativity of $g,\ \Psi$ yields:

$$\sum_{N\left(\frac{R}{2},b\right)\leq j=\ell<k-1}\int_{\mathcal{I}_{j,\ell,k}(b)}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi\left(\epsilon_1,\epsilon_2,\epsilon_3\right)\leq$$

$$\leq\sum_{N\left(\frac{R}{2},b\right)\leq j\leq\ell<k-1}\int_{\mathcal{I}_{j,\ell,k}(b)}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi\left(\epsilon_1,\epsilon_2,\epsilon_3\right). \tag{5.31}$$

Using the definition of the sets $\mathcal{I}_{j,\ell,k}(b)$ in (5.19), as well as (5.20), we obtain that in the integral term on the right-hand side of (5.31) we have, using that $j=\ell<k-1$:

$$\epsilon_-=\epsilon_3\quad,\quad\frac{1}{b^{\frac{3}{2}}}\leq\frac{(\epsilon_0)^{\frac{3}{2}}}{(\epsilon_+)^{\frac{3}{2}}}\leq 1\quad,\quad\left(\frac{\epsilon_0-\epsilon_-}{\epsilon_0}\right)^2\geq\left(1-\frac{1}{b}\right)^2=\rho^2. \tag{5.32}$$

We define:

$$\int_{\mathcal{I}_j(b)} g d\epsilon = I_j\quad,\quad j=0,1,2,... \tag{5.33}$$

Combining (5.27), (5.31), (5.32), (5.33) we then obtain:

$$\frac{\rho^2}{b^{\frac{3}{2}}}\sum_{N\left(\frac{R}{2},b\right)\leq j<k-1} I_j^2 I_k\leq\sum_{N\left(\frac{R}{2},b\right)\leq j\leq\ell<k-1}\int_{\mathcal{I}_{j,\ell,k}(b)}\left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\Psi\left(\epsilon_1,\epsilon_2,\epsilon_3\right).$$

Using this inequality, as well as (5.29), (5.30), (5.31) we arrive at:

$$\frac{\rho^2}{b^{\frac{3}{2}}}\sum_{N\left(\frac{R}{2},b\right)\leq j<k-1} I_j^2 I_k\leq\frac{bR}{3}f_{bR}(t).$$

Suppose that we write $M=N\left(\frac{R}{2},b\right)$. The definition of $N\left(\frac{R}{2},b\right)$ in (5.20) then implies:

$$\frac{\rho^2}{b^{\frac{3}{2}}}\sum_{M\leq j<k-1} I_j^2 I_k\leq\frac{2b^{1-M}}{3}f_{bR}(t),$$

whence, keeping only the terms in the sum with $j = M$ and choosing R_M as the solution of $M = \left[\frac{\log\left(\frac{2}{R}\right)}{\log(b)}\right]$ we obtain:

$$I_M^2 \sum_{M<k} I_k \leq \left[\frac{2b^{\frac{5}{2}}}{3\rho^2} f_{bR_M}(t)\right] b^{-M}.$$

By Lemma 5.2, we deduce:

$$\sum_{M_0\leq j\leq \ell<k-1} I_k(b)\, I_\ell(b)\, I_j(b) \leq \sum_{M_0\leq j\leq \ell} \sqrt{\left[\frac{2b^{\frac{5}{2}}}{3\rho^2} f_{bR_j}(t)\right] b^{-j} \left[\frac{2b^{\frac{5}{2}}}{3\rho^2} f_{bR_\ell}(t)\right] b^{-\ell}}$$
$$= \frac{2b^{\frac{5}{2}}}{3\rho^2} \sum_{M_0\leq j\leq \ell} \frac{\sqrt{f_{bR_j}(t)\, f_{bR_\ell}(t)}}{\sqrt{b^{j+\ell}}},$$

for any $M_0 \geq \left[\frac{\log\left(\frac{2}{R}\right)}{\log(b)}\right]$. We choose then $M_0 = \left[\frac{\log\left(\frac{2}{R}\right)}{\log(b)}\right]$. We now notice that, due to (5.25):

$$\int_{\mathcal{S}_{\frac{R}{2},\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right] \leq 6 \sum_{M_0\leq j\leq \ell<k-1} I_k(b)\, I_\ell(b)\, I_j(b)$$

and therefore:

$$\int_{\mathcal{S}_{\frac{R}{2},\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right] \leq \frac{4b^{\frac{5}{2}}}{\rho^2} \sum_{M_0\leq j\leq \ell} \frac{\sqrt{f_{bR_j}(t)\, f_{bR_\ell}(t)}}{\sqrt{b^{j+\ell}}}. \tag{5.34}$$

Integrating (5.34) in $[0,T]$ and using Young's inequality we obtain:

$$\int_0^T dt \int_{\mathcal{S}_{\frac{R}{2},\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right] \leq \frac{2b^{\frac{5}{2}}}{\rho^2} \sum_{M_0\leq j\leq \ell} \frac{1}{\sqrt{b^{j+\ell}}} \int_0^T \left[f_{bR_j}(t) + f_{bR_\ell}(t)\right] dt.$$

Using the estimate (5.24) that is uniform in R we obtain:

$$B\int_0^T dt \int_{\mathcal{S}_{\frac{R}{2},\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right] \leq \frac{4b^{\frac{5}{2}}}{\rho^2} \left[2\pi \int_0^T dt \left(\int_{[0,1]} g(\epsilon)\, d\epsilon\right)^2 + M\right] \sum_{M_0\leq j\leq \ell} \frac{1}{\sqrt{b^{j+\ell}}}.$$

Using then the inequalities

$$\sum_{M_0\leq j\leq \ell} \frac{1}{\sqrt{b^{j+\ell}}} \leq \frac{b^{-M_0}}{\left(1-\frac{1}{\sqrt{b}}\right)^2} \leq \frac{bR}{2\left(\sqrt{b}-1\right)^2},$$

where we have used the definition of M_0 we obtain:

$$B\int_0^T dt \int_{\mathcal{S}_{\frac{R}{2},\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right] \leq \frac{2b^{\frac{5}{2}}}{\rho^2} \left[2\pi \int_0^T dt \left(\int_{[0,1]} g(\epsilon)\, d\epsilon\right)^2 + M\right] \frac{bR}{\left(\sqrt{b}-1\right)^2},$$

and (5.22) follows. ■

6 A Measure Theory result.

We now prove the following measure theory result which will play a crucial role in all the remaining part of the argument.

Lemma 6.1 *Suppose that $b>1$ and let us define the intervals $\{\mathcal{I}_k(b)\}$, $\left\{\mathcal{I}_k^{(E)}(b)\right\}$ as in (5.10), (5.12). Let $\mathcal{P}_b$ as in (5.14) and $A^{(E)}$ as in (5.13) for $A\in\mathcal{P}_b$. Given $0<\delta<\frac{2}{3}$, we define $\eta=\min\left\{\left(\frac{1}{3}-\frac{\delta}{2}\right),\frac{\delta}{6}\right\}>0$. Then, for any $g\in\mathcal{M}^+[0,1]$ satisfying*

$$\int_{\{0\}} gd\epsilon=0, \tag{6.1}$$

at least one of the following statements is satisfied:

(i) There exists an interval $\mathcal{I}_k(b)$ such that:

$$\int_{\mathcal{I}_k^{(E)}(b)} gd\epsilon \geq (1-\delta)\int_{[0,1]} gd\epsilon, \tag{6.2}$$

(ii) There exists two sets $\mathcal{U}_1,\mathcal{U}_2\in\mathcal{P}_b$ such that $\mathcal{U}_2\cap\mathcal{U}_1^{(E)}=\varnothing$ and:

$$\min\left\{\int_{\mathcal{U}_1} gd\epsilon,\int_{\mathcal{U}_2} gd\epsilon\right\}\geq\eta\int_{[0,1]} gd\epsilon. \tag{6.3}$$

Moreover, in the case (ii) the set $\mathcal{U}_1$ can be written in the form:

$$\mathcal{U}_1=\bigcup_{j=1}^{L}\mathcal{I}_{k_j}(b), \tag{6.4}$$

for some set of integers $\{k_j\}\subset\{1,2,3,...\}$ and some finite L. We have:

$$\mathcal{I}_{k_m}(b)\cap\left(\bigcup_{j=1}^{m-1}\mathcal{I}_{k_j}^{(E)}(b)\right)=\varnothing \quad , \quad m=2,3,...L \tag{6.5}$$

and also:

$$\sum_{j=1}^{L}\left(\int_{\mathcal{I}_{k_j}(b)} gd\epsilon\right)^2\leq\left(\int_{\mathcal{I}_{k_1}(b)} gd\epsilon\right)^2+\sum_{j=2}^{L}\int_{\mathcal{I}_{k_1}(b)} gd\epsilon\int_{\mathcal{I}_{k_j}(b)} gd\epsilon, \tag{6.6}$$

$$\int_{\mathcal{I}_{k_1}(b)} gd\epsilon<(1-\delta)\int_{[0,1]} gd\epsilon. \tag{6.7}$$

Remark 6.2 *The choice of the sets $\mathcal{U}_1$, $\mathcal{U}_2$ is not entirely symmetric. The property (6.5) holds for $\mathcal{U}_1$ but not for $\mathcal{U}_2$.*

Proof. Since the result is trivial if $\int_{[0,1]} g d\epsilon = 0$ we can assume without loss of generality that $\int_{[0,1]} g d\epsilon = 1$ replacing, if needed, g by $\frac{g}{\int_{[0,1]} g d\epsilon}$. We will denote as $\mathcal{G}_1$ the family of intervals $\{\mathcal{I}_k(b)\}$. Using that $\int_{\{0\}} g d\epsilon = 0$ we have:

$$\sum_{\mathcal{I} \in \mathcal{G}_1} \int_{\mathcal{I}} g d\epsilon = \int_{[0,1]} g d\epsilon = 1. \tag{6.8}$$

We define:

$$a_1 = \max\left\{\int_{\mathcal{I}} g d\epsilon : \mathcal{I} \in \mathcal{G}_1\right\} = \int_{\mathcal{I}_{(1)}} g d\epsilon \quad , \quad \mathcal{I}_{(1)} \in \mathcal{G}_1.$$

Since the sum on the left of (6.8) is finite, it follows that this maximum exists. The interval $\mathcal{I}_{(1)}$ does not need to be unique. Since $\int_{\mathcal{I}_{(1)}^{(E)}} g d\epsilon \geq a_1$, if $a_1 \geq (1-\delta)$, we would have (6.2) with $\mathcal{I}_k(b) = \mathcal{I}_{(1)}$. Suppose then that $a_1 < (1-\delta)$. We define $\mathcal{G}_2$ as:

$$\mathcal{G}_2 = \left\{\mathcal{I} \setminus \mathcal{I}_{(1)}^{(E)} : \mathcal{I} \in \mathcal{G}_1\right\}.$$

Notice that $\mathcal{G}_2 \subset \mathcal{G}_1$. We define now:

$$a_2 = \max\left\{\int_{\mathcal{I}} g d\epsilon : \mathcal{I} \in \mathcal{G}_2\right\} = \int_{\mathcal{I}_{(2)}} g d\epsilon \quad , \quad \mathcal{I}_{(2)} \in \mathcal{G}_2.$$

If $\int_{\mathcal{I}_{(1)}^{(E)} \cup \mathcal{I}_{(2)}^{(E)}} g d\epsilon < (1-\delta)$ we continue the iteration procedure and define sequentially sets $\mathcal{G}_k$, values a_k and intervals $\mathcal{I}_{(k)}$ as long as we have $\int_{\bigcup_{j=1}^{k-1} \mathcal{I}_{(j)}^{(E)}} g d\epsilon < (1-\delta)$:

$$\mathcal{G}_k = \left\{\mathcal{I} \setminus \mathcal{I}_{(k-1)}^{(E)} : \mathcal{I} \in \mathcal{G}_{k-1}\right\} \quad , \quad k = 2, 3, ...,$$

$$a_k = \max\left\{\int_{\mathcal{I}} g d\epsilon : \mathcal{I} \in \mathcal{G}_k\right\} = \int_{\mathcal{I}_{(k)}} g d\epsilon \quad , \quad \mathcal{I}_{(k)} \in \mathcal{G}_k. \tag{6.9}$$

Due to (6.8), iterating the procedure, we eventually arrive to some integer value $M \geq 2$ such that:

$$\int_{\bigcup_{j=1}^{M} \mathcal{I}_{(j)}^{(E)}} g d\epsilon \geq (1-\delta) \quad \text{and} \quad \int_{\bigcup_{j=1}^{M-1} \mathcal{I}_{(j)}^{(E)}} g d\epsilon < (1-\delta). \tag{6.10}$$

We define $\mathcal{U}_1 = \bigcup_{j=1}^{M-1} \mathcal{I}_{(j)}$, $\mathcal{U}_2 = [0,1] \setminus \mathcal{U}_1^{(E)}$. Notice that $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{P}_b$ and $\mathcal{U}_2 \cap \mathcal{U}_1^{(E)} = \varnothing$. We prove now that (6.3) holds for some $\eta > 0$. To this end we consider two different possibilities. Either $a_M \geq \frac{\delta}{6}$ or $a_M < \frac{\delta}{6}$. Notice that in both cases the second inequality in (6.10) implies:

$$\int_{\mathcal{U}_2} g d\epsilon \geq \delta. \tag{6.11}$$

Suppose first that $a_M \geq \frac{\delta}{6}$. Then $\int_{\mathcal{U}_1} gd\epsilon \geq \int_{\mathcal{I}_{(1)}} gd\epsilon = a_1 \geq a_M \geq \frac{\delta}{6}$. Therefore (6.3) holds with $\eta = \frac{\delta}{6}$. Suppose now that $a_M < \frac{\delta}{6}$. Then, using the first inequality of (6.10) we obtain:

$$(1-\delta) \leq \int_{\bigcup_{j=1}^{M} \mathcal{I}_{(j)}^{(E)}} gd\epsilon \leq \int_{\mathcal{U}_1^{(E)}} gd\epsilon + \int_{\mathcal{I}_{(M)}^{(E)} \setminus \mathcal{U}_1^{(E)}} gd\epsilon. \tag{6.12}$$

Notice that the definitions of a_M, $\mathcal{I}_{(M)}^{(E)}$, $\mathcal{U}_1$ and the families $\mathcal{G}_k$ imply:

$$\int_{\mathcal{I}_{(M)}^{(E)} \setminus \mathcal{U}_1^{(E)}} gd\epsilon \leq 3a_M < \frac{\delta}{2},$$

because $\mathcal{I}_{(M)}^{(E)} \setminus \mathcal{U}_1^{(E)}$ contains at most three intervals of the family $\mathcal{G}_M$. Due to the definition of a_M, the integral of g over these intervals is at most a_M. Therefore, (6.12) yields:

$$\left(1 - \frac{3\delta}{2}\right) \leq \int_{\mathcal{U}_1^{(E)}} gd\epsilon. \tag{6.13}$$

We now remark that:

$$\int_{\mathcal{U}_1^{(E)}} gd\epsilon = \sum_{k=1}^{M-1} \int_{\mathcal{I}_{(k)}^{(E)} \setminus \bigcup_{j=1}^{k-1} \mathcal{I}_{(j)}^{(E)}} gd\epsilon, \tag{6.14}$$

where we assume that $\bigcup_{j=1}^{0} \mathcal{I}_{(j)}^{(E)} = \varnothing$. Notice that, by definition of the sequence $\{a_k\}$ and the extended intervals $\mathcal{I}_{(k)}^{(E)}$ we have:

$$\int_{\mathcal{I}_{(k)}^{(E)} \setminus \bigcup_{j=1}^{k-1} \mathcal{I}_{(j)}^{(E)}} gd\epsilon \leq 3a_k = 3 \int_{\mathcal{I}_{(k)}} gd\epsilon. \tag{6.15}$$

Combining (6.13) and (6.15) as well as (6.9) and the fact that $\mathcal{I}_{(k)} \cap \mathcal{I}_{(j)} = \varnothing$ if $k \neq j$ we obtain:

$$\left(1 - \frac{3\delta}{2}\right) \leq 3 \sum_{k=1}^{M-1} \int_{\mathcal{I}_{(k)}} gd\epsilon = 3 \int_{\bigcup_{k=1}^{M-1} \mathcal{I}_{(k)}} gd\epsilon = 3 \int_{\mathcal{U}_1} gd\epsilon. \tag{6.16}$$

Combining (6.11) and (6.16) we obtain (6.3) with $\eta = \min \left\{ \left(\frac{1}{3} - \frac{\delta}{2}\right), \delta \right\}$. Using the result obtained if $a_M \geq \frac{\delta}{6}$ we then obtain that (6.3) is valid in all cases with $\eta = \min \left\{ \left(\frac{1}{3} - \frac{\delta}{2}\right), \frac{\delta}{6} \right\}$.

It remains to prove (6.4)-(6.7). Note that (6.4), (6.5) just follow from the definition of the set $\mathcal{U}_1$. In order to prove (6.6) we use the fact that the sequence $\{a_k\}_{k=1}^{M}$ is nonincreasing and:

$$\int_{\mathcal{I}_{k_j}(b)} gd\epsilon = a_j \ , \quad j = 1, 2, ..., (M-1) \, ,$$

whence (6.6) follows. Finally (6.7) is a consequence of the construction of the sequence of intervals $\int_{\mathcal{I}_{k_1}(b)} gd\epsilon = a_1$ as well as the fact that in this case $a_1 < (1-\delta)$.
■

We will need a rescaled version of Lemma 6.1:

Lemma 6.3 *Suppose that* $b > 1,\ 0 < R \leq 1$. *We define intervals* $\{\mathcal{I}_k(b,R)\}$, $\left\{\mathcal{I}_k^{(E)}(b,R)\right\}$ *as in (5.15). Let* $\mathcal{P}_b(R)$ *as in (5.16) and* $A^{(E)}$ *as in (5.17) for* $A \in \mathcal{P}_b(R)$. *Given* $0 < \delta < \frac{2}{3}$, *we define* $\eta = \min\left\{\left(\frac{1}{3}-\frac{\delta}{2}\right), \frac{\delta}{6}\right\} > 0$. *Then, for any* $g \in \mathcal{M}^+[0,R]$ *satisfying* $\int_{\{0\}} gd\epsilon = 0$, *at least one of the following statements is satisfied:*

(i) Either there exist an interval $\mathcal{I}_k(b,R)$ *such that:*

$$\int_{\mathcal{I}_k^{(E)}(b,R)} gd\epsilon \geq (1-\delta)\int_{[0,R]} gd\epsilon, \tag{6.17}$$

(ii) or, either there exist two sets $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{P}_b(R)$ *such that* $\mathcal{U}_2 \cap \mathcal{U}_1^{(E)} = \varnothing$ *and:*

$$\min\left\{\int_{\mathcal{U}_1} gd\epsilon, \int_{\mathcal{U}_2} gd\epsilon\right\} \geq \eta \int_{[0,R]} gd\epsilon. \tag{6.18}$$

Moreover, in the case (ii) the set $\mathcal{U}_1$ *can be written in the form:*

$$\mathcal{U}_1 = \bigcup_{j=1}^{L} \mathcal{I}_{k_j}(b,R) \tag{6.19}$$

for some sequence $\{k_j\}$ *and some finite* L. *We have:*

$$\mathcal{I}_{k_m}(b,R) \cap \left(\bigcup_{j=1}^{m-1} \mathcal{I}_{k_j}^{(E)}(b,R)\right) = \varnothing \quad , \quad m = 2,3,...L, \tag{6.20}$$

and also:

$$\sum_{j=1}^{L}\left(\int_{\mathcal{I}_{k_j}(b,R)} gd\epsilon\right)^2 \leq \left(\int_{\mathcal{I}_{k_1}(b,R)} gd\epsilon\right)^2 + \sum_{j=2}^{L}\int_{\mathcal{I}_{k_1}(b,R)} gd\epsilon \int_{\mathcal{I}_{k_j}(b,R)} gd\epsilon, \tag{6.21}$$

$$\int_{\mathcal{I}_{k_1}(b,R)} gd\epsilon < (1-\delta)\int_{[0,d]} gd\epsilon. \tag{6.22}$$

Proof. It is essentially a rescaling of Lemma 6.1. It can be obtained just defining a new set of variables $\tilde{\epsilon} = \frac{\epsilon}{R}$. The sets $\mathcal{I}_k(b), \mathcal{U}_1, \mathcal{U}_2$ in Lemma 6.1 are then transformed in the sets stated in this Lemma by means of the inverse transform $\epsilon = R\tilde{\epsilon}$. ■

7 Lower estimate for the triples in $[0,R]^3$ in terms of those which are separated from the diagonal.

We now prove that for the times in which the second alternative in Lemma 6.1 holds (cf. 6.3), it is possible to estimate the total number of triples contained in $[0,R]^3$ by means of the triples which are separated from the diagonal $\{\epsilon_1 = \epsilon_2 = \epsilon_3\}$.

Lemma 7.1 *Let $0 < \delta < \frac{2}{3}$, $0 < \rho < 1$. For any $R \in (0,1)$ we define $\mathcal{S}_{R,\rho}$ as in (5.18). Let us assume also that $b = \frac{1}{(1-\rho)}$. Then there exists $\nu = \nu(\delta) > 0$ independent on R and ρ such that, for any $g \in \mathcal{M}^+[0,R]$ satisfying $\int_{\{0\}} gd\epsilon = 0$ if the alternative (ii) in Lemma 6.3 takes place we have:*

$$\int_{\mathcal{S}_{bR,\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right] \geq \nu \left(\int_{[0,R]} gd\epsilon\right)^3 . \tag{7.1}$$

Proof. Using the second inclusion in (5.20) of Lemma 5.3 we obtain:

$$\int_{\mathcal{S}_{bR,\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right] \geq \sum_{\sigma\in S^3} \left[\sum_{N(R,b)\leq j\leq \ell<k-1} \int_{\mathcal{I}_{\sigma(j,\ell,k)}(b,R)} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right]\right], \tag{7.2}$$

where:

$$\mathcal{I}_{\sigma(j,\ell,k)}(b,R) = \mathcal{I}_{\sigma(j)}(b,R) \times \mathcal{I}_{\sigma(\ell)}(b,R) \times \mathcal{I}_{\sigma(k)}(b,R).$$

We define the action of the permutations semigroup S^3 in $[0,R]^3$ by means of the mapping $(\epsilon_1,\epsilon_2,\epsilon_3) \to (\epsilon_{\sigma(1)},\epsilon_{\sigma(2)},\epsilon_{\sigma(3)})$. Notice that the subsets of the family $\{\mathcal{I}_k(b,R)\}$ are ordered by means of the order relation which says that $\mathcal{J}_m < \mathcal{J}_k$ if for any $\epsilon_1 \in \mathcal{J}_m$ and $\epsilon_2 \in \mathcal{J}_k$ we have that $\epsilon_1 < \epsilon_2$.

We now restrict ourselves to the case in which at least two of the intervals are contained in $\mathcal{U}_1$. The third one can be either in $\mathcal{U}_1$ or $\mathcal{U}_2$. More precisely, we define the following sets:

$$\mathcal{Y}_0 = \left[\bigcup_{\left[\mathcal{J}_{m_1}<\mathcal{J}_{m_2}\leq\mathcal{J}_{m_3};\mathcal{J}_{m_1}\subset\mathcal{U}_1,\mathcal{J}_{m_2}\subset\mathcal{U}_2,\mathcal{J}_{m_3}\subset\mathcal{U}_2\right]} \bigcup_{m_1}\bigcup_{m_2}\bigcup_{m_3} \sigma\left(\mathcal{J}_{m_1}\times\mathcal{J}_{m_2}\times\mathcal{J}_{m_3}\right)\right]$$

$$\mathcal{Y}_1 = \left[\bigcup_{\left[\mathcal{J}_{m_1}<\mathcal{J}_{m_2}\leq\mathcal{J}_{m_3};\mathcal{J}_{m_1}\subset\mathcal{U}_2,\mathcal{J}_{m_2}\subset\mathcal{U}_1,\mathcal{J}_{m_3}\subset\mathcal{U}_1\right]} \bigcup_{m_1}\bigcup_{m_2}\bigcup_{m_3} \sigma\left(\mathcal{J}_{m_1}\times\mathcal{J}_{m_2}\times\mathcal{J}_{m_3}\right)\right]$$

$$\mathcal{Y}_2 = \left[\bigcup_{\left[\mathcal{J}_{m_1}<\mathcal{J}_{m_2}\leq\mathcal{J}_{m_3};\mathcal{J}_{m_1}\subset\mathcal{U}_1,\mathcal{J}_{m_2}\subset\mathcal{U}_2,\mathcal{J}_{m_3}\subset\mathcal{U}_1\right]} \bigcup_{m_1}\bigcup_{m_2}\bigcup_{m_3} \sigma\left(\mathcal{J}_{m_1}\times\mathcal{J}_{m_2}\times\mathcal{J}_{m_3}\right)\right]$$

$$\mathcal{Y}_3 = \left[\bigcup_{\left[\mathcal{J}_{m_1} < \mathcal{J}_{m_2} \leq \mathcal{J}_{m_3}; \mathcal{J}_{m_1} \subset \mathcal{U}_1, \mathcal{J}_{m_2} \subset \mathcal{U}_1, \mathcal{J}_{m_3} \subset \mathcal{U}_2\right]} \bigcup_{m_1} \bigcup_{m_2} \bigcup_{m_3} \sigma \left(\mathcal{J}_{m_1} \times \mathcal{J}_{m_2} \times \mathcal{J}_{m_3} \right) \right]$$

$$\mathcal{Y} = \mathcal{Y}_0 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3. \tag{7.3}$$

We now claim that the set $\mathcal{Y}$ is contained in the union of sets $\mathcal{I}_{\sigma(j,\ell,k)}(b, R)$ appearing in the right-hand side of (7.2). (Notice that we impose there that $N(R, b) \leq j \leq \ell < k - 1$ in that union). Indeed, we have to consider several possibilities. Suppose first that $\mathcal{J}_{m_1} \in \mathcal{U}_1$ and $\mathcal{J}_{m_2} \in \mathcal{U}_2$. Then we have $\mathcal{J}_{m_1} \cap \mathcal{J}_{m_2}^{(E)} = \varnothing$ whence we would have $\ell < k - 1$ as in (7.2). If $\mathcal{J}_{m_1} \in \mathcal{U}_2$ and $\mathcal{J}_{m_2} \in \mathcal{U}_1$ we argue similarly. Suppose finally that $\mathcal{J}_{m_1} \in \mathcal{U}_1$ and $\mathcal{J}_{m_2} \in \mathcal{U}_1$. Then, the property (6.20) yields also $\mathcal{J}_{m_1} \cap \mathcal{J}_{m_2}^{(E)} = \varnothing$.

Therefore:

$$\int_{\mathcal{Y}} \left[\prod_{m=1}^{3} g_m d\epsilon_m \right] \leq \sum_{\sigma \in S^3} \left[\sum_{N(R,b) \leq j \leq \ell < k-1} \int_{\mathcal{I}_{\sigma(j,\ell,k)}(b,R)} \left[\prod_{m=1}^{3} g_m d\epsilon_m \right] \right],$$

whence, using also (7.2):

$$\int_{\mathcal{Y}} \left[\prod_{m=1}^{3} g_m d\epsilon_m \right] \leq \int_{\mathcal{S}_{bR,\rho}} \left[\prod_{m=1}^{3} g_m d\epsilon_m \right]. \tag{7.4}$$

Notice that, due to (6.18), since alternative (ii) holds, we have:

$$\left(\int_{[0,R]} g d\epsilon \right)^3 \leq \frac{1}{\eta^3} \left(\int_{\mathcal{U}_1} g d\epsilon \right)^2 \int_{\mathcal{U}_2} g d\epsilon, \tag{7.5}$$

where η is as in Lemma 6.3.

We then need to prove that it is possible to estimate the right-hand side of (7.5) by means of the integral on the left-hand side of (7.4). In order to check this we just notice that the product $\left(\int_{\mathcal{U}_1} g d\epsilon \right)^2 \int_{\mathcal{U}_2} g d\epsilon$ can be written as the sum:

$$\left[\sum_{\mathcal{J}_{m_1} \in \mathcal{U}_1, \mathcal{J}_{m_2} \in \mathcal{U}_1} \left(\int_{\mathcal{J}_{m_1}} g d\epsilon \right) \left(\int_{\mathcal{J}_{m_2}} g d\epsilon \right) \right] \int_{\mathcal{U}_2} g d\epsilon.$$

We decompose this sum in two types of terms, namely:

$$S_1 = \left[\sum_{\mathcal{J}_{m_1} \in \mathcal{U}_1, \mathcal{J}_{m_3} \in \mathcal{U}_2} \left(\int_{\mathcal{J}_{m_1}} g d\epsilon \right)^2 \left(\int_{\mathcal{J}_{m_3}} g d\epsilon \right) \right] = \sum_{\mathcal{J}_{m_1} \in \mathcal{U}_1} \left(\int_{\mathcal{J}_{m_1}} g d\epsilon \right)^2 \int_{\mathcal{U}_2} g d\epsilon,$$

$$S_2 = \left[\sum_{\mathcal{J}_{m_1} \in \mathcal{U}_1, \mathcal{J}_{m_2} \in \mathcal{U}_1, \mathcal{J}_{m_3} \in \mathcal{U}_2; \mathcal{J}_{m_1} \neq \mathcal{J}_{m_2}} \left(\int_{\mathcal{J}_{m_1}} g d\epsilon \right) \left(\int_{\mathcal{J}_{m_2}} g d\epsilon \right) \left(\int_{\mathcal{J}_{m_3}} g d\epsilon \right) \right].$$

The integral S_2 can be estimated by means of $\int_{\mathcal{Y}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right]$. To check this we just notice that all the sets of the form $\mathcal{J}_{m_1} \times \mathcal{J}_{m_2} \times \mathcal{J}_{m_3}$ with $\mathcal{J}_{m_1} \in \mathcal{U}_1, \mathcal{J}_{m_2} \in \mathcal{U}_1, \mathcal{J}_{m_3} \in \mathcal{U}_2$, $\mathcal{J}_{m_1} \neq \mathcal{J}_{m_2}$ are contained in $\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3$. Therefore:

$$S_2 \leq \int_{\mathcal{Y}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right]. \tag{7.6}$$

It only remains to estimate S_1. To this end we use (6.21). Then:

$$\begin{aligned} S_1 \leq & \left(\int_{\mathcal{I}_{k_1}(b,R)} g d\epsilon\right)^2 \int_{\mathcal{U}_2} g d\epsilon + \sum_{\mathcal{J}_{m_3} \in \mathcal{U}_2} \sum_{j=2}^{L} \int_{\mathcal{I}_{k_1}(b,R)} g d\epsilon \int_{\mathcal{I}_{k_j}(b,R)} g d\epsilon \int_{\mathcal{J}_{m_3}} g d\epsilon \\ \equiv & \; S_{1,1} + S_{1,2}, \end{aligned} \tag{7.7}$$

where the meaning of the sets $\mathcal{I}_{k_j}(b)$ is the same as in the Proof of Lemma 6.1.

The term $S_{1,2}$ in (7.7) consists of the sum of integrals in sets of the form $\mathcal{I}_{k_1}(b,R) \times \mathcal{I}_{k_j}(b,R) \times \mathcal{J}_{m_3}$ with $j = 2, 3, \ldots$ and $\mathcal{J}_{m_3} \in \mathcal{U}_2$. These sets are contained in $\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3$. Then:

$$S_{1,2} \leq \int_{\mathcal{Y}} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right]. \tag{7.8}$$

We now estimate the term $S_{1,1}$. We write $\mathcal{U}_2 = \bigcup_m \mathcal{I}_m^*$. Due to (6.18), as well as the fact that $\mathcal{I}_m^* \cap \mathcal{I}_{k_1}^{(E)}(b,R) = \varnothing$, we have at least one of the two following possibilities:

$$\int_{\bigcup_m \mathcal{I}_m^*; \mathcal{I}_m^* < \mathcal{I}_{k_1}(b,R)} g d\epsilon \geq \frac{1}{2} \int_{\mathcal{U}_2} g d\epsilon \tag{7.9}$$

or:

$$\int_{\bigcup_m \mathcal{I}_m^*; \mathcal{I}_m^* > \mathcal{I}_{k_1}(b,R)} g d\epsilon \geq \frac{1}{2} \int_{\mathcal{U}_2} g d\epsilon. \tag{7.10}$$

Suppose that (7.9) takes place. Then:

$$S_{1,1} \leq 2 \left(\int_{\mathcal{I}_{k_1}(b,R)} g d\epsilon\right)^2 \int_{\bigcup_m \mathcal{I}_m^*; \mathcal{I}_m^* < \mathcal{I}_{k_1}(b,R)} g d\epsilon.$$

The right-hand side of this inequality can be estimated by $2\int_{\mathcal{Y}_1} \left[\prod_{m=1}^{3} g_m d\epsilon_m\right]$, since it is possible to write the term on the right as the sum of integrals on sets with the form $\mathcal{I}_m^* \times \mathcal{I}_{k_1}(b,R) \times \mathcal{I}_{k_1}(b,R)$.

Suppose now that we have (7.10). Combining this formula with (6.22) and (6.18) we obtain:

$$\int_{\mathcal{I}_{k_1}(b,R)} g d\epsilon \leq (1-\delta) \int_{[0,R]} g d\epsilon \leq \frac{(1-\delta)}{\eta} \int_{\mathcal{U}_2} g d\epsilon \leq \frac{2(1-\delta)}{\eta} \int_{\bigcup_m \mathcal{I}_m^*; \mathcal{I}_m^* > \mathcal{I}_{k_1}(b,R)} g d\epsilon,$$

whence:

$$S_{1,1} \leq \frac{2(1-\delta)}{\eta}\left(\int_{\mathcal{I}_{k_1}(b,R)} gd\epsilon\right)\left(\int_{\mathcal{U}_2} gd\epsilon\right)\left(\int_{\bigcup_m \mathcal{I}_m^*;\mathcal{I}_m^*>\mathcal{I}_{k_1}(b,R)} gd\epsilon\right)$$
$$\leq \frac{4(1-\delta)}{\eta}\left(\int_{\mathcal{I}_{k_1}(b,R)} gd\epsilon\right)\left(\int_{\bigcup_m \mathcal{I}_m^*;\mathcal{I}_m^*>\mathcal{I}_{k_1}(b,R)} gd\epsilon\right)^2 .$$

The right hand side of this inequality can be estimated by $\frac{4(1-\delta)}{\eta}\int_{\mathcal{Y}_0}\left[\prod_{m=1}^3 g_m d\epsilon_m\right]$. We have then obtained that:

$$S_{1,1} \leq \max\left\{2, \frac{4(1-\delta)}{\eta}\right\}\int_{\mathcal{Y}}\left[\prod_{m=1}^3 g_m d\epsilon_m\right],$$

which combined with (7.8) yields:

$$S_1 \leq \max\left\{3, \frac{4(1-\delta)}{\eta}\right\}\int_{\mathcal{Y}}\left[\prod_{m=1}^3 g_m d\epsilon_m\right].$$

Combining this formula with (7.6) and using (7.4) we conclude the Proof of the Lemma. ■

8 Estimating the rate of formation of particles with small energy.

In this Section we prove several estimates whose meaning is the following. If we have a weak solution f of (1.10), (1.11) on $0 \leq t \leq T_0$ in the sense of Definition 2.7 such that $g(t) = 4\pi\sqrt{2\epsilon}f(t)$ satisfies condition (6.1) for all $0 \leq t \leq T_0$ for some suitable T_0, then either the alternative (ii) in Lemma 6.3 takes place during most of the time for small R, something that contradicts (5.22), or the alternative (i) in Lemma 6.3 takes place for most times with b sufficiently close to one. In this second case, if the initial density of particles is not too small near $\epsilon = 0$, there would be a large transfer of particles towards small energies and this would contradict the conservation of the total number of particles. The consequence of this contradiction is that the maximal time of existence for the solution f must be smaller than T_0.

The precise way of obtaining this contradiction is to estimate the measure of some subsets of $[0, T_0]$ for which precise information about the concentration properties of g over them are available. We will then prove that the total measure of these sets, which cover the whole interval $[0, T_0]$, is strictly smaller than T_0.

8.1 Defining some subsets of $[0, T_0]$.

In the remaining of this Section we assume that f is a weak solution of (1.10) (1.11) on $(0, T)$ in the sense of Definition 2.7, with initial data f_0 satisfying (2.3) and consider g as defined by (1.12).

For all $n=0,1,2,...$ let $R_n=2^{-n}$. For any $\theta_1>0,\ \theta_2>0$ and $0\leq T_0<T_{max}$ we define the following sets:

$$B_\ell=\left\{t\in[0,T_0]:\int_{[0,R_\ell]}g(\epsilon,t)\,d\epsilon\geq(R_\ell)^{\theta_1}\right\}\quad,\ \ell=0,1,2,... \tag{8.1}$$

We also define the sequence $b_\ell=1+(R_\ell)^{\theta_2}$, $\ell=0,1,2,...$ and the sets:

$$A_{n,\ell}=\left\{t\in[0,T_0]:\ \text{such that}\ \int_{\mathcal{I}_n^{(E)}(b_\ell,R_\ell)}g(t,\epsilon)\,d\epsilon\geq(R_{\ell+1})^{\theta_1}\right\}, \tag{8.2}$$

for $\ell=0,1,2,..,\ n=1,2,3,...$ We recall that $\mathcal{I}_n(b_\ell,R_\ell)$ has been defined in (5.15).

Notice that we have $\mathcal{I}_n^{(E)}(b_\ell,R_\ell)\subset[0,R_\ell]$ for all $n=1,2,...$. This is the motivation of the definitions of the sets above.

The following result is basically a consequence of Lemma 5.4 and Lemma 6.1.

Lemma 8.1 *Let f be a weak solution of (1.10), (1.11) on $[0,T]$ with initial data f_0 such that $g_0=4\pi\sqrt{2\epsilon}f_0\in\mathcal{M}_+(\mathbb{R}^+;1+\epsilon)$ and satisfying (2.3). Suppose also that $g(t)=4\pi\sqrt{2\epsilon}f(t)$ satisfies condition (6.1) for all $t\in[0,T]$. Given $\theta_1>0,\ \theta_2>0$, let us define the sets $B_n,\ A_{n,\ell}$ as in (8.1), (8.2) and Ω_ℓ as:*

$$\Omega_\ell=B_\ell\setminus\bigcup_{n\geq1}A_{n,\ell}\subset[0,T]\quad,\quad\ell=0,1,2,...$$

Then, there exists $\theta_0>0$ such that, if $\min\{\theta_1,\theta_2\}<\theta_0$, we have:

$$|\Omega_\ell|\leq K(1+T)\,R_\ell^{1-3\theta_1-4\theta_2},$$

for some $K=K(M,\theta_1)$ and for any $\ell=0,1,2,..$.

Proof. We apply Lemma 6.3 with $(1-\delta)=2^{-\theta_1}$ and $b=b_\ell$. The definitions of the sets B_ℓ and $A_{n,\ell}$ show that Ω_ℓ is the set of times t in $[0,T]$ for which the alternative (i) in Lemma 6.3 does not take place. Therefore, the alternative (ii) takes place. We can then apply, for such times, Lemma 7.1 which combined with Lemma 5.4 (cf. also (8.1)) gives the following estimate:

$$(R_\ell)^{3\theta_1}\int_0^{T_{\max}}\chi_{\Omega_\ell}dt\leq\frac{2b_\ell^{\frac{7}{2}}b_\ell R_\ell}{B\nu\rho_\ell^2\left(\sqrt{b_\ell}-1\right)^2}\left[2\pi\int_0^{T_{\max}}dt\left(\int_{[0,1]}g(\epsilon)\,d\epsilon\right)^2+M\right],$$

where ρ_ℓ is related with b_ℓ as in Lemma 7.1, whence $\rho_\ell=1-\frac{1}{b_\ell}$. We estimate the terms between brackets in the right-hand side in terms of the total number of particles. Therefore, using Taylor's expansion, it follows that there exists $K=K(M,\theta_1)$ such that:

$$\int_0^{T_{\max}}\chi_{\Omega_\ell}dt\leq KR_\ell^{1-3\theta_1-4\theta_2}(1+T)$$

whence the result follows. ■

We define a new family of sets $\mathcal{A}_\ell$ by means of:

$$\mathcal{A}_\ell = \bigcup_{n=1}^{\left[\frac{\log(2)}{\log(b_\ell)}\right]+1} A_{n,\ell}.$$

Lemma 8.2 *Under the assumptions of Lemma 8.1 we have:*

$$(B_\ell \setminus B_{\ell+1}) \cap \left(\bigcup_{n\geq 1} A_{n,\ell} \diagdown \mathcal{A}_\ell \right) = \varnothing,$$

for $\ell = 0, 1, 2, ...$

Proof. Notice that for $n \geq \left[\frac{\log(2)}{\log(b_\ell)}\right] + 2 > \frac{\log(2)}{\log(b_\ell)} + 1$ the extended intervals $\mathcal{I}_n^{(E)}(b_\ell, R_\ell)$ which appear in the definition of the sets $A_{n,\ell}$ are contained in

$$\left\{ \epsilon \leq b_\ell^{-\left(\frac{\log(2)}{\log(b_\ell)}+1\right)} R_\ell = \frac{b_\ell^{-1}}{2} R_\ell < R_{\ell+1} \right\}.$$

Then, if $t \in \bigcup_{n\geq 1} A_{n,\ell} \diagdown \mathcal{A}_\ell$ we have:

$$\int_{[0,R_{\ell+1}]} g(\epsilon, t)\, d\epsilon \geq \int_{\mathcal{I}_{n_0}^{(E)}(b_\ell, R_\ell)} g(\epsilon, t)\, d\epsilon,$$

for some $n_0 \geq \left[\frac{\log(2)}{\log(b_\ell)}\right] + 2$. Therefore, due to the definition of $A_{n_0,\ell}$:

$$\int_{[0,R_{\ell+1}]} g(\epsilon, t)\, d\epsilon \geq (R_{\ell+1})^{\theta_1}.$$

On the other hand, if $t \in (B_\ell \setminus B_{\ell+1})$ we have:

$$\int_{[0,R_\ell]} g(\epsilon, t)\, d\epsilon \geq (R_\ell)^{\theta_1} \quad , \quad \int_{[0,R_{\ell+1}]} g(\epsilon, t)\, d\epsilon < (R_{\ell+1})^{\theta_1},$$

but this gives a contradiction unless $(B_\ell \setminus B_{\ell+1}) \cap \left(\bigcup_{n\geq 1} A_{n,\ell} \diagdown \mathcal{A}_\ell\right) = \varnothing$. ■

Lemma 8.3 *Under the assumptions of Lemma 8.1 we have:*

$$|(B_\ell \setminus B_{\ell+1}) \setminus \mathcal{A}_\ell| \leq K (R_\ell)^\alpha,$$

where $K = K(E, M, \theta_1)$ *and* α *are as in Lemma 8.1.*

Proof. Due to Lemma 8.2 we have:

$$(B_\ell \setminus B_{\ell+1}) \setminus \mathcal{A}_\ell = (B_\ell \setminus B_{\ell+1}) \setminus \bigcup_{n=1} A_{n,\ell}.$$

Then, since $(B_\ell \setminus B_{\ell+1}) \setminus \bigcup_{n\geq 1} A_{n,\ell} \subset B_\ell \setminus \bigcup_{n\geq 1} A_{n,\ell}$, we obtain:

$$(B_\ell \setminus B_{\ell+1}) \setminus \mathcal{A}_\ell \subset B_\ell \setminus \bigcup_{n\geq \ell} A_{n,\ell} = \Omega_\ell.$$

Using Lemma 8.1 the result follows. ■

We now proceed to estimate $|\mathcal{A}_\ell|$. This is the crucial step where the properties of the kinetic equation are used. More precisely, we derive some detailed estimates for the lifetime of the possible concentrations of mass of g at regions of order R_ℓ. These estimates will be obtained using suitable test functions that solve some kind of adjoint equation of (1.10), (1.11). The choice of these test functions is made in order to show that, if the measure g is very concentrated, then the particles transported towards smaller sizes remain there for sufficiently long times. As a preliminary step we describe the construction of the test function.

We need to introduce some additional notation. Given $t \in \mathcal{A}_\ell$ there exists at least one integer $N = N(t) \in \left\{1, ..., \left[\frac{\log(2)}{\log(b_\ell)}\right] + 1\right\}$ such that $\int_{\mathcal{I}^{(E)}_{N(t)}(b_\ell,R_\ell)} g(t,\epsilon)\, d\epsilon \geq (R_{\ell+1})^{\theta_1}$. If different possible choices exist, it is possible to define a measurable function $N(t)$ with this property.

We then have the following result:

Lemma 8.4 *Suppose that the assumptions of Lemma 8.1 hold. Given* $\theta_1 > 0,\ \theta_2 > 0$ *such that* $(1 - 2\theta_1 - \theta_2) > 0$ *we define the sets* $B_n,\ A_{n,\ell}$ *as in (8.1), (8.2). Let us assume that there exists* $\tilde{T}_0 \in [0, T]$ *such that*

$$\int_0^{\tilde{T}_0} \chi_{\mathcal{A}_\ell}(t) \left(\int_{\mathcal{I}^{(E)}_{N(t)}(b_\ell,R_\ell)} g(t,\epsilon)\, d\epsilon \right)^2 dt = K_2 (R_\ell)^{1-\theta_2},$$

with

$$K_2 = \left(\frac{\sqrt{2}-1}{2} \right). \tag{8.3}$$

Then, there exists a function $\varphi \in L^\infty\left(\left[0, \tilde{T}_0\right], C^1(\mathbb{R}^+)\right)$ *satisfying the following properties:*

(i) $0 \leq \varphi(t,\epsilon) \leq 1$ *for* $(t,\epsilon) \in \left[0, \tilde{T}_0\right] \times \mathbb{R}^+$.

(ii) $\varphi(t,\cdot)$ *is convex in* $\mathbb{R}^+$ *for each* $t \in \left[0, \tilde{T}_0\right]$.

(iii) $\operatorname{supp}(\varphi(t,\cdot)) \subset \left[0, \frac{R_\ell}{4}\right]$ *for each* $t \in \left[0, \tilde{T}_0\right]$.

(iv) $\varphi(\epsilon, t) \geq \frac{1}{2}$ *for* $0 \leq \epsilon \leq \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) \frac{R_\ell}{8}$, $0 \leq t \leq \tilde{T}_0$.

(v) The following inequality holds for $0 \le t \le \tilde{T}_0,\ \epsilon \ge 0$:

$$\partial_t \varphi + \frac{\chi_{\mathcal{A}_\ell}(t)}{2^{\frac{3}{2}} R_\ell} \int\int_{\left\{\epsilon_2 \le \epsilon_3,\ \epsilon_2, \epsilon_3 \in \mathcal{I}^{(E)}_{N(t)}(b_\ell, R_\ell)\right\}} g_2 g_3 \left[\varphi(\epsilon_1 + \epsilon_3 - \epsilon_2) - \varphi(\epsilon_1)\right] d\epsilon_2 d\epsilon_3 \ge 0. \tag{8.4}$$

Proof. We define the functions:

$$\Omega\left(t; \tilde{T}_0\right) = \int_t^{\tilde{T}_0} \chi_{\mathcal{A}_\ell}(s)\, \omega(s)\, ds,$$

$$\omega(t) = \frac{1}{2^{\frac{3}{2}} R_\ell} \int\int_{\left\{\epsilon_2 \le \epsilon_3,\ \epsilon_2, \epsilon_3 \in \mathcal{I}^{(E)}_{N(t)}(b_\ell, R_\ell)\right\}} (\epsilon_3 - \epsilon_2)\, g_2 g_3 d\epsilon_2 d\epsilon_3, \tag{8.5}$$

$$\Psi(\zeta) = \frac{16}{(R_\ell)^2} \left[\left(\frac{R_\ell}{4} - \zeta\right)_+\right]^2,$$

where $(s)_+ = \max\{s, 0\}$. We then define the function $\varphi(\epsilon, t)$ by means of the formula:

$$\varphi(t, \epsilon) = \Psi\left(\epsilon + \Omega\left(t; \tilde{T}_0\right)\right). \tag{8.6}$$

Properties (i), (ii), (iii) can be immediately checked. In order to check (iv) we notice that the definition of K_2 (cf. 8.3) implies:

$$\Omega\left(t; \tilde{T}_0\right) \le \left(\frac{\sqrt{2} - 1}{\sqrt{2}}\right) \frac{R_\ell}{8} \quad , \quad 0 \le t \le \tilde{T}_0 \tag{8.7}$$

Indeed, we have (cf. (8.5)):

$$\begin{aligned} \Omega\left(t; \tilde{T}_0\right) &\le \frac{(b_\ell - 1)}{2^{\frac{5}{2}}} \int_0^{\tilde{T}_0} \chi_{\mathcal{A}_\ell}(t) \left(\int_{\left\{\epsilon \in \mathcal{I}^{(E)}_{N(t)}(b_\ell, R_\ell)\right\}} g(t, \epsilon)\, d\epsilon\right)^2 dt \\ &= \frac{(b_\ell - 1)}{2^{\frac{5}{2}}} K_2 (R_\ell)^{1-\theta_2} = \frac{1}{2^{\frac{5}{2}}} K_2 R_\ell, \end{aligned} \tag{8.8}$$

where we have used the definition of $\mathcal{A}_\ell$ (cf. (8.2)) and $N(t)$. Therefore:

$$\int_0^{\tilde{T}_0} \chi_{A_{\ell,\ell}}(t) \left(\int_{\left\{\epsilon \in \mathcal{I}^{(E)}_{N(t)}(b_\ell, R_\ell)\right\}} g(t, \epsilon)\, d\epsilon\right)^2 dt = K_2 (R_\ell)^{1-\theta_2}.$$

We notice also that $\Psi(\zeta) \ge \frac{1}{2}$ if $\zeta \le \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) \frac{R_\ell}{4}$. If $0 \le \epsilon \le \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) \frac{R_\ell}{8}$ we have, using (8.7):

$$\epsilon + \Omega\left(t; \tilde{T}_0\right) \le \left(\frac{\sqrt{2} - 1}{\sqrt{2}}\right) \frac{R_\ell}{8} + \left(\frac{\sqrt{2} - 1}{\sqrt{2}}\right) \frac{R_\ell}{8} \le \left(\frac{\sqrt{2} - 1}{\sqrt{2}}\right) \frac{R_\ell}{4},$$

whence (iv) follows.

It only remains to check (v). The convexity of $\varphi(t,\cdot)$ implies:

$$\varphi(\epsilon+\epsilon_3-\epsilon_2,t)-\varphi(\epsilon,t)\geq(\epsilon_3-\epsilon_2)\frac{\partial\varphi}{\partial\epsilon}(\epsilon,t)\quad\text{for }0\leq\epsilon_2\leq\epsilon_3.$$

Then, using also (8.5), (8.6):

$$\begin{aligned}&\partial_t\varphi(\epsilon,t)+\frac{1}{2^{\frac{3}{2}}R_\ell}\int\int_{\left\{\epsilon_2\leq\epsilon_3,\ \epsilon_2,\epsilon_3\in\mathcal{I}_{N(t)}^{(E)}(b_\ell,R_\ell)\right\}}g_2g_3\left[\varphi(\epsilon+\epsilon_3-\epsilon_2,t)-\varphi(\epsilon,t)\right]d\epsilon_2d\epsilon_3\\&\geq\partial_t\varphi(\epsilon,t)+\frac{1}{2^{\frac{3}{2}}R_\ell}\frac{\partial\varphi}{\partial\epsilon}(\epsilon,t)\int\int_{\left\{\epsilon_2\leq\epsilon_3,\ \epsilon_2,\epsilon_3\in\mathcal{I}_{N(t)}^{(E)}(b_\ell,R_\ell)\right\}}g_2g_3(\epsilon_3-\epsilon_2)\,d\epsilon_2d\epsilon_3\\&=\Psi'\left(\epsilon+\Omega\left(t;\tilde{T}_0\right)\right)\left[-\omega(t)+\omega(t)\right]=0,\end{aligned}$$

whence the Lemma follows. ■

We now prove the following result.

Proposition 8.5 *Suppose that the assumptions of Lemma 8.1 hold. Given* $\theta_1>0$, $\theta_2>0$ *such that* $(1-2\theta_1-\theta_2)>0$ *we define the sets* B_n, $A_{n,\ell}$ *as in (8.1), (8.2). Let us assume also that* $\nu>0$. *Then, there exists* $\rho=\rho(E,M,\nu,\theta_1,\theta_2)$ *such that, if* $f(\epsilon,0)=f_0(\epsilon)\geq\nu$ *in* $\epsilon\in[0,\rho]$ *for some* $0<\rho<1$ *we have:*

$$|\mathcal{A}_\ell|\leq K_2(R_\ell)^{1-2\theta_1-\theta_2}, \tag{8.9}$$

for $\ell>\frac{\log\left(\frac{1}{\rho}\right)}{\log(2)}$ *and where* K_2 *is as in (8.3).*

Proof. We must consider separately two cases. Suppose first that:

$$\int_0^{T_0}\chi_{\mathcal{A}_\ell}(t)\left(\int_{\left\{\epsilon\in\mathcal{I}_{N(t)}^{(E)}(b_\ell,R_\ell)\right\}}g(t,\epsilon)\,d\epsilon\right)^2dt\leq K_2(R_\ell)^{1-\theta_2}, \tag{8.10}$$

where K_2 is as in (8.3).

Then, using the definition of $\mathcal{A}_\ell$ (cf. (8.2)) we obtain:

$$|\mathcal{A}_\ell|\leq K_2(R_\ell)^{1-2\theta_1-\theta_2}$$

and (8.9) follows.

Let us assume now that:

$$\int_0^{T_0}\chi_{\mathcal{A}_\ell}(t)\left(\int_{\left\{\epsilon\in\mathcal{I}_{N(t)}^{(E)}(b_\ell,R_\ell)\right\}}g(t,\epsilon)\,d\epsilon\right)^2dt>K_2(R_\ell)^{1-\theta_2}.$$

Then, the continuity of the integral with respect to the domain of integration implies that there exists $\tilde{T}_0 \in [0,T] \cap A_{\ell,\ell}$ such that:

$$\int_0^{\tilde{T}_0} \chi_{\mathcal{A}_\ell}(t) \left(\int_{\left\{ \epsilon \in \mathcal{I}_{N(t)}^{(E)}(b_\ell, R_\ell) \right\}} g(t,\epsilon)\, d\epsilon \right)^2 dt = K_2 (R_\ell)^{1-\theta_2} . \tag{8.11}$$

Using (3.24), and symmetrizing the integral containing the cubic terms in the same way as in the derivation of (4.1), we obtain:

$$\begin{aligned}\partial_t \left(\int_{\mathbb{R}^+} g\varphi d\epsilon \right) &= \int_{\mathbb{R}^+} g \partial_t \varphi d\epsilon + \frac{1}{2^{\frac{5}{2}}} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{g_1 g_2 g_3 \Phi}{\sqrt{\epsilon_1 \epsilon_2 \epsilon_3}} \mathcal{G}_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 + \\ &\quad + \frac{\pi}{2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{g_1 g_2 \Phi}{\sqrt{\epsilon_1 \epsilon_2}} Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 \ , \quad a.e.\ t \in [0,T] ,\end{aligned} \tag{8.12}$$

where $\mathcal{G}_\varphi$ is as in (4.2), (4.3) and, symmetrizing in $\epsilon_1, \ \epsilon_2$ in the quadratic integral we can take:

$$Q_\varphi = [\varphi(t,\epsilon_3) + \varphi(t, \epsilon_1 + \epsilon_2 - \epsilon_3) - 2\varphi(t,\epsilon_1)] .$$

Using the symmetry of the function $\mathcal{G}_\varphi$ we can write the cubic term in the equivalent manner:

$$\frac{1}{2^{\frac{5}{2}}} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{g_1 g_2 g_3 \Phi}{\sqrt{\epsilon_1 \epsilon_2 \epsilon_3}} \mathcal{G}_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 = \frac{6}{2^{\frac{5}{2}}} \int \int \int_{\{\epsilon_1 \leq \epsilon_2 \leq \epsilon_3\}} \frac{g_1 g_2 g_3 \Phi}{\sqrt{\epsilon_1 \epsilon_2 \epsilon_3}} \mathcal{G}_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3,$$

with (cf. (4.9)):

$$\mathcal{G}_\varphi = \mathcal{G}_\varphi^{(1)} + \mathcal{G}_\varphi^{(2)} \ \text{ in } \ \{\epsilon_1 \leq \epsilon_2 \leq \epsilon_3\} ,$$

$$\mathcal{G}_\varphi^{(1)}(\epsilon_1,\epsilon_2,\epsilon_3) = \frac{\sqrt{\epsilon_1}}{3} [\varphi(\epsilon_1 + \epsilon_3 - \epsilon_2) + \varphi(\epsilon_3 + \epsilon_2 - \epsilon_1) - 2\varphi(\epsilon_3)] ,$$

$$\mathcal{G}_\varphi^{(2)}(\epsilon_1,\epsilon_2,\epsilon_3) = \frac{\sqrt{(\epsilon_2 + \epsilon_1 - \epsilon_3)_+}}{3} [\varphi(\epsilon_3) + \varphi(\epsilon_2 + \epsilon_1 - \epsilon_3) - \varphi(\epsilon_1) - \varphi(\epsilon_2)] ,$$

where the dependence of the function φ on t will not be made explicit unless it is needed. We now select the function φ as in Lemma 8.4. Since $\varphi(t,\cdot)$ is convex we have, arguing as in the Proof of Proposition 4.1:

$$\mathcal{G}_\varphi^{(1)}(\epsilon_1,\epsilon_2,\epsilon_3) \geq 0 \quad , \quad \mathcal{G}_\varphi^{(2)}(\epsilon_1,\epsilon_2,\epsilon_3) \geq 0. \tag{8.13}$$

Then, since $g \geq 0$ we obtain, using (8.12):

$$\begin{aligned}\partial_t \left(\int_{\mathbb{R}^+} g\varphi d\epsilon \right) &\geq \int_{\mathbb{R}^+} g \partial_t \varphi d\epsilon + \\ &\quad + \frac{\chi_{\mathcal{A}_\ell}(t)}{2^{\frac{5}{2}}} \int \int \int_{\left\{ \epsilon_1 \leq \frac{R_\ell}{4} \right\} \cap \left\{ \epsilon_2 \leq \epsilon_3, \ \epsilon_2, \epsilon_3 \in \mathcal{I}_{N(t)}^{(E)}(b_\ell, R_\ell) \right\}} \frac{g_1 g_2 g_3}{\sqrt{\epsilon_1 \epsilon_2 \epsilon_3}} \mathcal{G}_\varphi^{(1)} d\epsilon_1 d\epsilon_2 d\epsilon_3 + \\ &\quad + \frac{\pi}{2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{g_1 g_2 \Phi}{\sqrt{\epsilon_1 \epsilon_2}} Q_\varphi d\epsilon_1 d\epsilon_2 d\epsilon_3 ,\end{aligned} \tag{8.14}$$

a.e. $t \in [0,T]$. Using now that $\mathcal{G}_{\varphi}^{(1)}(\epsilon_1,\epsilon_2,\epsilon_3) \geq \varphi(\epsilon_1+\epsilon_3-\epsilon_2)\sqrt{\epsilon_1}$. We now add and substract $\varphi(\epsilon_1)\sqrt{\epsilon_1}$. Since $(\epsilon_3-\epsilon_2) > 0$ it follows that the support of $\varphi(\epsilon_1+\epsilon_3-\epsilon_2)$ is contained in the region where $\epsilon_1 \leq \frac{R_\ell}{4}$. Therefore:

$$
\begin{aligned}
&\frac{\chi_{\mathcal{A}_\ell}(t)}{2^{\frac{5}{2}}}\int\int\int_{\left\{\epsilon_1\leq\frac{R_\ell}{4}\right\}\cap\left\{\epsilon_2\leq\epsilon_3,\ \epsilon_2,\epsilon_3\in\mathcal{I}_{N(t)}^{(E)}(b_\ell,R_\ell)\right\}}\frac{g_1g_2g_3}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}}\mathcal{G}_{\varphi}^{(1)}d\epsilon_1d\epsilon_2d\epsilon_3\\
&\geq\frac{\chi_{\mathcal{A}_\ell}(t)}{2^{\frac{3}{2}}R_\ell}\int_{\left\{\epsilon_1\leq\frac{R_\ell}{4}\right\}}g_1d\epsilon_1\times\\
&\qquad\times\int\int_{\left\{\epsilon_2\leq\epsilon_3,\ \epsilon_2,\epsilon_3\in\mathcal{I}_{N(t)}^{(E)}(b_\ell,R_\ell)\right\}}g_2g_3\left[\varphi(\epsilon_1+\epsilon_3-\epsilon_2)-\varphi(\epsilon_1)\right]d\epsilon_2d\epsilon_3+\\
&\qquad+\frac{\chi_{\mathcal{A}_\ell}(t)}{2^{\frac{3}{2}}R_\ell}\int_{\left\{\epsilon_1\leq\frac{R_\ell}{4}\right\}}g_1\varphi_1d\epsilon_1\int\int_{\left\{\epsilon_2\leq\epsilon_3,\ \epsilon_2,\epsilon_3\in\mathcal{I}_{N(t)}^{(E)}(b_\ell,R_\ell)\right\}}g_2g_3d\epsilon_2d\epsilon_3. \qquad (8.15)
\end{aligned}
$$

On the other hand we can estimate the quadratic term in (8.14) as:

$$
\int\int\int\frac{g_1g_2}{\sqrt{\epsilon_1\epsilon_2}}\Phi Q_\varphi d\epsilon_1d\epsilon_2d\epsilon_3\geq-2\int\int\int\frac{g_1g_2}{\sqrt{\epsilon_1\epsilon_2}}\Phi\varphi(\epsilon_1)\,d\epsilon_1d\epsilon_2d\epsilon_3. \qquad (8.16)
$$

Using the definition of Φ we obtain:

$$
\begin{aligned}
&\int\int\int\frac{g_1g_2}{\sqrt{\epsilon_1\epsilon_2}}\Phi\varphi(\epsilon_1)\,d\epsilon_1d\epsilon_2d\epsilon_3\\
&\leq\int g(\epsilon_1)\varphi(\epsilon_1)\frac{d\epsilon_1}{\sqrt{\epsilon_1}}\int_0^{\epsilon_1}g_2d\epsilon_2\int_0^{\epsilon_1+\epsilon_2}d\epsilon_3+\int g(\epsilon_1)\varphi(\epsilon_1)\,d\epsilon_1\int_{\epsilon_1}^{\infty}\frac{g_2d\epsilon_2}{\sqrt{\epsilon_2}}\int_0^{\epsilon_1+\epsilon_2}d\epsilon_3\\
&\leq2\int g(\epsilon_1)\varphi(\epsilon_1)\sqrt{\epsilon_1}d\epsilon_1\int_0^{\epsilon_1}g_2d\epsilon_2+2\int g(\epsilon_1)\varphi(\epsilon_1)\,d\epsilon_1\int_{\epsilon_1}^{\infty}g_2\sqrt{\epsilon_2}d\epsilon_2\\
&\leq4(E+M)\int g(\epsilon_1)\varphi(\epsilon_1)\,d\epsilon_1.
\end{aligned}
$$

Taking into account that φ satisfies (8.4) as well as (8.14), (8.15) and (8.16) we obtain:

$$
\begin{aligned}
\partial_t\left(\int_{\mathbb{R}^+}g\varphi d\epsilon\right)\geq&\frac{\chi_{\mathcal{A}_\ell}(t)}{2^{\frac{3}{2}}R_\ell}\int g_1\varphi_1d\epsilon_1\int\int_{\left\{\epsilon_2\leq\epsilon_3,\ \epsilon_2,\epsilon_3\in\mathcal{I}_{N(t)}^{(E)}(b_\ell,R_\ell)\right\}}g_2g_3d\epsilon_2d\epsilon_3-\\
&-2\pi(E+M)\int_{\mathbb{R}^+}g\varphi d\epsilon
\end{aligned}
$$

and after a symmetrization argument:

$$
\partial_t\left(\int_{\mathbb{R}^+}g\varphi d\epsilon\right)\geq\frac{\chi_{\mathcal{A}_\ell}(t)}{2^{\frac{5}{2}}R_\ell}\left(\int_{\mathcal{I}_{N(t)}^{(E)}(b_\ell,R_\ell)}g(\epsilon)\,d\epsilon\right)^2\int_{\mathbb{R}^+}g\varphi d\epsilon-2\pi(E+M)\int_{\mathbb{R}^+}g\varphi d\epsilon. \qquad (8.17)
$$

We recall that the construction of the function φ implies:

$$\varphi(\epsilon, t) \geq \frac{1}{2} \quad \text{for} \quad 0 \leq \epsilon \leq \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\frac{R_\ell}{8} \quad , \quad 0 \leq t \leq \tilde{T}_0. \tag{8.18}$$

Then, Since $f(\epsilon, 0) = f_0(\epsilon) \geq \nu$ in $\epsilon \in [0, \rho]$ and using also that $\ell > \frac{\log\left(\frac{1}{\rho}\right)}{\log(2)}$ (whence $R_\ell < \rho$), we obtain, using also (8.18):

$$\int g_0(\epsilon)\, \varphi(\epsilon, 0)\, d\epsilon \geq 2\nu \sqrt{\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\frac{R_\ell}{8}} = \nu\sqrt{R_\ell}\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}. \tag{8.19}$$

Integrating the differential inequality (8.17) we then obtain:

$$\int g\left(\epsilon, \tilde{T}_0\right) \varphi\left(\epsilon, \tilde{T}_0\right) d\epsilon \geq \nu\sqrt{R_\ell}\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \exp\left(-2\pi(E+M)\tilde{T}_0\right) \times$$
$$\times \exp\left(\frac{1}{2^{\frac{3}{2}} R_\ell} \int_0^{\tilde{T}_0} \chi_{\mathcal{A}_\ell}(t) \left(\int_{\mathcal{I}^{(E)}_{N(t)}(b_\ell, R_\ell)} g(\epsilon)\, d\epsilon\right)^2 dt\right),$$

whence, using the definition of $\tilde{T}_0$ (cf. (8.11)):

$$\int g\left(\epsilon, \tilde{T}_0\right) \varphi\left(\epsilon, \tilde{T}_0\right) d\epsilon \geq \nu\sqrt{R_\ell}\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \exp\left(-2\pi(E+M)\tilde{T}_0\right) \times$$
$$\times \exp\left(\frac{1}{2^{\frac{3}{2}} R_\ell} K_2 (R_\ell)^{1-\theta_2}\right),$$

$$\int g\left(\epsilon, \tilde{T}_0\right) \varphi\left(\epsilon, \tilde{T}_0\right) d\epsilon \geq \nu\sqrt{R_\ell}\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \times$$
$$\times \exp\left(-2\pi(E+M)\tilde{T}_0\right) \exp\left(\frac{K_2}{2^{\frac{3}{2}} (R_\ell)^{\theta_2}}\right).$$

Choosing $\rho = \rho(M, E, \nu, \theta_2)$ sufficiently small satisfying

$$\nu\sqrt{\rho}\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}} \exp\left(-2\pi(E+M)\, T_0(M, E)\right) \exp\left(\frac{K_2}{2^{\frac{3}{2}} (R_\ell)^{\theta_2}}\right) > M,$$

we obtain a contradiction, since $R_\ell < \rho$ and $\varphi \leq 1$. This implies (8.10) and the result follows. ■

We can estimate now the measure of the sets $(B_\ell \setminus B_{\ell+1})$.

Lemma 8.6 *Suppose that the assumptions of Proposition 8.5 hold. Then, there exists* $\beta > 0$ *and* $K_3 = K_3(M, E, \theta_1) > 0$ *such that:*

$$|(B_\ell \setminus B_{\ell+1})| \leq K_3 (R_\ell)^\beta$$

for $\ell > \frac{\log\left(\frac{1}{\rho}\right)}{\log(2)}$.

Proof. The result is a consequence of Lemma 8.3 and Proposition 8.5. We choose $\beta = \min\{\alpha, 1 - 2\theta_1 - \theta_2\}$. Then:

$$|(B_\ell \setminus B_{\ell+1})| = |(B_\ell \setminus B_{\ell+1}) \setminus \mathcal{A}_\ell| + |\mathcal{A}_\ell| \leq (K + K_2)(R_\ell)^\beta,$$

whence the result follows with $K_3 = (K + K_2)$. ■

We can obtain then the following estimate.

Lemma 8.7 *Suppose that the assumptions of Proposition 8.5 are satisfied. Suppose that* $L \geq \frac{\log\left(\frac{1}{\rho}\right)}{\log(2)}$. *Then:*

$$|B_L| \leq \frac{K_3}{1 - 2^{-\beta}} (R_L)^\beta,$$

where $K_3 = K_3(M, E, \theta_1)$ *and* β *are as in Lemma 8.6.*

Proof. We write:

$$B_L = \bigcup_{\ell=L}^{\infty} (B_\ell \setminus B_{\ell+1}),$$

whence, using Lemma 8.6:

$$|B_L| = \sum_{\ell=L}^{\infty} |(B_\ell \setminus B_{\ell+1})| \leq K_3 \sum_{\ell=L}^{\infty} (R_\ell)^\beta = \frac{K_3}{1 - 2^{-\beta}} (R_L)^\beta.$$

■

8.2 A lower estimate for the mass in a given region.

We prove now that the mass in a small interval containing the origin cannot decay too fast.

Lemma 8.8 *Suppose that* $\int_{[0,\frac{\rho}{2}]} g_0 d\epsilon \geq m_0 > 0$, $\int_0^\infty g_0 d\epsilon = M \geq m_0$, $\int_0^\infty \epsilon g_0 d\epsilon = E > 0$ *where* $0 < \rho \leq 1$. *There exists* $T_0 = T_0(M, E) > 0$, *independent on* ρ *and* m_0, *such that for every weak solution* g *of (1.13), (1.14) on* $[0, T_0]$ *in the sense of Definition 2.6, with initial data* g_0 *and for which* $g(t)$ *satisfies (6.1) for all* $t \in [0, T_0]$, *we have*

$$\int_{[0,\rho]} g(\epsilon, t)\, d\epsilon \geq \frac{m_0}{4},$$

for $t \in [0, T_0]$.

Remark 8.9 *Notice that this Lemma assumes the existence of a weak solution f of (1.10), (1.11) on $[0,T_0]$ satisfying condition (6.1) for all $t\in[0,T_0]$. The "goal" of the whole argument is to prove that such solution cannot exist.*

Proof. Let us denote as $\varphi=\varphi(\epsilon)$ the test function:

$$\varphi(\epsilon)=\frac{1}{\rho}(\rho-\epsilon)_+ .$$

Using (3.24), as well as Theorem 4.1, we obtain:

$$\frac{d}{dt}\left(\int_0^\infty g(\epsilon)\varphi(\epsilon)\,d\epsilon\right)=\frac{1}{2^{\frac{5}{2}}}\int_0^\infty\int_0^\infty\int_0^\infty\frac{g_1g_2g_3}{\sqrt{\epsilon_1\epsilon_2\epsilon_3}}\mathcal{G}_\varphi(\epsilon_1,\epsilon_2,\epsilon_3)\,d\epsilon_1d\epsilon_2d\epsilon_3+$$
$$+\frac{\pi}{2}\int_0^\infty\int_0^\infty\int_0^\infty\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}Q_\varphi d\epsilon_1d\epsilon_2d\epsilon_3,$$

where:

$$Q_\varphi=[\varphi(\epsilon_3)+\varphi(\epsilon_1+\epsilon_2-\epsilon_3)-2\varphi(\epsilon_1)],$$
$$\Phi=\min\left(\sqrt{\epsilon_1},\sqrt{\epsilon_2},\sqrt{\epsilon_3},\sqrt{(\epsilon_1+\epsilon_2-\epsilon_3)_+}\right).$$

Theorem 4.1 yields $\mathcal{G}_\varphi(\epsilon_1,\epsilon_2,\epsilon_3)\geq 0$. Therefore:

$$\frac{d}{dt}\left(\int_0^\infty g(\epsilon)\varphi(\epsilon)\,d\epsilon\right)\geq-\pi\int_0^\infty\int_0^\infty\int_0^\infty\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}\varphi(\epsilon_1)\,d\epsilon_1d\epsilon_2d\epsilon_3. \tag{8.20}$$

We estimate the right-hand side of (8.20) splitting the integral as follows:

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}\varphi(\epsilon_1)\,d\epsilon_1d\epsilon_2d\epsilon_3=\int_0^\infty d\epsilon_1\int_0^{\epsilon_1}d\epsilon_2\int_0^\infty\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}\varphi(\epsilon_1)\,d\epsilon_3+$$
$$+\int_0^\infty d\epsilon_1\int_{\epsilon_1}^\infty d\epsilon_2\int_0^\infty\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}\varphi(\epsilon_1)\,d\epsilon_3.$$

Using the definition of Φ we obtain:

$$\int_0^\infty d\epsilon_1\int_0^{\epsilon_1}d\epsilon_2\int_0^\infty\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}\varphi(\epsilon_1)\,d\epsilon_3\leq 2\int_0^\infty\varphi(\epsilon_1)\sqrt{\epsilon_1}g_1d\epsilon_1\int_0^{\epsilon_1}g_2d\epsilon_2$$
$$\leq M\sqrt{2\rho}\int_0^\infty\varphi(\epsilon_1)\,g_1d\epsilon_1.$$

We have also:

$$\int_0^\infty d\epsilon_1\int_{\epsilon_1}^\infty d\epsilon_2\int_0^\infty\frac{g_1g_2\Phi}{\sqrt{\epsilon_1\epsilon_2}}\varphi(\epsilon_1)\,d\epsilon_3\leq\int_0^\infty d\epsilon_1\int_{\epsilon_1}^\infty d\epsilon_2\int_0^{\epsilon_1+\epsilon_2}\frac{g_1g_2}{\sqrt{\epsilon_2}}\varphi(\epsilon_1)\,d\epsilon_3$$
$$\leq\int_0^\infty\varphi(\epsilon_1)\,g_1d\epsilon_1\int_0^\infty g_2(1+\epsilon_2)\,d\epsilon_2=(M+E)\int_0^\infty\varphi(\epsilon_1)\,g_1d\epsilon_1.$$

Combining (8.20) with these estimates we obtain, using also $0 < \rho \leq 1$:

$$\frac{d}{dt}\left(\int_0^\infty g(\epsilon)\varphi(\epsilon)\,d\epsilon\right) \geq -\pi\left[M\sqrt{2}+(M+E)\right]\int_0^\infty g(\epsilon)\varphi(\epsilon)\,d\epsilon.$$

Integrating this inequality we obtain:

$$\int_0^{2\rho} g(\epsilon)\,d\epsilon \geq \int_0^\infty g(\epsilon)\varphi(\epsilon)\,d\epsilon \geq \frac{m_0}{2}\exp\left(-\pi\left[M\sqrt{2}+(M+E)\right]t\right),$$

whence the result follows if we assume that $T_0 = \frac{\log(2)}{\pi\left[M\sqrt{2}+(M+E)\right]}$. ■

We now prove the following:

Lemma 8.10 *Suppose that the assumptions of Lemma 8.8 are satisfied. Let* $\rho = 2^{-L}$ *for some* $L = 0, 1, 2, \ldots$ *Let us assume also that* $\theta_1 > 0$, $\frac{m_0}{4} \geq (\rho)^{\theta_1}$. *Then:*

$$[0, T_0] = B_L,$$

with B_L *defined as in (8.1) and* T_0 *as in Lemma 8.8.*

Proof. It is just a Corollary of Lemma 8.8. ■

9 End of the Proof of Theorem 2.3.

Proof of Theorem 2.3. Let $T_0(M,E)$ be as in Lemma 8.8. Suppose that the maximal time of existence T of the mild solution of (1.10), (1.11) whose existence has been shown in Theorem 3.4 satisfies $T_{\max} > T_0(M,E)$.

By Lemma 3.13, this solution is also a weak solution on $(0, T_{max})$ in the sense of Definition 2.7. Moreover, since $f \in L^\infty_{loc}([0,T_{\max}); L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma))$ and $T_{max} > T_0$, the function $g(t) = 4\pi\sqrt{2\epsilon}f(t)$ defined in (1.12) satisfies condition (6.1) for all $t \in [0, T_0]$. Suppose that we choose $\theta_1 > 0,\ \theta_2 > 0$ compatible with Lemma 8.1 and Proposition 8.5. We then choose ρ as in Proposition 8.5 and K_3, β as in Lemma 8.7. Suppose that we choose K^* in order to have:

$$K^*(\rho)^{\theta_*} \geq 4(\rho)^{\theta_1}.$$

We can then apply Lemma 8.10 to obtain $|B_L| = T_0$. Using Lemma 8.7 we obtain $|B_L| \leq \frac{K_3}{1-2^{-\beta}}(R_L)^\beta \leq \frac{K_3}{\left(1-2^{-\beta}\right)}\rho^\beta$. Therefore, if ρ is chosen smaller than $\left(\left(1-2^{-\beta}\right)\frac{T_0}{K_3}\right)^{\frac{1}{\beta}}$ we would obtain a contradiction, whence the result follows. ■

10 Finite time condensation.

It is usual in the physical literature to relate Bose-Einstein condensation phenomena, at the level of the kinetic equation (1.10), (1.11) with the onset of a macroscopic fraction of particles at the energy level $\epsilon = 0$. More precisely, the papers [15], [16], [27], [28] suggest that for some particle distributions with initially bounded f, finite time blow-up takes place, but where the resulting distribution $g(t^*, \cdot)$ at the blow-up time t^* does not contain any positive fraction of particles at the point $\epsilon = 0$. Numerical simulations suggest that near the blow-up time and for small values of ϵ, the solutions of (1.10), (1.11) behave in a self-similar manner, and eventually develops an integrable power law singularity $g(t^*, \epsilon) \sim \frac{K}{\epsilon^{\nu - \frac{1}{2}}}$, where the exponent ν, numerically computed, takes the value $\nu = 1.234...$.

The results of this paper prove that initial particle distributions with bounded f are able to develop singularities in finite time, as suggested in [15], [16], [27], [28]. However, our construction does not give much detail about the shape of the particle distribution g near the blow-up time t^*. Nevertheless, the techniques that we use allow to prove that suitable weak solutions of (1.10), (1.11), which f initially bounded, have, under suitable conditions a positive fraction of particles at $\epsilon = 0$, after a finite, positive time. More precisely, we can prove that there exists a finite $t^{**} > 0$ such that $\int_{\{0\}} g(\epsilon, t^{**})\, d\epsilon > 0$, even if $f_0 \in L^\infty(\mathbb{R}^+; (1+\epsilon)^\gamma)$, with $\gamma > 3$. We will term this phenomenon as finite time condensation. It is worth noticing that the onset of Dirac measures for $g(t, \cdot)$ at some $\epsilon = \epsilon_0 > 0$, will not be consider as a condensation. The reason, motivated by the physics of the problem, is that there are not stationary solutions of (1.10), (1.11) containing a positive fraction of mass outside the origin. The only isotropic stationary solutions of the family (1.7) containing a positive amount of mass at some value $\epsilon \geq 0$, are the supercritical Bose-Einstein distributions which contain a positive fraction of mass at $\epsilon = 0$.

Our results on finite time condensation for solutions of (1.10), (1.11) are Theorem 2.9 and Theorem 2.10. We start proving the first.

Proof of Theorem 2.9. Suppose that

$$\sup_{0<t\leq T_0} \int_{\{0\}} g(t, \epsilon)\, d\epsilon = 0 \tag{10.1}$$

where T_0 is defined as in Lemma 8.8. We can apply then Lemma 8.1, Proposition 8.5, Lemma 8.7, and Lemma 8.10 as in the proof of Theorem 2.3, to show that:

$$T_0 \leq \frac{K_3}{(1 - 2^{-\beta})} \rho^\beta.$$

Therefore, this gives a contradiction if ρ is chose sufficiently small. This concludes the proof of Theorem 2.9. ■

Remark 10.1 *It is worth noticing the specific point where the condition concerning* $\sup_{0<t\leq T_0}\int_{\{0\}} g(t,\epsilon)\,d\epsilon$ *appears in the Proof of Theorems 2.3 and 2.9. This condition plays a crucial role in the proof of Lemma 6.1. Indeed, an essential ingredient in this Lemma is the assumption* $\int_{\{0\}} g(t,\epsilon)\,d\epsilon = 0$.

Proof of Theorem 2.10. We construct a weak solution of (1.13), (1.14) in the sense of Definition 2.6 as follows. We first use Theorem 3.4 to obtain a bounded mild solution f of (1.10), (1.11) in the sense of Definition 2.1 in a time interval $0 \leq t \leq T_*$. We then use the approach in [19] to obtain a weak solution $\tilde{g}$ of (1.13), (1.14) defined for $T_* \leq t < \infty$ with initial datum $g(T_*,) = 4\pi\sqrt{2\epsilon} f(T_*,)$, with f as obtained in the previous step. We then construct a global weak solution g defined in $0 \leq t < \infty$ by means of $g(t,\cdot) = 4\pi\sqrt{2\epsilon} f(t,)$ for $0 \leq t \leq T_*$ and $g(t,\cdot) = \tilde{g}(t,\cdot)$ for $t \geq T_*$. Due to Lemma 3.13, g is a weak solution in $0 \leq t \leq T_*$. Using the continuity of $g(t,\cdot)$, in the weak topology for $t = T_*$ it follows that the constructed measure $g \in C([0,T);\mathcal{M}_+(\mathbb{R}^+;1+\epsilon))$ is a weak solution of (1.13), (1.14) defined in $0 \leq t < \infty$ in the sense of Definition 2.6. It is readily seen by construction as well as Theorem 2.9 that g satisfies (2.27) whence the result follows. ■

Remark 10.2 *The existence of initial data* f_0 *satisfying (2.25) and (2.26) has been shown in Proposition 2.4, just after the statement of Theorem 2.3.*

Remark 10.3 *Suppose that the hypothesis of Theorem 2.3 and Theorem 2.10 are satisfied. By Theorem 2.3, there exists* $T_{max} \in (0, T_0(M,E))$ *such that*

$$\lim_{t\to T_{max}^-} ||f(t)||_{L^\infty([0,\infty))} = \infty$$

On the other hand we can define

$$T_{cond} = \inf\{t > 0, \int_{\{0\}} g(t,\epsilon)d\epsilon > 0\}.$$

Since f *is bounded for* $t < T_{max}$ *it immediately follows that* $T_{max} \leq T_{cond}$. *However the possibility that* $T_{max} < T_{cond}$ *can not be immediately ruled out. It has been conjectured in the physical literature that* $T_{max} = T_{cond}$ *(cf. [15], [16], [27], [28]). However, in any case that does not follows from the above arguments.*

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