Multiple Parallel Concatenated Codes with Optimal Puncturing and Energy Distribution

Fredrik Brännström
Department of Signals and Systems
Chalmers University of Technology
SE-412 96 Göteborg, Sweden
Email: fredrikb@s2.chalmers.se

Lars K. Rasmussen
Institute for Telecommunications Research
University of South Australia
Mawson Lakes, SA 5095, Australia
Email: lars.rasmussen@unisa.edu.au

Abstract—in this paper we show how to find optimal energy distribution together with optimal puncturing ratios for parallel concatenated codes with two or more constituent codes. The energy distribution and the puncturing ratios are optimal in terms of minimizing the average signal-to-noise ratio convergence threshold. The extrinsic information transfer functions of the constituent codes are used for the optimization. Using this technique we obtain additional degrees of freedom for constructing codes with low convergence thresholds over a large range of code rates.

I. INTRODUCTION

Since the invention of parallel concatenated turbo codes [1], the turbo principle has been extended to symmetric multiple parallel concatenated codes (MPCCs)\(^1\) [2, 3] where all constituent codes are identical, and asymmetric MPCCs [4] where the constituents can be different. The original turbo code [1] transmits all uncoded (systematic) bits and punctures half of the coded bits from the two constituents to raise the code rate from 1/3 to 1/2. MPCCs can have systematic doping [5], where some of the parity bits are replaced by systematic bits preserving the code rate. The ratio between the number of systematic bits and the number of coded bits is usually referred to as the doping ratio, \(d\), which means that all constituents have the same fraction of punctured bits. Usually the doping ratio is fixed to a small number [3–5], chosen to achieve some satisfactory level of performance. In [6], the performance for different doping ratios, \(0 \leq d \leq 1\), is evaluated using approximated error-floors for maximum likelihood decoding.

Parallel concatenated codes (PCCs) with two constituent codes can be analyzed using two-dimensional extrinsic information transfer (EXIT) charts [7]. The signal-to-noise ratio (SNR) convergence threshold is estimated by tracking the evolution of mutual information (MI) in the EXIT chart. The EXIT chart for an MPCC with \(N\) constituents is \(N\)-dimensional, and thus, conventional EXIT chart analysis is complicated. Such \(N\)-dimensional charts have been used for analysis of \(N = 3\) symmetric and asymmetric MPCCs [8]. Convergence analysis for codes with \(N \geq 3\) constituents is simplified by “projecting” the multi-dimensional EXIT charts onto a single two-dimensional chart as shown in [9–11].

Each of the \(N\) constituent codes in an MPCC can have their own puncturing ratio independent of each other and independent of the puncturing ratio of the systematic bits [11, 12]. In all of the examples above, BPSK symbols with equal symbol energy is used (uniform energy distribution). However, the constituent codes can have other energy distributions. In this paper the EXIT functions for the constituent codes are used to jointly optimize the energy distribution and the \(N + 1\) puncturing ratios. The objective is to minimize the convergence threshold. A specific choice of \(N\) constituents will then have an achievable SNR-rate region within which it is possible to construct MPCCs with low decoding complexity for any desired code rate.

II. SYSTEM MODEL

Consider a system with \(N\) parallel concatenated constituent codes \(C_n\), transmitting binary data over an additive white Gaussian noise (AWGN) channel. Fig. 1 shows an example with three components concatenated in parallel. The source bits are divided into blocks of \(L\) bits, \(x \in \{-1, +1\}^L\) and \(N + 1\) interleavers\(^2\) permute the source sequence into \(N + 1\) different sequences \(x_n = \pi_n(x)\), \(n = 0, 1, \ldots, N\). Encoder \(C_n\) maps a sequence of \(L\) input bits \(x_n \in \{-1, +1\}^L\) to a sequence of \(L/R_n\) output bits \(y_n \in \{-1, +1\}^{L/R_n}\), where \(R_n\) is the rate of code \(n = 1, 2, \ldots, N\). The uncoded (systematic) bit sequence is denoted \(x_0\) and \(R_0 = 1\), since it corresponds to the “code rate” of the systematic bits. Individual elements of these sequences are denoted \(x_{n,i}\), \(i = 1, 2, \ldots, L\), and \(x_n = [x_{n,1}, x_{n,2}, \ldots, x_{n,N}]\). This notation is naturally extended to all other sequences.

With reference to Fig. 1, \(U_n\) denotes a random puncturer\(^3\) for the sequence \(y_n\) with a puncturing ratio \(\delta_n \in [0, 1]\), for \(n = 0, 1, \ldots, N\). \(1 - \delta_n\) denotes the fraction of bits in \(y_n\) that are punctured [11]. If \(\delta_n = 0.8\), it means that 20% of the bits in \(y_n\) are removed, i.e., \(z_n\) contains only 80% of the bits from \(y_n\), namely \(z_n \in \{-1, +1\}^{L\delta_n}\), where \(L_n \triangleq L\delta_n/R_n\).

In Fig. 1, \(M\) is a memoryless modulator/multiplexer, mapping bits from \(z_n\) to BPSK symbols \(s_n \in \{-1, +1\}\) according to some puncturing pattern. This pattern is determined by the relative interleaver between the encoders that is important [11].

\(^1\)An MPCC contains an arbitrary number \((N \geq 2)\) of constituent codes concatenated in parallel.

\(^2\)Note that \(\pi_0\) and \(\pi_1\) can be removed since it is the relative interleaver between the encoders that is important [11].

\(^3\)A random puncturer has a randomly chosen puncturing pattern, known to the receiver.

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The modulated symbols are transmitted in serial over the 
AWGN channel, \( s = [s_0, s_1, \ldots, s_N] \). The overall rate code of the punctured parallel system is [11]

\[
R = \frac{L}{\sum_{n=0}^{N} L_n} \left( \sum_{n=0}^{N} \frac{\delta_n}{R_n} \right)^{-1}. \tag{1}
\]

The receiver’s matched filter output is \( r = s + w \), where the elements in \( w \) is zero-mean Gaussian noise with variance \( \sigma_w^2 = N_0/2 \). The average bit energy of the source bits is

\[
E_b = \frac{1}{L} \sum_{n=0}^{N} L_n E_{s,n} = \sum_{n=0}^{N} R_n \delta_n E_{s,n}, \tag{2}
\]

and the average SNR is defined as

\[
\gamma_b \triangleq \frac{E_b}{N_0} = \sum_{n=0}^{N} \frac{\delta_n}{R_n} \gamma_{s,n}, \tag{3}
\]

where \( \gamma_{s,n} = E_{s,n}/N_0 \). Define

\[
\psi_n \triangleq \frac{E_{s,n}}{\sum_{k=0}^{N} E_{s,k}} = \frac{\gamma_{s,n}}{\sum_{k=0}^{N} \gamma_{s,k} \Gamma}, \tag{4}
\]

as the fraction of transmitted average symbol energy for the output bits from \( U_n \), where \( \Gamma \triangleq \sum_{k=0}^{N} \gamma_{s,k} \). Note that \( \psi_n < 1 \) for all \( n = 0, 1, \ldots, N \) and \( \sum_{n=0}^{N} \psi_n = 1 \). From now on this will be referred to as the energy distribution. Using (3) and (4), the average SNR can be rewritten as

\[
\gamma_h = \Gamma \sum_{j=0}^{N} \frac{\psi_j}{F_j} \gamma_{f,j}. \tag{5}
\]

Combining (4) and (5) and solving for \( \gamma_{s,n} \) gives

\[
\gamma_{s,n} = \frac{\psi_n}{\sum_{j=0}^{N} \frac{\delta_j}{F_j} \psi_j} \gamma_b. \tag{6}
\]

For later use, collect the code rates, the puncturing ratios, and the energy distributions in three vectors with \( N + 1 \) elements each, \( R \triangleq [R_0, R_1, \ldots, R_N] \), \( \Delta \triangleq [\delta_0, \delta_1, \ldots, \delta_N] \), and \( \Psi \triangleq [\psi_0, \psi_1, \ldots, \psi_N] \).

Let \( E(x_n) \) denote a sequence of extrinsic values corresponding to \( x_n \). Likewise, \( A(x_n) \) is a sequence of priors for \( x_n \). Priors and extrinsics will be represented as sequences of log-likelihoods, \( A(x_n) \in \mathbb{R}_+^L \) and \( E(x_n) \in \mathbb{R}_+^L \) [11]. This notation is naturally extended to other sequences, \( x, y_n \) and \( z_0, n = 0, 1, \ldots, N \).

The demapper/demultiplexer, \( M^{-1} \) produces \( A(z_n), n = 0, 1, 2, \ldots, N \), which are constant during the decoding process

\[
A(z_{n,i}) = \ln \left( \frac{p_r(r_i|z_{n,i} = +1)}{p_r(r_i|z_{n,i} = -1)} \right) = \frac{\sigma_w^2}{2} z_{n,i} + v_n, \tag{7}
\]

where \( \sigma_w^2 = 8 \gamma_{s,n} \) is the variance of the zero-mean Gaussian \( v_n \) [7] and \( p_r(r_i|z_{n,i}) \) is the conditional probability density function (PDF) of the matched filter output \( r_i = \sqrt{E_{s,n}} z_{n,i} + w_i \). The depuncturer \( U_n^{-1} \) inserts zeros at the positions in \( A(z_{n,i}) \) where the punctured bits are located, to create \( A(y_n) \), \( n = 0, 1, \ldots, N \). The decoder consists of \( N \) a posteriori probability (APP) decoders \( C_n^{-1} \) [13], interconnected by interleavers \( \pi_n \) and deinterleavers \( \pi_n^{-1} \), \( n = 1, 2, \ldots, N \). Upon activation4, decoder \( C_n^{-1} \) uses its code constraint and the most recent priors \( A(x_n) = \pi_n(A(x_0)) \) and \( A(y_n) \) [14] to update the extrinsics on the source bits [13], \( E_n(x) = \pi_n^{-1}(E(x_n)) \), where

\[
A_n(x) = \sum_{j=0}^{N} E_j(x). \tag{8}
\]

Prior to decoding, the extrinsic values are set to zero, \( E_n(x) = \{0\}^L, n = 1, 2, \ldots, N \). The only extrinsic sequence with non-zero elements is \( E_0(x) = \pi_0^{-1}(A(y_0)) \).

Let \( D(x) \in \mathbb{R}_+^L \) denote the decision statistics for the source bits \( x \). The decision statistics are updated after each activation

\[4\]The term activation is used instead of iteration. In a system with two constituents, one iteration is the same as two activations, one for each of the two decoders.

Fig. 1. Three parallel concatenated codes with puncturing.
according to

\[ D(x) = \sum_{j=0}^{N} E_j(x). \]  

(9)

The hard decision \( \hat{x}_i \) on source bit \( x_i \) is \( D(x_i) \), \( \hat{x}_i = 0 \) and the system performance is measured in bit-error rate (BER), i.e., the probability \( P_b = \Pr(\hat{x}_i \neq x_i) \).

III. MUTUAL INFORMATION AND EXIT FUNCTIONS

Let \( I_{A}(x_n) = \frac{1}{T} \sum_{i=1}^{L} I(x_n; A(x_{n,i})) \) and \( I_{E}(x_n) = \frac{1}{T} \sum_{i=1}^{L} I(x_n; E(x_{n,i})) \) denote the average MI between the input bits and the prior values, and the extrinsic values, respectively, (similarly for \( y_n \) and \( x_n \)). We shall refer to these as prior and extrinsic MIs [11]. This section shows how the prior MI for punctured sequences are simple functions of the prior MI of the un-punctured sequences. It will also be shown here that the extrinsic MI can be expressed by the EXIT functions of the un-punctured constituent codes. Since we are only dealing with bits, all MIs are between zero and one, i.e., \( 0 \leq I_{A}(x_n) \leq 1 \) and \( 0 \leq I_{E}(x_n) \leq 1 \), for all \( n = 0, 1, \ldots, N \).

A code is characterized by its EXIT function, \( I_{A}(x_n) \), \( \Theta_{A} = [0, 1]^2 \rightarrow [0, 1] \), \( I_{E}(x_n) = T_{x_n}(I_{A}(x_n), I_{E}(x_n)) \), where the extrinsic MI \( I_{E}(x_n) \) for decoder \( n = 1, 2, \ldots, N \) is a function of the prior MIs \( I_{A}(x_n) \) and \( I_{A}(y_n) \). In practice, this function is obtained by Monte-Carlo simulations\(^3\) of the constituent code for all values of \( 0 \leq I_{A}(x_n) \leq 1 \) and \( 0 \leq I_{A}(y_n) \leq 1 \) by modelling the priors as Gaussian [7], similar to (7). It can be shown that

\[ I_{A}(z_n) = J(\sqrt{8\gamma z_n}), \quad \text{where} \]  

\[ J(\sigma) = 1 - \frac{1}{2^{2\pi \sigma}} \int_{-\infty}^{+\infty} e^{-\frac{(e^{-\xi} - 2\sigma)^2/4\sigma^2}{\xi}} \log_2(1 + e^{-\xi}) \, d\xi, \]  

(11)

according to [7]. \( J(\sigma) \) is monotonically increasing and therefore has a unique inverse, \( \sigma = J^{-1}(J) \). Unfortunately, \( J \) and \( J^{-1} \) can not be expressed in closed form, but they can be closely approximated by [11]

\[ J(\sigma) \approx \left( 1 - 2 - H_1 e^{2\sigma H_2} \right)^{H_3}, \]  

(12)

\[ J^{-1}(J) \approx \left( \frac{1}{H_1} \log_2 \left( 1 - I \left( \sqrt{\frac{J}{H_1}} \right) \right) \right)^{\frac{1}{H_2}}. \]  

(13)

Numerical optimization, using the Nelder-Mead simplex method [17], to minimize the total squared difference between (11) and (12) gives the parameters \( H_1 = 0.3073 \), \( H_2 = 0.8935 \), and \( H_3 = 1.1064 \). Note that (10) is also the constellation-constrained capacity, \( \delta \leq C_{BPSK} \leq 1 \), of a system using BPSK modulation with symbol energy \( E_{s,n} \) over an AWGN channel, \( C_{BPSK} = J(\sqrt{8\gamma z_n}) \).

Since the system in Fig. 1 uses \( N + 1 \) different symbol energies \( (L_n \text{ symbols with } E_{s,n} \text{ for } n = 0, 1, \ldots, N) \), the maximum rate of the whole system can be expressed as

\[ C(R, \Delta, \Psi, \gamma_b) = \frac{1}{N} \sum_{k=0}^{N} \frac{1}{L_k} \sum_{n=0}^{N} L_n J(\sqrt{8\gamma_{s,n}}) \]  

(14)

\[ = \frac{1}{N} \sum_{k=0}^{N} \sum_{n=0}^{N} \frac{\delta_n}{R_n} \sum_{j=0}^{N} \sqrt{\frac{\Psi_n}{\sqrt{N}} \gamma_{j,n}} \gamma_{j,n}. \]  

A special case of (14) is when the energy distribution is uniform, \( E_{s,n} = \text{RE}_{\text{b}}, \) i.e., \( \psi_n = \frac{1}{N} \) for all \( n = 0, 1, \ldots, N \). This uniform energy distribution will be denoted by \( \Psi_0 \).

\[ \Psi_0 = \left[ \frac{1}{N+1}, \frac{1}{N+1}, \ldots, \frac{1}{N+1} \right]. \]  

(15)

The system treated in [11, 12] uses \( \Psi_0 \), where the maximum rate is equal to \( C_{BPSK} \)

\[ C(R, \Delta, \Psi_0, \gamma_b) = J(\sqrt{\frac{1}{N} \sum_{j=0}^{N} \frac{\delta_j}{R_j}} \gamma_{b}) = J(\sqrt{\frac{\gamma_b}{8R}}). \]  

(16)

Let \( \gamma_{b} \) denote the minimum required SNR for a given \( R, \Delta \) and \( \Psi \); \( \gamma_{b} \) can be found by letting \( C(R, \Delta, \Psi, \gamma_{b}) = R \) and solving for \( \gamma_{b} \). If the energy distribution is uniform, \( \gamma_{b} \) can be expressed in closed form using (16)

\[ \gamma_{b} = \frac{J^{-1}(R)^2}{8R}. \]  

(17)

The puncturer, \( U_n \), only removes some bits from certain randomly chosen positions of the sequence \( y_n \), and \( U_n^{-1} \) adds zeros at these positions, therefore the relationship between \( I_{A}(y_n) \) and \( I_{A}(z_n) \) is linear [11]

\[ I_{A}(y_n) = \delta_n I_{A}(z_n) = \delta_n J(\sqrt{8\gamma_{y,n}}), \]  

(18)

for all \( n = 0, 1, \ldots, N \). The linear relationship in (18) assumes random puncturing and infinitely big interleavers, similar to the assumptions for the EXIT chart analysis [7]. The average MI is not affected by an interleaver or a deinterleaver. Therefore, \( I_{E}(x_n) = I_{E_{x,n}} \) and \( I_{A}(z_n) = I_{A_{y,n}} \). Since the prior values are sums of \( N \) extrinsic values \( (8) \), they are modelled as sums of \( N \) biased Gaussian random variables [8]. Using (11) and its inverse,

\[ I_{A}(x_n) = J \left( \sum_{j=0}^{N} \frac{J^{-1}(I_{E}(x_j))}{2} \right), \]  

(19)

since they need to be added in the variance domain [8, 11]. The approximations (12) and (13) are used in all calculations. Let \( I_{D}(x) = \frac{1}{T} \sum_{i=1}^{L} I(x_i; D(x_i)) \), then similarly

\[ I_{D}(x) = J \left( \sum_{j=0}^{N} \frac{J^{-1}(I_{E}(x_j))}{2} \right). \]  

(20)

\( I_{D}(x) = 1.0 \) means full information about the source bits, i.e., \( P_b \) is close to zero. If the decision statistics are close

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\(^3\)More recently it has been shown that for certain codes and simple channel models, it is possible to compute the EXIT functions [15, 16].
to a Gaussian model as in (7), the BER can be approximated by $P_b = Q(J^{-1}(I_{D(n)})/2)$ [7], where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-\beta^2/2)\,d\beta$ is the Gaussian Q-function.

The $N$ EXIT functions for a parallel concatenated system, as in Fig. 1, can therefore be expressed as (21) for all $n = 1, 2, \ldots, N$, using (6), (18), (19) and the fact that $I_{E(x_n)} = \delta_n J[\sqrt{B \gamma_0}]$ [11]. Note that if $\Delta, \Psi, \gamma_b$, together with the EXIT functions of the codes, $T_{x_n}$, are known, $I_{E(x_n)}$ in (21) depends only on all the other $I_{E(x_j)}, j \neq n$.

When the energy distribution is uniform, $\psi_n = \frac{1}{\sqrt{\frac{1}{N}+1}}$, $n = 0, 1, \ldots, N$, a fixed $\gamma_b$ gives $N+1$ degrees of freedom in $\Delta$ to change the mutually dependence of the $N$ EXIT functions in (21) over a range of code rates $R$ [11, 12]. From (21) it is also clear that the energy distribution $\Psi$ gives an additional $N$ degrees of freedom in $\Delta$ to change the mutually dependence of the EXIT functions. The next section shows how to utilize these additional degrees of freedom to improve the performance of the system.

### IV. Optimal Puncturing and Energy Distribution

If the $N$ functions in (21) are applied for a given $\Delta, \Psi, \gamma_b$, all extrinsic MIIs will converge to fixed values, independent of activation schedule, as long as an unlimited number of decoder activations is allowed [9–11, 14]. This implies that the MI on the decision statistics will also converge to a fixed value according to (20). This fixed value is defined as the convergence point, $I_{D}^*$. For a specific set of constituent codes (implying that $R$ is known) the convergence point is a function of $\Delta, \Psi, \gamma_b$, which can be stated as $I_{D}^* = f(\Delta, \Psi, \gamma_b)$.

For a given code rate $R$, the minimum SNR, $\gamma_{\text{b, min}}$, together with the optimal puncturing, $\Delta^*$, and the optimal energy distribution, $\Psi^*$, can be found by

$$[\Delta^*, \Psi^*, \gamma_{\text{b, min}}] = \arg \min_{[\Delta, \Psi, \gamma_b]} g(\Delta, \Psi, \gamma_b)$$

subject to

$$\Delta \in [0, 1]^{N+1}, \quad R = \left( \frac{N}{\sum_{n=0}^{N} \frac{\delta_n}{R_n}} \right)^{-1},$$

$$\Psi \in (0, 1)^{N+1}, \quad \sum_{n=0}^{N} \psi_n = 1,$$

$$\gamma_b \geq \bar{\gamma}_b, \quad \text{or equivalent } R \leq C(R, \Delta, \Psi, \gamma_b).$$

The objective function in (22) is defined as

$$g(\Delta, \Psi, \gamma_b) = \begin{cases} \gamma_b & f(\Delta, \Psi, \gamma_b) \geq J(2Q^{-1}(P_b)) \, \beta \gamma_b \, J[\sqrt{B \gamma_0}] \\ \infty & \text{otherwise.} \end{cases}$$

The threshold for $f(\Delta, \Psi, \gamma_b)$ in (26) corresponds to a BER $P_b$ if the decision statistics were Gaussian [7, 11].

The optimization can be solved for a specific $R$ by initializing $\gamma_b = Q^{-1}(P_b)^2 / 2$ (uncoded BPSK). Then $f(\Delta, \Psi, \gamma_b)$ is evaluated for all values of $\Delta$ and $\Psi$ that satisfy the constraints in (23)–(24) for the chosen code rate $R$. Whenever $f(\Delta, \Psi, \gamma_b)$ is above the threshold, $\gamma_b$ can be decreased with an arbitrary small step size until $f(\Delta, \Psi, \gamma_b)$ is below the threshold. This search is continued until there is no $\Delta$ or $\Psi$ that satisfies the constraints at the same time as $f(\Delta, \Psi, \gamma_b)$ is above the threshold. See [11] for a more detailed description of the optimization procedure when $\psi_n = \frac{1}{\sqrt{\frac{1}{N}+1}}$.

For a specific set of constituent codes, $\Delta^*, \Psi^*$, and the corresponding $\gamma^*_{\text{b}}$ is found by applying (22). Here, only convolutional codes (CCs) with $R_n = 1$ for $n = 0, 1, \ldots, N$, are considered as constituent codes, i.e., $R = 1$. By changing $\Delta$ and $\Psi$ for a fixed $\gamma_b$, each combination of $N$ constituents can have a code rate between $\frac{1}{N+1} \leq R \leq C(1, \Delta, \Psi, \gamma_b)$, according to (1) and (14). The optimization procedure can easily be applied to other component codes and code rates.

A code search using CCs with memory $\nu \leq 2$ and uniform energy distribution is performed in [11, 12]. The results show that the parallel concatenation of CC(5/7), CC(7/6), and CC(7/4) is a PCC with a low $\gamma_{\text{b, min}}$ for all $R \geq 1/3$. This PCC is denoted by PCC(1 + 5/7 + 7/6 + 7/4), where ‘1’ corresponds to the systematic bits. The entire SNR-rate region for PCC(1 + 5/7 + 7/6 + 7/4) is presented in [12]. It shows that PCC(1 + 5/7 + 7/6 + 7/4) has an increasing $\gamma_{\text{b, min}}$ when $R < 1/3$ is decreasing. When $R = 1/4$, there is only one choice of puncturing ratios, $\Delta = \Delta^* = 1$, and $\gamma_{\text{b, min}} = 0.16$ dB. However, this result is with uniform energy distribution $\Psi_0 = [0.25, 0.25, 0.25, 0.25]$. Optimizing the energy distribution using (22) gives $\Psi^* = [0.08, 0.34, 0.24, 0.34]$ and $\gamma_{\text{b, min}}^* = -0.36$ dB. This gain of 0.52 dB compared to uniform energy distribution. The optimal puncturing and energy distribution for $R = 0.25$ and $R = 0.3$ are listed in Table I. The same table also lists $\gamma_{\text{b, min}}^*_{\Psi_0}$ for $\Psi_0$ and $\gamma_{\text{b, min}}^*_{\Psi_0}$ for $\Psi_0$. The predicted gain for $0.25 \leq R \leq 0.30$ is in the range of 0.08–0.52 dB using $\Psi^*$ instead of $\Psi_0$. At the same time, the minimum required SNR, $\gamma_{\text{b, min}}$, increases with 0.03–0.14 dB when $\Psi^*$ is used instead of $\Psi_0$.

The EXIT chart projection [9–11] for PCC(1 + 5/7 + 7/6 + 7/4) at $R = 1/4$ is shown in Fig. 2. Changing the energy distribution affects the EXIT chart. Fig. 2 shows that using $\Psi^*$ the tunnel is open at $\gamma_b = -0.36$ dB, while it is closed using $\Psi_0$. Fig. 3 shows the simulated performance of PCC(1 + 5/7 + 7/6 + 7/4) in BER using both uniform and optimal energy distribution at $R = 1/4$. The gain in performance (around 0.5 dB), corresponds well with the predicted gain listed in Table I.
We considered the problem of finding the optimal energy distribution together with the optimal puncturing ratios for multiple parallel concatenated codes, minimizing the average signal-to-noise ratio convergence threshold. For such codes, the individual puncturing ratios for the constituent codes can be chosen freely within a desired code rate. The energy distribution can also be chosen arbitrary without affecting the code rate, while preserving the average bit energy. We have shown how to jointly optimize the energy distribution and the puncturing ratios by utilizing the extrinsic information transfer functions of the constituents. The result is an SNR-rate region for any combination of an arbitrary number of constituent codes. The example here shows that a gain in the performance can be obtained using optimal distribution compared to uniform distribution of the symbol energy, even though the maximum rate given by information theory decreases.

REFERENCES


