Transversal Hypergraphs to Perfect Matchings in Bipartite Graphs: Characterization and Generation Algorithms

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Abstract
A minimal blocker in a bipartite graph $G$ is a minimal set of edges the removal of which leaves no perfect matching in $G$. We give an explicit characterization of the minimal blockers of a bipartite graph $G$. This result allows us to obtain a polynomial delay algorithm for finding all minimal blockers of a given bipartite graph. Equivalently, this gives a polynomial delay algorithm for listing the anti-vertices of the perfect matching polytope of $G$. We also give generation algorithms for other related problems, including $d$-factors in bipartite graphs, and perfect 2-matchings in general graphs.

1 Introduction
Let $G = (A, B, E)$ be a bipartite graph with bipartition $A \cup B$ and edge set $E$. For subsets $A' \subseteq A$ and $B' \subseteq B$, denote by $G[A', B']$ the subgraph of $G$ induced by $A' \cup B'$. A perfect matching in $G$ is a subgraph in which every vertex of $G$ has degree exactly one, or equivalently a subset of $|A| = |B|$ pairwise disjoint edges of $G$. Throughout the paper, we shall assume that a graph with no vertices has a perfect matching. A blocker in $G$ is a subset of edges $X \subseteq E$ such that the graph $G = (A, B, E \setminus X)$ does not have a perfect matching. A blocker $X$ is minimal if, for every edge $e \in X$, the set $X \setminus \{e\}$ is not a blocker. Denote respectively by $\mathcal{M}(G)$ and $\mathcal{B}(G)$ the families of all perfect matchings and all minimal blockers in $G$. Note that in any bipartite graph $G$, the set $\mathcal{B}(G)$ is the family of minimal transversals to the family $\mathcal{M}(G)$, i.e. $\mathcal{B}(G)$ is the family of minimal edge sets containing an edge from every perfect matching in $G$. Clearly, $\mathcal{B}(G) = \{\emptyset\}$ if $|A| \neq |B|$.

The main problem we consider in this paper is the following.

$\text{GEN}(\mathcal{B}(G))$: Given a bipartite graph $G$, generate all minimal blockers of $G$.

The analogous problem $\text{GEN}(\mathcal{M}(G))$ of enumerating perfect matchings for bipartite graphs was considered in [9, 10, 25, 26]. These problems are special cases of the more
general open problems of generating vertices and anti-vertices of a polytope given by its linear description. Indeed, let $H$ be the vertex-edge incidence matrix of $G = (A, B, E)$, and consider the polytope $P(G) = \{ x \in \mathbb{R}^E \mid Hx = e, \ x \geq 0 \}$, where $e \in \mathbb{R}^{|A|+|B|}$ is the vector of all ones. Then the perfect matchings of $G$ are in one-to-one correspondence with the vertices of $P(G)$, which in turn, are in one-to-one correspondence with the minimal collections of columns of $H$ containing the vector $e$ in their conic hull. On the other hand, the minimal blockers of $G$ are in one-to-one correspondence with the anti-vertices of $P(G)$, i.e. the maximal collections of columns of $H$, not containing $e$ in their conic hull. Both corresponding generating problems are open for general polytopes, see [2] for details.

In the special case, when $H$ is the incidence matrix of a bipartite graph $G$, the problem of generating the vertices of $P(G)$ can be solved with polynomial delay [9, 10, 25, 26], i.e. given a graph $G = (A, B, E)$ and a collection $\mathcal{X}$ of vertices of $P(G)$, a new vertex can be found in time polynomial in $|A|$, $|B|$ and $|E|$ (but independent of $|\mathcal{X}|$). In this note, we obtain an analogous result for anti-vertices of $P(G)$.

**Theorem 1** Problem GEN($\mathcal{B}(G)$) can be solved with polynomial delay.

For non-bipartite graphs $G$, it is well-know that the vertices of $P(G)$ are half-integral [20] (i.e. the components of each vertex are in $\{0, 1, 1/2\}$), and that they correspond to the basic 2-matchings of $G$, i.e. subsets of edges that cover the vertices with vertex-disjoint edges and vertex-disjoint odd cycles. Polynomial delay algorithms exist for listing the vertices of 0/1-polytopes [5], simple and simplicial polytopes (e.g. [1]), but the status of the problem for general polytopes is still open. We show here that for the special case of the polytope $P(G)$, the problem can be solved in incremental polynomial time, i.e. given a collection $\mathcal{X}$ of vertices of $P(G)$, a new vertex can be found in time polynomial in $|A|$, $|B|$, $|E|$ and $|\mathcal{X}|$.

**Theorem 2** All basic perfect 2-matchings of a graph $G$ can be generated in incremental polynomial time.

The proof of Theorem 1 is based on a nice characterization of minimal blockers, which may be of independent interest. This characterization will be given in Section 2. The method used for enumeration is the supergraph method which will be outlined in Section 3.1. A special case of this method, when applicable, can provide enumeration algorithms with stronger properties, such as enumeration in lexicographical ordering, or enumeration with polynomial space. This special case will be described in Section 3.2 and will be used in Section 5 to solve two other related problems of enumerating generalized $d$-factors in bipartite graphs, and enumerating perfect 2-matchings in general graphs. The latter result will be used to prove Theorem 2.

It should be mentioned that there exists a polynomial-time algorithm for near-uniformly sampling from within the perfect matchings of a bipartite graph [16]. The analogous problem for minimal blockers of bipartite graphs remains open.
2 Characterization of Minimal Blockers and Some of their Properties

2.1 The neighborhood function

Let $G = (A, B, E)$ be a bipartite graph. For a subset $X \subseteq A \cup B$, denote by

$$\Gamma(X) = \Gamma_G(X) = \{ v \in (A \cup B) \setminus X \mid \{v, x\} \in E \text{ for some } x \in X \}$$

the set of neighbors of $X$. It is known (see, e.g., [19]), and also easy to verify, that the set-function $|\Gamma(\cdot)|$ is submodular, i.e.

$$|\Gamma(X \cup Y)| + |\Gamma(X \cap Y)| \leq |\Gamma(X)| + |\Gamma(Y)|$$

holds for all subsets $X, Y \subseteq A \cup B$. A simple but useful observation about submodular set-functions is the following (see, e.g., [21]):

**Proposition 1** Let $V$ be a finite set and $f : 2^V \to \mathbb{R}$ be a submodular set-function. Then the family of sets $X$ minimizing $f(X)$ forms a sub-lattice $\mathcal{L}$ of the Boolean lattice $2^V$. If $f(X)$ is polynomial-time computable for every $X \subseteq V$, then the unique minimum and maximum of this sub-lattice can be found in polynomial time.

**Proof.** Let $\alpha = \min\{f(X) : X \subseteq V\}$ and $\mathcal{F} = \{X \subseteq V : f(X) = \alpha\}$. If $X, Y \in \mathcal{F}$, then

$$2\alpha \leq f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y) = 2\alpha,$$

from which the first statement of the proposition follows. To see the second statement, note that $X$ is the unique minimum (maximum) of the sub-lattice $\mathcal{L}$ if and only if $X$ is a minimum (maximum) cardinality subset with the property that $f(X) = \alpha$. Thus to find such a minimum (maximum) set, we first find any set $X \in \mathcal{F}$ (which can be computed in polynomial time, see [11, 15, 24]). Then we greedily reduce (extend) $X$ to a minimum (maximum) cardinality set as follows. Iteratively, for $v \in X$ (respectively, for $v \notin X$), we compute a minimizer $X' \leftarrow \arg\min\{f(Y) : Y \subseteq X \setminus \{v\}\}$ (respectively, $X' \leftarrow \arg\min\{f(Y) : Y \subseteq V, Y \supseteq X \cup \{v\}\}$) and if $f(X') = \alpha$ we set $X \leftarrow X'$.

2.2 Matchable graphs

Matchable graphs, introduced in [14], play an important role in the characterization of minimal blockers. In this subsection, we define them and give some of their useful properties.

**Definition 1 ([14])** A bipartite graph $G = (A, B, E)$ with $|A| = |B| + 1$ (or $|B| = |A| + 1$) is said to be matchable if the graph $G - v$ has a perfect matching for every vertex $v \in A$ (respectively, $v \in B$).

The smallest matchable graph is one in which $|A| = 1$ and $|B| = 0$ (or $|A| = 0$ and $|B| = 1$). The following is a useful equivalent characterization of matchable graphs.

**Proposition 2 ([14])** For each bipartite graph $G = (A, B, E)$ with $|A| = |B| + 1$, the following statements are equivalent:
(M1) $G$ is matchable;

(M2) $G$ has a matchable spanning tree, that is, a spanning tree $T$ in which every vertex $v \in B$ has degree 2;

It can be shown that for bipartite graphs $G = (A, B, E)$ with $|A| = |B| + 1$, properties (M1) and (M2) are further equivalent with: bipartite graphs $G = (A, B, E)$ with $|A| = |B| + 1$, properties (M1) and (M2) are further equivalent with:

(M3) $G$ contains a unique independent set of size $|A|$, namely the set $A$.

Given a bipartite graph $G$, let us say that a matchable induced subgraph of $G$ is maximal, if it is not a proper subgraph of any other matchable induced subgraph of $G$. The following extension procedure, which we shall call $\text{EXTEND}(G_1, M)$, was proposed in [14] in the proof of (2), and will be used occasionally in the paper: Given a matchable induced subgraph $G_1 = G[A_1, B_1]$ of a bipartite graph $G = (A, B, E)$, where $A_1 \subseteq A$ and $B_1 \subseteq B$, and a perfect matching $M$ of $G_2 = G[A_2 = A \setminus A_1, B_2 = B \setminus B_1]$, we can extend $G_1$ into a maximal matchable induced subgraph of $G$ as follows: Until there exist adjacent vertices $a \in A_1$ and $b \in B_2$, add $b$ to $B_1$ and $a'$ to $A_1$, where $\{a', b\}$ is an edge of $M$. When $G = (A, B, E)$ is a matchable graph, this procedure can be used to construct a matchable spanning subtree of $G$ in polynomial time [14]. Indeed, let $v$ be an arbitrary vertex of $A$, $G_1 = \{\{v\}, \emptyset\}$ and $M$ be a matching in $G - v$, then it easy to see that the above extension procedure will end up in a matchable spanning subtree of $G$.

We shall also need the following further properties of matchable graphs.

**Proposition 3** Let $G = (A, B, E)$ be a matchable graph with $|A| = |B| + 1$, and let $a \in A$ and $b \in B$ be given vertices. Then there exist partitions $A_1 \cup A_2 = A \setminus \{a\}$ and $B_1 \cup B_2 = B \setminus \{b\}$ such that (i) $G_1 = G[A_1, B_1]$ is a maximal matchable induced subgraph of $G' = G - \{a, b\}$, (ii) $G_2 = G[A_2, B_2]$ has a perfect matching, and (iii) the graph $G[A_3, B_2]$ is empty (i.e. edgeless). Furthermore, this decomposition of $G'$ is unique, and can be found in polynomial time.

**Proof.** First observe that

$$|\Gamma_{G'}(X)| \geq |X| - 1,$$

for all $X \subseteq A \setminus \{a\}$. (1)

This follows from the matchability of $G$. Indeed, if $\Gamma_{G'}(X) \leq |X| - 2$ for some $X \subseteq A \setminus \{a\}$, then $|\Gamma_{G'}(X)| \leq |X| - 1$, and consequently, the graph $G - a$ does not have a perfect matching.

Since $G'$ does not have a perfect matching, it follows by Hall’s theorem (see, e.g., [13, 20]) that there must exist a subset $X$ of $A \setminus \{a\}$ such that $|\Gamma_{G'}(X)| = |X| - 1$. Let $A_1$ be a subset of $A \setminus \{a\}$ with minimum cardinality such that $|\Gamma_{G'}(A_1)| = |A_1| - 1$.

Then the induced graph $G_1 \equiv (A_1, \Gamma_{G'}(A_1))$ is matchable. To see this, note that for any vertex $v \in A_1$ and any subset $X \subseteq A_1 \setminus \{v\}$, we must have $|\Gamma_{G_1}(X)| \geq |X|$ by the minimality of $|A_1|$. Thus it follows by Hall’s theorem that the graph $G_1 - v$ has a perfect matching.

Now suppose that the graph $G_2 = G[A_2, B_2]$, where $A_2 \equiv A \setminus (A_1 \cup \{a\})$ and $B_2 \equiv B \setminus (\Gamma_{G'}(A_1) \cup \{b\})$, is not empty. Since $G$ is matchable, the graph $G - a$ has a perfect matching $M$. Note that there are no edges in the original graph between $A_1$ and $B_2$, i.e. the graph $G[A_1, B_2]$ is empty. It follows that $M$ must match all
vertices in $B_2$ to those in $A_2$, i.e. $G_2$ has a perfect matching. Furthermore, the fact that $G[A_1, B_2]$ is empty implies also that $G_1$ cannot be a proper subgraph of any other matchable subgraph of $G'$.

Let us now show that the above decomposition of $G'$ is unique. It is enough to show that the set $A_1$ defined above is unique. But the latter claim follows from the following facts: (i) the function $f : 2^A \rightarrow \mathbb{Z}$ defined by $f(X) = |\Gamma_G(X)| - |X|$ is submodular, (ii) by (1), $\min \{ f(X) : X \subseteq A \setminus \{a\} \} = -1$, and (iii) by Proposition 1, the set $\mathcal{F} = \{ X \subseteq A \setminus \{a\} : f(X) = -1 \}$ forms a lattice, and therefore the minimality of $|A_1|$ implies that $A_1$ is the unique minimum of this lattice. Finally, Proposition 1 also implies that the set $A_1$, and hence the above decomposition of $G'$, can be constructed in polynomial time.

The next proposition acts, in a sense, as the opposite to Proposition 3: Given a matchable graph into a matchable subgraph and a subgraph with a perfect matching, we can reconstruct in a unique way the original matchable graph.

**Proposition 4** Let $G = (A, B, E)$ be a bipartite graph and let $G_1 = G[A_1, B_1]$ and $G_2 = G[A_2, B_2]$ be vertex-disjoint subgraphs of $G$ such that $G_1$ is matchable and $G_2$ has a perfect matching. Let $G' = G[A_1 \cup A_2, B_1 \cup B_2]$. Then there is a unique extension of $G_1$ into a maximal matchable induced subgraph $G_1' = G[A_1' \supseteq A_1, B_1' \supseteq B_1]$ of $G'$, such that the graph $G_2' \overset{\text{def}}{=} G[(A_1 \cup A_2) \setminus A_1', (B_1 \cup B_2) \setminus B_1']$ has a perfect matching. Such an extension yields the unique possible decomposition of $G'$ into a maximal matchable induced subgraph and a subgraph with a perfect matching, and can be constructed in polynomial time.

**Proof.** Fix a perfect matching $M$ in $G_2$, and apply the extension procedure EXTEND($G_1, M$) given after Proposition 2. When the procedure terminates, we get two graphs $G_1' = G[A_1', B_1']$ and $G_2' = G[(A_1 \cup A_2) \setminus A_1', (B_1 \cup B_2) \setminus B_1']$ such that $G_1'$ is (maximal) matchable and $G_2'$ has a perfect matching. So it remains to verify that such an extension is unique. Suppose on the contrary that there is another decomposition of $G'$ given by the graphs $G_1'' = G[A_1'', B_1'']$ and $G_2'' = G[A_2'', B_2'']$, where $G_1''$ is a maximal matchable induced subgraph of $G'$, and $G_2''$ has a perfect matching. By maximality, the graphs $G_1'[A_1', B_1']$ and $G_1''[A_1'', B_1'']$ must be empty (otherwise, the extension procedure can be used to extend $G_1'$ and $G_1''$ into strictly bigger matchable subgraphs). This, together with $A_1' \cup A_1'' = A_1' \cup A_1''$ and $B_1' \cup B_1'' = B_1' \cup B_1''$, implies that (i) $|A_1' \cap A_1''| \leq |B_1' \cap B_1''|$, since $G_2''$ has a perfect matching and there is no way to match the vertices of $A_1' \cap A_1''$ in $G_2''$ except using vertices from $B_1' \cap B_1''$ (there are no edges between $A_1' \cap A_1''$ and $B_1' \cap B_1''$). (ii) if $B_1' \cap B_1''$ is not empty then $|A_1' \cap A_1''| > |B_1' \cap B_1''|$: This follows from the fact that for any $v \in A_1'$, the graph $G_1' - v$ has a perfect matching and there is no way to match the vertices of $B_1' \cap B_1''$ in $G_1' - v$ except using vertices from $A_1' \cap A_1''$ (note that the graph $G[A_1' \cap A_1'', B_1''_2]$ is empty). Thus $|B_1' \cap B_1''| > 0$ implies that $A_1' \cap A_1''$ is not empty, so it contains a vertex, say $v'$. But then again $G_1' - v'$ has a perfect matching which matches the set $B_1' \cap B_1''$ with the set $(A_1' \cap A_1'') \setminus \{v'\}$.

Clearly (i) and (ii) above imply that the sets $A_1' \cap A_1''$ and $B_1' \cap B_1''$ are both empty. By symmetry we can also conclude that the sets $A_1'' \cap A_1'$ and $B_1'' \cap B_1'$ are empty and hence complete the proof the proposition.

As a simple application of Proposition 4, we obtain the following corollary.
Corollary 1 Let $G = (A, B, E)$ be a bipartite graph and $G' = G[A', B']$ be a matchable subgraph of $G$. Given two vertices $a \in A \setminus A'$ and $b \in B \setminus B'$ such that there is an edge $\{a, b\}'$ for some $a' \in A'$ (but $\{a, b\}$ is not necessarily an edge of $G$), there exists a unique decomposition of $G'' = G[A' \cup \{a\}, B' \cup \{b\}]$ into a maximal matchable induced subgraph and a subgraph with a perfect matching.

Proof. Start with the graphs $G_1 = (\{a\}, \emptyset)$ which is matchable, and $G_2 = (A', B' \cup \{b\})$ which has a perfect matching, and apply Proposition 4. □

Proposition 5 Let $T = (A, B, E)$ and $T' = (A', B', E')$ be two matchable trees such that $|A| = |B| + 1$, $|A'| = |B'| + 1$, $A \subset A'$ and $B \subset B'$. Then (i) there exists a vertex $b \in B' \setminus B$ such that $|\Gamma_T(\{b\}) \cap A| \geq 1$, (ii) for any such vertex $b$, there exists a vertex $a \in A' \setminus A$, such that the graph $G[a, b] \overset{\text{def}}{=} (A \cup \{a\}, B \cup \{b\}, E'')$, where $E'' = (E \cup E') \cap ((A \cup \{a\}) \times (B \cup \{b\}))$ is matchable.

Proof. Note that $|A' \setminus A| = |B' \setminus B|$. Thus if there no vertex $b \in B' \setminus B$ which has a neighbor in $A$, then $|\Gamma_T(B' \setminus B)| = |A' \setminus A'| = |B'| \setminus B|$. But then the number of edges in the sub-forest of $T'$ induced by the set $(A' \setminus A) \cup (B' \setminus B)$ is $2|B' \setminus B| > |A' \setminus A| + |B' \setminus B| - 1$, which is a contradiction. This shows (i). To prove (ii), let $b_0$ be a vertex in $B' \setminus B$, with a neighbor $x \in A$. Let $y$ be the other neighbor of $b_0$ in $T'$ (recall that $|\Gamma_{T'}(b_0)| = 2$ since $T'$ is matchable). If $y \in A' \setminus A$, then the tree $T \cup \{b_0, x\} \cup \{b_0, y\}$ is a matchable spanning tree for $G[y, b_0]$. Otherwise ($y \in A$), the following procedure finds a vertex $a \in A' \setminus A$ and a matchable spanning tree $T''$ for the graph $G[a, b_0]$:

1. Let $C_0 = (A(C_0), B(C_0), E(C_0))$ be the cycle in $T \cup T'$ composed of the edges $\{b_0, x\}, \{b_0, y\}$, and the path in $T$ from $x$ to $y$, where $A(C_0) \subseteq A$, $B(C_0) \subseteq B$, and $E(C_0) \subseteq E \cup E'$ is the set of edges of $C_0$. Initialize $S \leftarrow A(C_0) \cup (B(C_0) \setminus \{b_0\})$, and $i \leftarrow 1$. In what follows, we let $A(S) = A \cap S$ and by $B(S) = B \cap S$, and note that the invariant that $T[A(S), B(S)]$ is a matchable spanning tree of $S$ is maintained throughout the procedure. Initialize $a_0$ to any vertex in $A$, and repeat Steps 1.1 and 1.2 while $a_{i-1} \notin A' \setminus A$.

1.1. Find vertices $b_i \in B(S)$ and $a_i \notin A(S)$ such that the edge $\{a_i, b_i\} \in E' \setminus E$; such a pair of vertices must exist since not all vertices $v \in B(S)$ have both neighbors in $T'$ belonging to $A(S)$, as this would imply by $|A(S)| = |B(S)| + 1$ that there is a cycle in $T'$. If $a_i \in A' \setminus A$, then continue with Step 2.

1.2. Otherwise, the graph $T \cup \{a_i, b_i\}$ contains a unique cycle $C_i = (A(C_i), B(C_i), E(C_i))$. Set $S \leftarrow S \cup (A(C_i) \cup B(C_i))$, and $i \leftarrow i + 1$.

Clearly, the above loop must terminate after $i \leq |A|$ iterations, and we get a vertex $a_i \in A' \setminus A$. Let $i_0 = i$, $i_1$ be the smallest index $j$ such that $b_{i_0} \in B(C_j)$, $i_2$ be the smallest index $j$ such that $b_{i_1} \in B(C_j)$, and so on. This way we obtain indices $i = i_0 > i_1 > \ldots > i_s = 0$, for some $s \geq 0$. Now we modify the tree $T$ iteratively as follows.

2. For $j = i_{s-1}, \ldots , i_1$ we do the following: Let $v_j$ be the neighbor of $b_j$ in the unique cycle $C_j'$ in the graph $T \cup \{b_j, a_j\}$. Set $T \leftarrow T \cup \{b_j, a_j\} \setminus \{b_j, v_j\}$.

Note that, for $j \in \{i_1, \ldots , i_{s-1}\}$, the cycle $C_j'$ found in Step 2 may be different from the corresponding cycle $C_j$ found in Step 1.2 above. However, it is not difficult to see that at the end of each iteration $j = i_k$ in Step 1.2, it holds that the path between $x$
Figure 1: Illustration of the proof of Proposition 5: White vertices belong to $A \cup A'$ and black vertices belong to $B \cup B'$; solid lines indicate edges of $T$ and dotted lines indicate edges of $T'$. On the left we show the original set of cycles obtained by the procedure described in the proof (here $i_0 = 4$, $i_1 = 3$, $i_2 = 1$, and $i_3 = 0$), and on the right we show the modified tree $T$.

and $y$ in the current tree $T$ passes through $b_{i_k+1}$. In particular at the end of Step 5, we know that $b_{i_0}$ lies on the path between $x$ and $y$ in $T$. Let $a'$ be one of the two neighbors of $b_{i_0}$ on that path. Finally we set $T \leftarrow T \cup \{\{b_0, x\}, \{b_0, y\}, \{b_{i_0}, a_{i_0}\}\} \setminus \{b_{i_0}, a'\}$. To see that the final tree $T$ is indeed a matchable spanning tree, observe that by deleting the edge $\{b_{i_0}, a'\}$ we break $T$ into two components, one of which will be connected to $a_{i_0}$ after adding the edge $\{a_{i_0}, b_{i_0}\}$, and both of which will connected to $b_0$ after adding the edges $\{b_0, x\}$ and $\{b_0, y\}$. Thus $T$ is connected and each vertex in $B \cup \{b_0\}$ has degree 2 in $T$. Thus we can now conclude that the graph $G[a_{i_0}, b_0]$ is matchable since it has a matchable spanning tree. □

Proposition 5 implies that matchable graphs are “continuous” in the sense that we can reach from a given matchable graph to any matchable graph containing it by repeatedly appending pairs of vertices, one at a time, always remaining within the class of matchable graphs. See Figure 2 for an Example.

**Corollary 2** Let $G = (A, B, E)$ and $G' = (A', B', E')$ be two matchable graphs such that $|A| = |B| + 1$, $|A'| = |B'| + 1$, $A \subset A'$, and $B \subset B'$. Then there exist a pair of vertices $a \in A' \setminus A$ and $b \in B' \setminus B$, and a set of edges $E'' \subseteq E \cup E'$, such that the graph $(A \cup \{a\}, B \cup \{b\}, E'')$ is matchable.

**Proof.** Let $T$ and $T'$ be matchable spanning trees of $G$ and $G'$ respectively, and apply Proposition 5. □

Corollary 2 also implies that, given a matchable subgraph $G'$ of a bipartite graph $G(A, B, E)$, we can check in polynomial time whether $G'$ is a maximal matchable induced subgraph of $G$. 

7
Figure 2: Example for Proposition 5: White vertices belong to $A \cup A'$ and black vertices belong to $B \cup B'$; solid lines indicate edges of $T$ and dotted lines indicate edges of $T'$. On the left we show the initial configuration, and on the right we show the new one after the extension.

2.3 Characterization of minimal blockers

We start with the following definition.

**Definition 2** Let $G = (A, B, E)$ be a bipartite graph that has a perfect matching. A matchable split of $G$, denoted by $[(A_1, B_1), (A_2, B_2), (A_3, B_3)]$, is a partition of the vertex set of $G$ into sets $A_1, A_2, A_3 \subseteq A$ and $B_1, B_2, B_3 \subseteq B$ such that

(B1) $|A_1| = |B_1| + 1$, $|A_2| = |B_2| - 1$, $|A_3| = |B_3|$,

(B2) the graph $G_1 = G[A_1, B_1]$ is matchable,

(B3) the graph $G_2 = G[A_2, B_2]$ is matchable,

(B4) the graph $G_3 = G[A_3, B_3]$ has a perfect matching, and

(B5) the graphs $G[A_1, B_3]$ and $G[A_3, B_2]$ are empty.

Denote by $S(G)$ the family of all matchable splits of $G$. We are now ready to state our characterization for minimal blockers in bipartite graphs.

**Theorem 3** Let $G = (A, B, E)$ be a bipartite graph in which there exists a perfect matching. Then a subset of edges $X \subseteq E$ is a minimal blocker if and only if there is a matchable split $[(A_1, B_1), (A_2, B_2), (A_3, B_3)]$ of $G$, such that $X = \{\{a, b\} \in E : a \in A_1, b \in B_2\}$. This correspondence between minimal blockers and matchable splits is one-to-one. Given one representation we can construct the other in polynomial time.

**Proof.** The correspondence $\Phi : B(G) \leftrightarrow S(G)$ is defined as follows:

(i) Given a minimal blocker $X \in B(G)$, define the corresponding matchable split $\Phi(X) = [(A_1, B_1), (A_2, B_2), (A_3, B_3)]$ by the following (polynomial-time) procedure.

1. Let $G' = G - X$. By the minimality of $X$, we must have $|\Gamma_{G'}(Z)| \geq |Z| - 1$ for every $Z \subseteq A$. 2. Since $G'$ does not have a perfect matching, there exists a subset $Z \subseteq A$ with $|\Gamma_{G'}(Z)| = |Z| - 1$. Let $A_1 \subseteq A$ be a smallest such $Z$, and $B_1 = \Gamma_{G'}(A_1)$. 3. Similarly,
let $B_2$ be a minimum cardinality subset of $B$ such that $|\Gamma_{G'}(B_2)| = |B_2| - 1$, and let $A_2 = \Gamma_{G'}(B_2)$. 4. Finally, define $A_3 = A \setminus (A_1 \cup A_2)$ and $B_3 = B \setminus (B_1 \cup B_2)$.

Now let us verify that $\Phi(X)$ is indeed a matchable split. First, the sets $A_1$ and $A_2$ are disjoint: If $A_1 \cap A_2 \neq \emptyset$, then $B_1 \cap B_2 \neq \emptyset$ and the minimality of $A_1$ and $B_2$ gives $|\Gamma_{G'}(A_1 \setminus A_2)| \geq |A_1 \setminus A_2|$ and $|\Gamma_{G'}(B_2 \setminus B_1)| \geq |B_2 \setminus B_1|$. Note that $\Gamma_{G'}(A_1 \setminus A_2) \subseteq B_1 \setminus B_2$ and $\Gamma_{G'}(B_2 \setminus B_1) \subseteq A_2 \setminus A_1$, and thus we get $|A_1 \setminus A_2| \leq |B_1 \setminus B_2|$ and $|B_2 \setminus B_1| \leq |A_2 \setminus A_1|$. But adding-up the last two inequalities gives the contradiction $|A_1| + |B_2| \leq |B_1| + |A_2|$. This shows that $A_1$ and $A_2$ are disjoint sets, implying that there are no edges in $G'$ between the sets $A_1$ and $B_2$. Furthermore, since for any edge $e \in X$, the graph $G' + e$ has a perfect matching, $e$ must be incident to both $A_1$ and $B_2$ in $G$, since otherwise $|\Gamma_{G'+e}(A_1)| = |A_1| - 1$ or $|\Gamma_{G'+e}(B_2)| = |B_2| - 1$, in violation of Hall’s condition. Thus $X = \{(a, b) \in E : a \in A_1, b \in B_2\}$, which also implies (B5). Second, the graphs $G_1 = G[A_1, B_1]$ and $G_2 = G[A_2, B_2]$ are both matchable.

Finally, the fact that the sets $A_1$ and $B_2$ defined above, and hence the matchable split they define, are unique and can be found in polynomial time, follows from Proposition 1, since the function $\Gamma_{G'}(Z) - |Z|$ is submodular and $A_1$ and $B_2$ are its smallest minimizers over $A$ and $B$ respectively.

(ii) Given a matchable split $Y = [(A_1, B_1), (A_2, B_2), (A_3, B_3)]$ of $G$, define the corresponding minimal blocker to be $\Phi^{-1}(Y) = \{\{a, b\} \in E : a \in A_1, b \in B_2\}$. Clearly, $\Phi^{-1}(Y)$ is blocker since $A_1$ has neighborhood of size $|A_1| - 1$ in $G' \overset{\text{def}}{=} G - \Phi^{-1}(Y)$. Furthermore, if $e = \{a, b\} \in \Phi^{-1}(Y)$, where $a \in A_1$ and $b \in B_2$, then the graph $G' + e$ has a perfect matching. Indeed, by the matchability of the split $Y$, the graphs $(A_1 \setminus \{a\}, B_1), (A_2, B_2 \setminus \{b\})$ and $(A_3, B_3)$ have disjoint perfect matchings $M_1, M_2$ and $M_3$. Hence, $M = M_1 \cup M_2 \cup M_3 \cup \{e\}$ is a perfect matching for $G' + e$. Thus we conclude that $\Phi^{-1}(Y)$ is a minimal blocker.

Finally, let us verify that the above mapping is one-to-one. For this we need to verify that $\Phi^{-1}(\Phi(X)) = X$ for all $X \in \mathcal{B}(G)$ and $\Phi(\Phi^{-1}(Y)) = Y$ for all $Y \in \mathcal{S}(G)$. The former claim follows from the fact that, if $X \in \mathcal{B}(G)$ and $Y = \Phi(X) = [(A_1, B_1), (A_2, B_2), (A_3, B_3)]$, then $X = \{(a, b) \in E : a \in A_1, b \in B_2\} = \Phi^{-1}(Y)$. The latter claim follows from the uniqueness of the sets $A_1$ and $B_2$ defining the matchable split $\Phi(X)$ for any $X \in \mathcal{B}(G)$. \hfill $\square$

3 Generation Algorithms

Given a (bipartite) graph $G = (V,E)$, let $\pi : 2^E \rightarrow \{0,1\}$ be a monotone property defined on subsets of $E$: if $\pi(X) = 1$, i.e. the graph $(V,X \subseteq E)$ satisfies $\pi$, then any supergraph $(V,Y \supseteq X)$ also satisfies $\pi$. Consider the problem of listing all subgraphs of $G$, or correspondingly, the family $\mathcal{F}_\pi \subseteq 2^E$ of all minimal subsets of $E$, satisfying a monotone property $\pi$. For instance, if $\pi(X)$ is the property that the subgraph with
edge set $X \subseteq E$ is connected, then $\mathcal{F}_\pi$ is the family of spanning trees of $G$; if $\pi(X)$ is the property that the subgraph $G(A, B, X \subseteq E)$, of a bipartite graph $G = (A, B, E)$, has a perfect matching, then $\mathcal{F}_\pi$ is the family of all perfect matchings of $G$. Note that, with each family of subgraphs $\mathcal{F}_\pi$ satisfying a monotone property $\pi$, we can associate the dual family

$$\mathcal{F}^d_\pi = \{ X \subseteq E \mid X \text{ is a minimal transversal of } \mathcal{F}_\pi \},$$

where $X \subseteq E$ is a transversal of $\mathcal{F}_\pi$ if and only if $X \cap Y \neq \emptyset$ for all $Y \in \mathcal{F}_\pi$. Thus if $\pi(X)$ is the property that $G(V, X)$ is connected, then $\mathcal{F}^d_\pi$ is the family of all minimal cuts of $G = (V, E)$, and if $\pi(X)$ is the property that the subgraph $G(A, B, X \subseteq E)$, of a bipartite graph $G = (A, B, E)$, has a perfect matching then $\mathcal{F}^d_\pi$ is the family of all minimal blockers of $G$.

Enumeration algorithms for listing subgraphs satisfying a number of monotone properties are well known. For instance, it is known [23] that the problems of listing all minimal cuts or all spanning trees of an undirected graph $G = (V, E)$ can be solved with delay $O(|E|)$ per generated cut or spanning tree. It is also known (see e.g., [7, 12, 22]) that all minimal $(s, t)$-cuts or $(s, t)$-paths, can be listed with delay $O(|E|)$ per cut or path. Furthermore, polynomial delay algorithms also exist for listing perfect matchings, maximal matchings, maximum matchings in bipartite graphs, and maximal matchings in general graphs, see e.g. [9, 10, 25, 26].

In the next subsections we give an overview of two commonly used techniques for solving such enumeration problems; see e.g. [17, 18, 23] for more details.

### 3.1 The supergraph approach

This technique works by building and traversing a directed graph $\mathcal{G} = (\mathcal{F}_\pi, \mathcal{E})$, defined somehow on the family to be generated $\mathcal{F}_\pi \subseteq 2^E$. The arcs of $\mathcal{G}$ are defined by a polynomial-time computable neighborhood function $\mathcal{N} : \mathcal{F}_\pi \mapsto 2^{\mathcal{F}_\pi}$ that defines, for any $X \in \mathcal{F}_\pi$, the set of its outgoing neighbors $\mathcal{N}(X)$ in $\mathcal{G}$. A special vertex $X_0 \in \mathcal{F}_\pi$ is identified from which all other vertices of $\mathcal{G}$ are reachable. The method works by traversing, say in breadth-first search order, the vertices of $\mathcal{G}$, starting from $X_0$. If $\mathcal{G}$ is strongly connected then $X_0$ can be any vertex in $\mathcal{F}_\pi$. The following is a basic fact that we will need about this approach (see e.g. [17]):

**S** If $|\mathcal{N}(X)| \leq p(|V|, |E|)$ for every $X \in \mathcal{F}_\pi$, where $p(|V|, |E|)$ is polynomial that depends only on the size of $G$, then the supergraph method yields a polynomial delay algorithm for enumerating $\mathcal{F}_\pi$.

We give two examples below.

**Generation of matchable spanning trees.** Let $G = (A, B, E)$ be a matchable bipartite graph with $|A| = |B| + 1$. For $X \subseteq E$, let $\pi(X)$ be the property that graph $(A, B, X)$ is matchable. Then, by (M2), $\mathcal{F}_\pi$ is the family of all matchable spanning trees of $G$. For any $X \in \mathcal{F}_\pi$, the outgoing neighbors of $X$ are defined as follows. For every edge $e = \{a, b\} \in E \setminus X$, where $a \in A$ and $b \in B$, let $e'(X, e)$ be the edge incident to $b$ on the unique path in $T = (A, B, X)$ between $a$ and $b$. Note that $(A, B, X \cup \{e\} \setminus \{e'(X, e)\})$ is a matchable spanning tree of $G$. Define $\mathcal{N}(X) = \{X \cup \{e\} \setminus \{e'(X, e)\} \mid e \in E \setminus X\}$. Clearly, for any $X \in \mathcal{F}_\pi$, $|\mathcal{N}(X)| = |E| - 2|A| + 2$ and each element of $\mathcal{N}(X)$ can be computed in $O(|A|)$ time. Given two vertices $X, Y \in \mathcal{F}_\pi$ of $\mathcal{G}$, we claim that there
exists a path from $X$ to $Y$ of length at most $|X \setminus Y||Y \setminus X|$. Namely, we can show that if $X \neq Y$, then in at most $|Y \setminus X|$ neighbor steps we can reduce the cardinality $|X \setminus Y|$ by one. This will prove the strong connectivity of $G$ (and in fact show that $G$ has diameter at most $O(|A|^2)$). To prove the claim, let us consider a vertex $b \in B$, and denote by $d(b)$ the degree of $b$ in the graph $G' \overset{\text{def}}{=} (A, B, X \cup Y)$. If $d(b) = 2$ for all $b \in B$, then we must have $X = Y$, and we are done. Otherwise, assume that $d(b) > 2$, and consider two cases:

Case I. $d(b) = 4$: Choose an arbitrary edge $e = \{a, b\} \in Y \setminus X$. Then the edge $e' = e'(X, e)$ on the unique cycle in the graph $(A, B, X \cup \{e\})$ is in $X \setminus Y$. Let $X' = X \cup \{e\} \setminus \{e'\}$ be the corresponding neighbor of $X$ in $G$. Then $|X' \setminus Y| = |X \setminus Y| - 1$.

Case II. $d(b) = 3$: Denote by $\{a, b\} \in X \cap Y$, $\{a', b\} \in X \setminus Y$ and $\{a'', b\} \in Y \setminus X$ the edges incident to $b$ in $G'$. Let $G'' = G' - b$ and denote by $Z$ the edge set of $G''$. Let us assign length 0 to all edges in $X \cap Z$ and length 1 to all edges in $(Y \setminus X) \cap Z$, and denote by $\text{dist}_Z(a', a'')$ the length of a shortest $(a', a'')$-path in $G''$. With this assignment, it is not difficult to see that $\text{dist}_Z(a', a'')$ is finite. If $\text{dist}_Z(a', a'') > 0$, then choose an arbitrary edge $e = \{x, y\} \in Y \setminus X$ on the shortest $(a', a'')$-path in $G''$, and consider the neighbor $X'$ of $X$ obtained by adding $e$ to $X$, giving it length 0, and then dropping $e' = e'(X, e)$. Since the endpoints of $e'$ are connected within the set $X \cup \{e\}$, we have $\text{dist}_Z(a', a'') + \text{dist}_Z(a', a'') < \text{dist}_Z(a', a''),$ where $Z' = Z \cup \{e\} \setminus \{e'\}$. Thus, repeating this operation for at most $|Y \setminus X|$ steps we can arrive to a vertex $X'$ in $G$, where $\text{dist}_Z(a', a'') = 0$ ($Z = X' \cap Y$ minus the edges incident to $b$), i.e., in which $a'$ and $a''$ are connected by a path $P$ within $X' \cap Z$. Now we add the edge $e = \{a'', b\}$ to $X'$. The unique cycle in $(A, B, X \cup \{e\})$ is formed then by $P \cup \{a'', b\}, \{a', b\}$, and hence we need to remove $e'(X', e) = \{a', b\}$ to get to a neighbor $X''$ of $X$. Hence, for this neighbor we have $|X'' \setminus Y| < |X \setminus Y|$.

It follows then from (S) that we get a polynomial delay algorithm for generating the family $F_\pi$.

**Generation of minima of submodular functions.** Let $f : 2^V \rightarrow \mathbb{R}$ be a submodular function, $\alpha = \min\{f(X) : X \subseteq V\}$, and $\pi(X)$ be the property that $X \subseteq V$ is a minimizer of $f$. Although $\pi$ is not monotone, we can still use the supergraph approach to generate the elements of the family $F \overset{\text{def}}{=} \{X \subseteq V : \pi(X) = 1\}$. By Proposition 1, the set $F$ forms a sub-lattice $\mathcal{L}$ of $2^V$, and the smallest element $X_0$ of this lattice can be computed in polynomial time. For $X \in F$, define $N(X)$ to be the set of subsets $Y \in \mathcal{L}$ that immediately succeed $X$ in the lattice order. The elements of $N(X)$ can be computed by finding, for each $v \in V \setminus X$, the smallest cardinality minimizer $X' = \text{argmin}\{f(Y) : Y \supseteq X \cup \{v\}\}$ (again using Proposition 1) and checking if $f(X') = \alpha$. Then $|N(X)| \leq |V|$, for all $X \in F$. Note also that the definition of $N(\cdot)$ implies that there is a path in the supergraph $\mathcal{G}$ from $X_0$ to any other vertex $X \in F$. Thus it follows from (S) that all the elements of $F$ can be enumerated with polynomial delay.

**3.2 The flashlight (backtracking) approach**

This can be regarded as an important special case of the supergraph method. Assume that we have fixed some order on the elements of $E$. Let $X_0$ be the lexicographically smallest element in $F_\pi$. For any $X \in F_\pi$, let $N(X)$ consist of a single element, namely,
the element next to $X$ in lexicographical ordering. Thus the supergraph $G$ in this case is a Hamiltonian path on the elements of $F_\pi$. The following is a sufficient condition for the operator $\mathcal{N}(\cdot)$ (and also for the element $X_0$) to be computable in polynomial time:

(F1) For any two disjoint subsets $S_1, S_2$ of $E$, we can check in polynomial time $p(|V|, |E|)$ if there is an element $X \in F_\pi$, such that $X \supseteq S_1$ and $X \cap S_2 = \emptyset$

(see [23] for general background on backtracking algorithms). The traversal of $G$, in this case, can be organized in a backtracking tree of depth at most $|E|$, whose leaves correspond to the elements of the family $F_\pi$, as follows. Each node of the tree is identified with an ordered pair $(S_1, S_2)$ of two disjoint subsets $S_1, S_2 \subseteq E$, and has at most two children. At the root of the tree, we have $S_1 = S_2 = \emptyset$. The two children of an internal node $(S_1, S_2)$ of the tree are defined as follows. We choose the smallest element $e \in E \setminus (S_1 \cup S_2)$ such that there is a subset $X \in F_\pi$, satisfying $X \supseteq S_1 \cup \{e\}$ and $X \cap S_2 = \emptyset$. If no such element can be found, then the current node is a leaf. Otherwise, the left child of the node $(S_1, S_2)$ is identified with $(S_1 \cup \{e\}, S_2)$. Analogously, the right child of the node $(S_1, S_2)$ is $(S_1, S_2 \cup \{e\})$, provided that there is a subset $X \in F_\pi$, such that $X \supseteq S_1$ and $X \cap (S_2 \cup \{e\}) = \emptyset$.

Clearly, for this method to work in polynomial time, we need (F1) to hold. In general, such a check is NP-hard, but in some cases it can be performed in polynomial time, provided that we do the extension from the set $S_1$ to $S_1 \cup \{e\}$ in a more careful way. More precisely, let $F'_\pi \subseteq 2^E$ be a family of sets, such that

(F2) $F'_\pi \supseteq F_\pi$,

(F3) for every non-empty $X \in F'_\pi$, there exists an element $e \in X$ such that $X \setminus \{e\} \in F'_\pi$ (in particular, $\emptyset \in F'_\pi$), and

(F4) we can test in polynomial time if a given set $X \in F'_\pi$.

In the backtracking procedure, we always maintain the invariant $S_1 \in F'_\pi$. The following is a weaker requirement than that of (F1), and provides a sufficient condition for this backtracking method to work in polynomial time:

(F5) For any two disjoint subsets $S_1 \in F'_\pi$ and $S_2 \subseteq E$, and for any $e \in E \setminus (S_1 \cup S_2)$ such that $S_1 \cup \{e\} \in F'_\pi$, we can check in polynomial time if there is an element $X \in F_\pi$, such that $X \supseteq S_1 \cup \{e\}$ and $X \cap S_2 = \emptyset$.

This way, under assumption (F5), we obtain a polynomial delay, polynomial space algorithm for enumerating the elements of $F_\pi$ in lexicographical order.

We give an example.

**Generation of perfect matchings in general graphs.** Let $G = (V, E)$ be a graph on vertex set $V$, and let $\pi(X)$ be the property that the graph $(V, X)$ has a perfect matching, for $X \subseteq E$. Given disjoint subsets $S_1, S_2 \subseteq E$, we can perform the check in (F1) by checking if there is a perfect matching in the graph $G' = G - V' - S_2$, where $V' = \{v \in V : v \in e$ for some $e \in S_1\}$, that is the graph obtained from $G$ by deleting the edges in $S_2$ and vertices incident to the edges in $S_1$. Thus all perfect matchings of $G$ can be generated with polynomial delay. A more efficient procedure, in the bipartite case, can be found in [9].
4 Polynomial Delay Generation of Minimal Blockers

In this section, we use the supergraph approach to show that minimal blockers can be enumerated with polynomial delay. Using Theorem 3, we may equivalently consider the generation of matchable splits. We start by defining the neighborhood function used for constructing the supergraph $G_S$ of matchable splits.

4.1 The supergraph

Let $G = (A, B, E)$ be a bipartite graph that has a perfect matching. Given a matchable split $X = [(A_1, B_1), (A_2, B_2), (A_3, B_3)] \in S(G)$, the set of outgoing neighbors of $X$ in $G_S$ are defined as follows. Denote by $G_1 = G[A_1, B_1]$, $G_2 = G[A_2, B_2]$ and $G_3 = G[A_3, B_3]$, the subgraphs of $G$ induced by $X$. For each edge $\{a, b\} \in E$ connecting a vertex $a \in A_2$ to a vertex $b \in B_2$, such that $b$ is also connected to some vertex $a' \in A_1$, there is a unique neighbor $X' = N(X, a, b)$ of $X$, obtained by the following procedure:

1. Let $A''_1 = A_1 \cup \{a\}$, $B''_1 = B_1 \cup \{b\}$, and $G''_1 = G[A''_1, B''_1]$. Because of the edges $\{a, b\}, \{a', b\} \in E$, the graph $G''_1$ is matchable; see Corollary 1.
2. Delete the vertices $a, b$ from $G_2$. This splits the graph $G_2 - \{a, b\}$ in a unique way into a matchable graph $G''_2 = G[A_2', B_2']$, with $|B'_2| = |A'_2| + 1$, and a graph with a perfect matching $G'''_2 = G[A_2'', B''_2]$, such that there are no edges in $E$ between $A''_2$ and $B''_2$; see Proposition 3.
3. Let $A''_3 = A_3 \cup A_2''$ and $B''_3 = B_3 \cup B''_2$. Then the graph $G''_3 = G[A''_3, B''_3]$ has a perfect matching. Using Proposition 4, extend $G''_3$ in the union $G[A''_3 \cup A_1', B''_3 \cup B_1']$ to a maximal matchable graph $G'_1 = G[A'_1, B'_1]$ such that the graph $G''_3 = G[A'_3, B'_3] = G[A''_3 \cup A_1' \backslash A'_1, B''_3 \cup B_1' \backslash B'_1]$ has a perfect matching.
4. Let $X' = [(A'_1, B'_1), (A_2', B_2'), (A'_3, B'_3)]$, and note that $X' \in S(G)$.

Similarly, for each edge $\{a, b\} \in E$ connecting a vertex $a \in A_1$ to a vertex $b \in B_1$, such that $a$ is also connected to some vertex $b' \in B_2$, there is a unique neighbor $X'$ of $X$, obtained by a procedure similar to the above.

4.2 Strong connectivity

For the purpose of generating minimal blockers in a bipartite graph $G = (A, B, E)$, we may assume without loss of generality that

(A1) The graph $G$ is connected.

(A2) Every edge in $G$ appears in a perfect matching.

Indeed, if $G$ has at least two connected components, then the set of minimal blockers of $G$ is the disjoint union of the sets of minimal blockers in the different components, computed individually for each component. Furthermore, all the edges in $G$ that do not appear in any perfect matching can be removed, since they do not appear in any minimal blocker. In this section, we prove the following.

Lemma 1 Under the assumptions (A1) and (A2), the supergraph $G_S$ is strongly connected.

In view of (S), Lemma 1 implies Theorem 1. Call a $v$-star, and denote by $v^*$, the set of edges incident to a given vertex $v \in A \cup B$. We need the following lemma.
Lemma 2 Let \( G = (A, B, E) \) be a bipartite graph satisfying (A1) and (A2). Then (i) Every star in \( G \) is a minimal blocker. (ii) The matchable split corresponding to an a-star, \( a \in A \), is \([\{a\}, \emptyset], (A \setminus \{a\}, B), (\emptyset, \emptyset)\). (iii) The matchable split corresponding to a b-star, \( b \in B \), is \([A \setminus \{b\}, (\emptyset, \{b\}), (\emptyset, \emptyset)]\).

Proof. (i) Clearly, \( v^* \) is a blocker for any \( v \in A \cup B \). Furthermore, since by (A2) any edge connected to \( v \) appears in some perfect matching (which of course does not contain any other edge incident to \( v \)), it follows that \( v^* \) is indeed a minimal blocker.

(ii) Consider any vertex \( a \in A \). Let \([A_1, B_1], (A_2, B_2), (A_3, B_3)\) be the matchable split corresponding to \( a^* \). Then \( a \in A_1 \), \( \Gamma_G(\{a\}) \subseteq B_2 \), and \( a^* \) is the set of edges in the graph \( G[A_1, B_2] \). Note that \( B_1 \) is empty. If not, then \( |A_1| \geq 2 \) and therefore, there exists a vertex \( a' \in A_1 \), \( a' \neq a \). But then the graph \( G[A_1 \setminus \{a'\}, B_1] \) does not have a perfect matching since \( a \in A_1 \setminus \{a'\} \) cannot be matched with any vertex from \( B_1 \), in contradiction to the matchability of \( G[A_1, B_1] \). Thus we conclude that \( A_1 = \{a\} \) and \( B_1 = \emptyset \). Next, suppose that \( B_3 \) is non-empty. Then the connectivity of \( G \) and the fact that there are no edges in \( G \) between \( A_3 \) and \( B_2 \) imply that there must exist an edge \( e = \{a', b\} \) where \( a' \in A_2 \) and \( b \in B_3 \). By (A2) there is a perfect matching \( M \), containing \( e \) in \( G \). Clearly, \( M \) must match \( a' \) to \( b \) and \( a \) to some vertex \( b' \in B_2 \). But then \( M \) must also match the set \( A_2 \setminus \{a'\} \) with \( B_2 \setminus \{b'\} \), which is impossible since \( |A_2 \setminus \{a'\}| = |B_2 \setminus \{b'\}| - 1 \). This contradiction shows (ii); (iii) can be shown by a similar argument. \( \square \)

Given two matchable splits \( X = [(A_1, B_1), (A_2, B_2), (A_3, B_3)] \) and \( X' = [(A'_1, B'_1), (A'_2, B'_2), (A'_3, B'_3)] \) of a graph \( G \), let us say that \( X \) and \( X' \) are perfectly nested (see Figure 3) if

(N1) \( A_1 \subseteq A'_1 \) and \( B_1 \subseteq B'_1 \),

(N2) \( A_2 \supseteq A'_2 \) and \( B_2 \supseteq B'_2 \), and

(N3) each of the graphs \( G[A_2 \cap A'_1, B_2 \cap B'_1], G[A_3 \cap A'_1, B_3 \cap B'_1], G[A_2 \cap A'_3, B_2 \cap B'_3], \) and \( G[A_3 \cap A'_3, B_3 \cap B'_3] \), has a perfect matching.

The strong connectivity of the supergraph \( G_S \) is a consequence of the following fact.

Lemma 3 There is a directed path in \( G_S \) between any pair of perfectly nested matchable splits.

Proof. Let \( X = [(A_1, B_1), (A_2, B_2), (A_3, B_3)] \) and \( X' = [(A'_1, B'_1), (A'_2, B'_2), (A'_3, B'_3)] \) be two matchable splits, satisfying (N1), (N2) and (N3) above. We show that there is a directed path in \( G_S \) from \( X \) to \( X' \). A path in the opposite direction can be found symmetrically. Fix a matching \( M \) in the graph \( G[A'_1 \cap A_2, B'_1 \cap B_2] \). By Proposition 5 (applied to any matchable spanning trees of the graphs \( G[A_1, B_1] \) and \( G[A'_1, B'_1] \)), there is a vertex \( b \in B'_1 \setminus B_1 \), such that \( b \) is connected by an edge \( \{a, b\} \) to some vertex \( a \in A_1 \) (see Figure 3). Note that \( b \notin B_3 \cap B'_1 \) since the graph \( G[A_1, B_3] \) is empty. Thus \( b \in B_2 \cap B'_1 \). Let \( a' \in A'_1 \setminus A_2 \) be the vertex matched by \( M \) to \( b \). Now consider the neighbor \( X'' = N(X, a', b) = [(A''_1, B''_1), (A''_2, B''_2), (A''_3, B''_3)] \) of \( X \) in \( G_S \). We claim that \( X'' \) is perfectly nested with respect to \( X' \). To see this observe, by Proposition 3, that the graph \( G[A_2, B_2] - \{a', b\} \) is decomposed, in a unique way, into a matchable graph \( G_2'' = G[A''_2, B''_2] \) and a graph with a perfect
Figure 3: Two matchable splits $X$ and $X'$: solid lines surround the sets of $X$ and dotted lines surround the set of $X'$. The right figure shows the configuration after a neighbor $N(X, a, b)$ of $X$ in $G_S$ is found.

matching $G'' = G[A'_2', B''_2']$. On the other hand, the graph $G[A'_2', B''_2']$ is matchable, while the graph $G[A_2 \cap (A'_1 \cup A'_3) \setminus \{a'\}, B_2 \cap (B'_1 \cup B'_3) \setminus \{b\}]$ has a perfect matching, and therefore by Proposition 4, the induced subgraph of $G$ defined by the union of these two graphs (which is $G[A_2, B_2] - \{a', b\}$) decomposes in a unique way into a maximal matchable induced subgraph $F_1$ containing $G[A'_2', B''_2']$ and a graph with a perfect matching $F_2$. The uniqueness of the decomposition implies that $F_1 = G''_2$ and $F_2 = G''$, and hence $A'_2 \supseteq A'_2'$ and $B'_2 \supseteq B'_2$. Note that the graph $G[A'_2 \cap A'_1, B''_2 \cap B'_1]$ still has a perfect matching since $F_1 = G''_2$ can be obtained from $G[A'_2', B''_2]$ by calling the procedure $\text{EXTEND}(G[A'_2', B''_2], M')$, where $M'$ is a matching containing $M \setminus \{a', b\}$. Note also that the graph $G[A'_1', B''_1]$ is obtained by extending the matchable graph $G[A_1 \cup \{a'\}, B_1 \cup \{b\}]$ with pairs of vertices from the graph $G[A'_2' \cup (A'_1', B''_1) \cup (B'_1 \cap B_3)]$, which has a perfect matching. Such an extension must stay within the graph $G[A'_1', B''_1]$, i.e. $A'_1 \subseteq A'_1'$ and $B''_1 \subseteq B'_1$, since there are no edges between $A'_1$ and $B''_1 \cap B_3$. Finally, since the extension leaves a perfect matching in the graph $G[A'_2' \cup (A'_1 \cap A_3) \setminus A''_1, B''_1 \cup (B'_1 \cap B_3) \setminus B''_1]$, we conclude that $X''$ and $X'$ are perfectly nested. This way, we obtained a neighbor $X''$ of $X$ that is closer to $X'$ in the sense that $|A'_1'| > |A_1|$. This implies that there is a path in $G_S$ from $X$ to $X'$ of length at most $|A'_1' \setminus A_1|$. $\square$

**Proof of Lemma 1.** Let $X = [(A_1, B_1), (A_2, B_2), (A_3, B_3)]$ be a matchable split. Let $a$ be any vertex in $A_1$, then the matchability of $A_1$ implies that the graph $G[A_1 \setminus \{a\}, B_1]$ has a perfect matching. Thus, with respect to $a^*$ and $X$, the conditions (N1)-(N3) hold, implying that they are perfectly nested. Lemma 3 hence implies that there are directed paths in $G_S$ from $a^*$ to $X$ and from $X$ to $a^*$. Similarly we can argue that there are directed paths in $G_S$ between $X$ and $b^*$ for any $b \in B_2$. In particular, there are directed paths in $G_S$ between $a^*$ and $b^*$ for any $a \in A$ and $b \in B$, and any other matchable split $X'$ is connected to $X$ through the stars. The lemma follows. $\square$
5 Some Generalizations and Related Problems

5.1 \(d\)-factors

Let \(G = (A, B, E)\) be a bipartite graph, and \(d : A \cup B \mapsto \{0, 1, \ldots, |A| + |B|\}\) be a non-negative function assigning integral weights to the vertices of \(G\). We shall assume in what follows that, for each vertex \(v \in A \cup B\), the degree of \(v\) in \(G\) is at least \(d(v)\). A \(d\)-factor in \(G\) is a subgraph \((A, B, X)\) covering all the vertices of \(G\), such that each vertex \(v \in A \cup B\) has degree \(d(v)\) (see Section 10 in [20]). Thus perfect matchings correspond to the case \(d \equiv 1\). Note that \(d\)-factors of \(G\) are in one-to-one correspondence with the vertices of the polytope \(P^*(G) = \{x \in \mathbb{R}^E \mid Hx = \bar{d}, \quad 0 \leq x \leq e\}\), where \(H\) is the incidence matrix of \(G\), and \(\bar{d} = (d(v) : \ v \in A \cup B)\). In particular, checking if a bipartite graph has a \(d\)-factor can be done in polynomial time by solving a capacitated transportation problem with edge capacities equal to 1, see e.g. [6].

Since the \(d\)-factors are the vertices of a 0/1-polytope, they can be computed with polynomial delay [5]. We present below a more efficient procedure.

Theorem 4 For any given bipartite graph \(G = (A, B, E)\), and any non-negative integer function \(d : A \cup B \mapsto \{0, 1, \ldots, |A| + |B|\}\), after computing the first \(d\)-factor, all other \(d\)-factors of \(G\) can be enumerated with delay \(O(|E|)\).

Proof. We use the flashlight method. Let \(M_d(G)\) be the set of \(d\)-factors of \(G\). It is enough to check that the condition (F1) is satisfied. Given \(S_1, S_2 \subseteq E\), we can check in polynomial time whether there is an \(X \in M_d(G)\) such that \(X \supseteq S_1\) and \(X \cap S_2 = \emptyset\) by checking the existence of a \(d'\)-factor in the graph \((A, B, E \setminus S_2)\), where for \(v \in A \cup B\), \(d'(v) = d(v) - \deg_{S_1}(v)\) and \(\deg_{S_1}(v)\) denotes the number of edges incident to \(v\) in the graph \((A, B, S_1)\).

A more efficient procedure can be obtained as a straightforward generalization of the one in [9]. It is a slight modification of the flashlight method that avoids checking the existence of a \(d\)-factor at each node in the backtracking tree. Given a \(d\)-factor \(X\) in \(G\), it is possible to find another one, if it exists, by the following procedure. Orient all the edges in \(X\) from \(A\) to \(B\) and all the other edges in \(E \setminus X\) from \(B\) to \(A\). Then it is not difficult to see that the resulting directed graph \(G'\) has a directed cycle if and only if \(G\) contains another \(d\)-factor \(X' \in M_d(G)\). Such an \(X'\) can be found by finding a directed cycle \(C\) in \(G'\), and taking the symmetric difference between the edges corresponding to the arcs of \(C\) and \(X\).

Now, the backtracking algorithm proceeds as follows. Each node \(w\) of the backtracking tree is identified with a weight function \(d_w\), a \(d_w\)-factor \(X_w\), and a bipartite graph \(G_w = (A, B, E_w)\). At the root \(r\) of the tree, we compute a \(d\)-factor \(X\) in \(G\), and let \(d_w \equiv d\) and \(G_w = G\). At any node \(w\) of the tree, we check if there is another \(d_w\)-factor \(X'_w\) using the above procedure, and if there is none, we define \(w\) as a leaf. Otherwise, we let \(e_w\) be an arbitrary edge in \(X_w \setminus X'_w\). This defines the two children \(u\) and \(z\) of \(w\) by: \(d_u(v) = d_w(v) - 1\) for \(v \in e_u\), and \(d_z(v) = d_w(v)\) for all other vertices \(v \in A \cup B\), \(X_u = X_w, G_u = G_w, d_z = d_w, X_z = X'_w\), and finally, \(G_z = G_w - e_w\). The \(d\)-factors computed at the nodes of the backtracking tree are distinct and they form the complete set of \(d\)-factors of \(G\). \(\square\)

Let us mention that the supergraph approach can also be used directly to generate the \(d\)-perfect matchings of \(G = (A, B, E)\) in incremental polynomial time. Given a \(d\)-perfect matching \(X \in M_d(G)\), we define the neighborhood function \(N(X)\) as follows.
Direct all the edges in \( X \) from \( A \) to \( B \) and all the other edges in \( E \setminus X \) from \( B \) to \( A \). For every directed cycle \( C \), with (corresponding) edge set \( Y \), in the resulting directed graph \( G(X) = (A, B, E(X)) \), let \( N(X, C) \in \mathcal{N}(X) \) be the \( d \)-factor obtained from \( X \) by deleting all the edges in \( X \cap Y \) and adding all the edges in \( Y \setminus X \). Now the set of neighbors of \( X \) is given by

\[
N(X) = \{ N(X, C) : C \text{ is a directed cycle in } G(X) \}.
\]

Note that two different cycles \( C \) and \( C' \) in \( G(X) \) give two different neighbors \( N(X, C) \) and \( N(X, C') \), since \( N(X, C) = X \Delta Y \) is the symmetric difference of \( X \) and the edges of \( C \). Furthermore, the computation of all directed cycles in \( G(X) \) can be carried out with polynomial delay, see \cite{23}. Finally, the the supergraph \( \mathcal{G} \) defined by the above neighborhood function is strongly connected. Indeed, consider any two \( d \)-factors \( X, X' \in \mathcal{M}_d(G) \). Consider the graph \( G' = (A, B, X \cup X') \). Delete all the edges in \( X \cap X' \) from \( G' \) and orient the edges in \( X \setminus X' \) from \( A \) to \( B \) and the edges in \( X' \setminus X \) from \( B \) to \( A \). Then every vertex in the resulting graph \( G'' \) has equal in- and out-degrees. Thus it follows that \( G' \) has an Eulerian decomposition into the union of arc-disjoint directed cycles. Taking any such cycle \( C \) and constructing the neighbor \( X'' = N(X, C) \), we obtain an element \( X'' \) in the neighborhood of \( X \) with a smaller difference \( |X' \setminus X''| \). This shows that \( \mathcal{G} \) is strongly connected and has diameter linear in \( |E| \).

### 5.2 2-Matchings in General Graphs

Let \( G = (V, E) \) be an undirected graph with vertex set \( V \) and edge set \( E \). Let \( H \) be the incidence matrix of \( G \). As stated in the introduction, if \( G \) is a bipartite graph then the perfect matchings of \( G \) correspond to the vertices of the polytope \( P(G) = \{ x \in \mathbb{R}^E | Hx = e, x \geq 0 \} \). In general, the vertices of \( P(G) \) are in one-to-one correspondence with the basic perfect 2-matchings of \( G \), i.e. subsets of edges which form a cover of the vertices with vertex-disjoint edges and odd cycles (see \cite{20}). A (not necessarily basic) perfect 2-matching of \( G \) is a subset of edges that covers the vertices of \( G \) with vertex-disjoint edges and (even or odd) cycles. Denote respectively by \( \mathcal{M}_2(G) \) and \( \mathcal{M}^2_2(G) \) the families of perfect 2-matchings and basic perfect 2-matchings of a graph \( G \). We show below that, the family \( \mathcal{M}_2(G) \) can be enumerated with polynomial delay, and the family \( \mathcal{M}^2_2(G) \) can be enumerated in incremental polynomial time.

Theorem 2 will follow from the following two lemmas.

**Lemma 4** All perfect 2-matchings of a graph \( G \) can be generated with polynomial delay.

**Proof.** We use the flashlight method with a slight modification. For \( X \subseteq E \), let \( \pi(X) \) be the property that the graph \( (V, X) \) has a perfect 2-matching. Then \( \mathcal{F}_\pi = \mathcal{M}_2(G) \).

Define \( \mathcal{F}_\pi' = \{ X \subseteq E : \text{ the graph } (V, X) \text{ is a vertex-disjoint union of some cycles, some edges, and a path possibly of length zero} \} \). It is easy to check that conditions (F2), (F3) and (F4) are satisfied. Given \( S_1 \subseteq \mathcal{F}_\pi', S_2 \subseteq E \), we modify the basic approach described in Section 3.2 in two ways. First, when we consider a new edge \( e \in E \setminus (S_1 \cup S_2) \) to be added to \( S_1 \), we first try an edge incident to an endpoint of the path in \( S_1 \), if this path has positive length. If the path has length zero, then any edge \( e \in E \setminus (S_1 \cup S_2) \) can be chosen and defined to be a path of length one in \( S_1 \cup \{ e \} \). Second, when we backtrack on an edge \( e \) defining a path of length one in \( S_1 \), we redefine \( S_1 \) by considering \( e \) as a single edge rather than a path of length one. Now it remains to check that (F5) is satisfied. Given \( S_1 \subseteq \mathcal{F}_\pi', S_2 \subseteq E \), and an edge
$e \in E \setminus (S_1 \cup S_2)$, chosen as above, such that $S_1 \cup \{e\} \in \mathcal{F}^*$, we can check in polynomial time whether there is an $X \in \mathcal{M}_2(G)$ such that $X \supseteq S_1 \cup \{e\}$ and $X \cap S_2 = \emptyset$ in the following way. First, we delete from $G$ all edges in $S_2$, and all vertices incident to edges in $S_1$, except the end-points $x$ and $y$ of the single path $P$ in $S_1$. Let us call the resulting graph $G'$. Then, we construct an auxiliary bipartite graph $G^b$ from $G'$ as follows (see [20]). For every vertex $v \neq x, y$ of $G'$ we define two vertices $v'$ and $v''$ in $G^b$. In addition, $G^b$ also contains two other vertices $x'$ and $y''$. For every edge $\{u, v\}$ in $G'$, with $\{u, v\} \cap \{x, y\} = 0$, we define two edges $\{u', v''\}$ and $\{u'', v'\}$ in $G^b$. For each edge $\{x, u\}$ in $G'$, we introduce an edge $\{u', x''\}$ in $G^b$, and for each edge $\{u, y\}$ in $G'$, we introduce an edge $\{u', y''\}$ in $G^b$. It is easy to see that there is an $X \in \mathcal{M}_2(G)$ such that $X \supseteq S_1 \cup \{e\}$ and $X \cap S_2 = \emptyset$ if and only if there is a perfect matching in $G^b$. □

Lemma 5 For a graph $G = (V, E)$, we have

$$|\mathcal{M}_2(G)| \leq \left(\frac{|\mathcal{M}_2^*(G)| + 1}{2}\right).$$ (2)

Proof. Note that each non-basic perfect 2-matching $M$ in $G$ can be decomposed into two distinct basic perfect 2-matchings $M'$ and $M''$. This can be done by decomposing each even cycle $C$ in $M$ into two edge-disjoint perfect matchings $C'$ and $C''$. The two basic perfect 2-matchings $M'$ and $M''$ are defined to contain the disjoint edges in $M$ and the edges of the odd cycles in $M$. In addition, $M'$ contains the edges of the perfect matching $C'$ for each even cycle $C$ in $M$ and $M''$ contains the edges of the perfect matching $C''$ for each even cycle $C$ in $M$. This decomposition implies (2). □

Proof of Theorem 2. By generating all perfect 2-matchings of $G$ and discarding the non-basic ones, we can get generate all basic perfect 2-matchings. By Lemma 5, the total time for this generation is polynomial in $|V|$, $|E|$, and $|\mathcal{M}_2^*(G)|$. Since the problem is self-reducible, we can convert this output polynomial generation algorithm to an incremental one, see [3, 4, 8] for more details. □

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References


18


