The Cell Structures of Certain Lattices*

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> And out of the ground the Lord God formed every beast of the field, and every fowl of the air; and brought them into Adam to see what he would call them: and whatsoever Adam called every living creature, that was the name thereof.
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> Genesis 2:19

> Kaum nennt man die Dinge beim richtigen Namen, so verlieren sie ihren gefährlichen Zauber. Der primitive Mensch benannte alles und jedes falsch. Ein einziger furchtbarer Zauberbann umgab ihn, wo und wann war er nicht gefährdet? Die Wissenschaft hat uns von Aberglauben and Glauben befreit. Sie gebraucht immer die gleichen Namen, mit Vorliebe griechisch-lateinische, und meint damit die wirklichen Dinge. Missverständnisse sind unmöglich.(**)
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> Elias Canetti, *Die Blendung*, Hanser, Munich, 1963, p. 425

Abstract. The most important lattices in Euclidean space of dimension $n \le 8$ are the lattices $A_n (n \ge 2)$, $D_n (n \ge 4)$, $E_n (n = 6, 7, 8)$ and their duals. In this paper we determine the cell structures of all these lattices and their Voronoi and Delaunay polytopes in a uniform manner. The results for E_6^* and E_7^* simplify recent work of Worley, and also provide what may be new space-filling polytopes in dimensions 6 and 7.

1. Introduction

The Coxeter-Dynkin diagrams of types A_n, D_n, E_6, E_7 and E_8 arise in surprisingly different parts of mathematics – see the discussions by Arnold [1] and Hazewinkel et al. [30]. In the present paper we study

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(**) From the English version *Auto-da-Fe´*(Continuum, New York, p. 385) as translated by C. V. Wedgwood: ‘‘You have but to know an object by its proper name for it to lose its dangerous magic. Primitive man called each and all by the wrong name. One single and terrible web of magic surrounded him; where and when did he not feel threatened? Knowledge has freed us from superstitions and beliefs. Knowledge makes use always of the same names, preferably Graeco-Latin, and indicates by these names actual things. Misunderstandings are impossible.’’

the lattices associated with these diagrams (the so-called root lattices) and their duals (the weight lattices), as well as some of the polytopes arising from consideration of the cell structure of these lattices. In accordance with our epigraphs we provide names for many of these polytopes.

The Coxeter-Dynkin diagrams provide an astonishing amount of information about these lattices, much of which can be found in Coxeter's book ''Regular Polytopes'' [20]. The lattices are discussed from the point of view of Lie algebras in references such as Bourbaki [5] and Humphreys [33], and are also extensively analyzed in our book [13]. In recent years they have been used to construct modulation schemes (called trellis codes) for high-speed transmission of digital data [7]-[9], [22]-[24].

However, in spite of their long history, there are still new things to be said about these lattices and their polytopes. We recently had occasion to determine the covering multiplicity of these lattices, that is, the maximal number of times the interiors of the covering spheres(1) can overlap [14]. We found that to do this we needed to understand the Voronoi and Delaunay polytopes(1) of these lattices in considerable detail. Much to our surprise, this information does not appear to be available anywhere in the literature, and we have decided to present it in this paper. Partial information is available in Coxeter's works (especially [15]-[20]) and in our book [13], and in a sense the present paper can be regarded as an extension to Chapter 21 of [13]. We shall offer no proofs, since the results, although hard to discover, are not difficult to verify once found. The results for the two hardest cases, the lattices E_6^* and E_7^*, depend on recent work of Worley [42], [43].

2. Lattices and polytopes

We begin with an informal discussion of some of the geometric notions associated with a lattice. (For further information see for example [13], [20], [36].) By an n-dimensional *lattice* Λ we mean a discrete abelian subgroup of $\mathbb{R}^n$, consisting of all integer combinations of n linearly independent vectors $v_1, \ldots, v_n \in \mathbb{R}^n$. The squared volume of the parallelepiped spanned by the v_i is called the *determinant* of Λ, denoted by det Λ. The *dual lattice* Λ^* is defined by

$$\Lambda^* = \{ x \in \mathbb{R}^n : x \cdot v \in \mathbb{Z} \text{ for all } v \in \Lambda \} ,$$

and $\det \Lambda^* = (\det \Lambda)^{-1}$. Many interesting lattices have the property that $\Lambda \subseteq \Lambda^*$; these are the *classically integral* lattices, and in this case the determinant of Λ equals the order of the quotient group Λ^*/Λ.

If the minimal distance between lattice points is $2r$, solid spheres of radius r drawn around the lattice points will just touch. The set of all such spheres forms an n-dimensional *sphere packing*, and r is called the *packing radius* of the lattice. The *density* Δ of this packing is the fraction of $\mathbb{R}^n$ occupied by these spheres, and is given by the formula

$$\Delta = \frac{V_n r^n}{\sqrt{\det \Lambda}} ,$$

where $V_n = \pi^{n/2}/(n/2)!$ is the volume of an n-dimensional sphere of radius 1. It is a classical problem to find the lattices that maximize Δ – we mention some of the results in Table III below.

If the radius of the spheres is increased (allowing them to overlap) until they just *cover* the whole space, we obtain a *sphere-covering* of $\mathbb{R}^n$. The radius R (say) of the spheres when this happens for the first time is called the *covering radius* of the lattice. The *covering density* or *thickness* $\boldsymbol{\theta}$ of Λ is

$$\boldsymbol{\theta} = \frac{V_n R^n}{\sqrt{\det \Lambda}} .$$

Another classical problem is to find the lattices that minimize $\boldsymbol{\theta}$. (Except for dimensions 1, 2 and 24 the answers to the two problems appear to be different – see [13] and Table III.) Figure 1 shows the packing and covering obtained from the familiar planar hexagonal lattice.

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(1) These terms are defined in the next section.

Figure 1. (a) The hexagonal lattice A_2 (small circles) and the associated packing of circles. (b) The corresponding covering.

The *Voronoi polytope* $V(u)$ centered at a lattice point $u \in \Lambda$ consists of the points

$$V(u) = \{ x \in \mathbb{R}^n : N(x-u) \le N(x-v) \text{ , all } v \in \Lambda \} ,$$

where $N(x) = x \cdot x$ denotes the squared length or *norm* of $x \in \mathbb{R}^n$. The sphere of radius r centered at u (a packing sphere) is the inscribed sphere in $V(u)$, while the sphere of radius R at u (a covering sphere) is the circumsphere around $V(u)$. If the lattice points are used as codewords in a communication system, the polytopes $V(u)$ are the decoding regions: any point $x \in V(u)$ should be decoded as u.

The Voronoi polytopes $V(u)$, $u \in \Lambda$, are all congruent, and we shall usually just study $V(0)$, the polytope containing the origin. $V(0)$ is a convex centrally-symmetric polytope, and the $V(u)$, $u \in \Lambda$, form a tessellation of $\mathbb{R}^n$ by copies of $V(0)$.

The vertices of the Voronoi polytopes are especially interesting points of $\mathbb{R}^n$: they are the *holes* in the lattice, i.e. the points of $\mathbb{R}^n$ that are locally maximally distant from the lattice. In particular the vertices of $V(u)$ at distance R from u are the *deep holes* in Λ: they are the points that are globally maximally distant from the lattice.

Figure 2. (a) The hexagonal lattice A_2 (small circles) and the Voronoi polygons (hexagons). (b) The corresponding Delaunay polygons (equilateral triangles in two orientations).

Let $h \in \mathbb{R}^n$ be a hole in Λ. The convex hull of the lattice points closest to h is called the *Delaunay polytope* containing h. The Delaunay polytopes form a second tessellation of $\mathbb{R}^n$ into convex polytopes (dual to the Voronoi tessellation). Figure 2 shows the two tessellations in the case of the hexagonal lattice.

Another important polytope associated with a lattice is its *contact polytope:* take one of the spheres in the sphere packing, find all the points where neighboring spheres touch it, and form their convex hull.[(2)]

The number of vertices of this contact polytope is thus the number of spheres that touch one sphere in the packing (the *kissing number* of the lattice). If the Voronoi polytope $V(0)$ has just τ walls (where τ is the kissing number), one wall bisecting the line from 0 to each of the neighboring lattice points, then the contact polytope and $V(0)$ are *dual* polytopes (cf. [29]).

The lattice itself determines an infinite polytope or *honeycomb*, in which the vertices are the lattice points, the edges join lattice points at distance $2r$ apart, etc., and the n-dimensional cells are the Delaunay polytopes.

There are many questions one can ask about lattices. In coding applications for example one wishes to maximize the probability of correct decoding, which is proportional to

$$\int_{V(0)} e^{-x^2/2\sigma^2}\, dx$$

for a given $\sigma > 0$ (see [6]), while for applications to quantization or data compression one wishes to minimize the second moment of $V(0)$, which is proportional to

$$\int_{V(0)} x \cdot x \, dx \, .$$

The latter expression has been evaluated for many lattices in [12], [13], [42], [43].

A problem of recent interest (Sullivan [40]) is to find the lattices of smallest covering multiplicity. The *covering multiplicity* $CM(\Lambda)$ is the maximal number of times that the interiors of the covering spheres overlap. As illustrated in Figure 2, the covering multiplicity of the hexagonal lattice is 2. The problem is to find the minimal covering multiplicity of any n-dimensional lattice. In [14] we show that this is equal to n for all $n \le 8$ – again see Table III. We conjecture that it exceeds n in all other cases.

In the present paper we are concerned with the lattices I_n, A_n, D_n, E_n and their duals. The lattice I_n (or $\mathbb{Z}^n$), $n \ge 1$, is the n-dimensional *cubic lattice*, and consists of all vectors $(u_1, \ldots, u_n)$ with integer coordinates u_i. This lattice is self-dual: $I_n^* = I_n$. The other lattices are defined in Sections 4-10, but for convenience we give a summary of their properties in Table I. For each lattice Λ the table gives the order g of its point group (factorized as $g_0 \cdot g_1$, where g_0 is the order of the corresponding Weyl group), the determinant det Λ, the squared packing and covering radii r^2 and R^2, and representative vectors for the cosets of Λ in Λ^*. All of I_n, A_n, D_n, E_n are classically integral, and we use $[i]$ to denote a minimal vector in the i^{th} coset of Λ in Λ^*, for $i = 0, 1, \ldots,$ det (Λ). Then the dual lattice is given by

$$\Lambda^* = \bigcup_{i=0}^{d-1} ([i] + \Lambda) \, , \quad \text{where} \quad d = \det \Lambda \, .$$

Further information about a lattice Λ is provided by its *theta series*, given by

$$\theta_\Lambda(q) = \sum_{u \in \Lambda} q^{N(u)} \, , \quad |q| < 1 \, ,$$

$$= \sum_r M_r \, q^r \, ,$$

where M_r is the number of lattice points of norm r. Analytical expressions (in terms of Jacobi theta functions) for the theta series of all these lattices can be found in Chapter 4 of [13]. Note that for I_n (the simplest of our lattices), the coefficient M_r is equal to the number of ways of writing r as a sum of n squares, a remark which hints at the enormous number–theoretic aspects of lattice theory. For further information see for example [10], [13], [27], [39].

For our present purposes the first few terms of the theta series will suffice: these are given in Table II. Note in particular that the second term, $M_\mu q^\mu$ say, supplies both the minimal nonzero norm μ in the lattice,

(2) By ‘‘*the*’’ *contact n-polytope* we mean the contact polytope for the densest lattice in $\mathbb{R}^n$. For $n = 1, \ldots, 8$ the contact n-polytope is respectively an interval, hexagon, cuboctahedron, regular polytope $\{3,4,3\}$, ambo-orthoplex, 1_{22}, 2_{31} and the Gosset polytope 4_{21}.

Table I. Basic properties of root lattices and their duals.

Λ	$g = g_0 \cdot g_1$	$\det \Lambda$	r^2	R^2	Coset representatives for Λ^* / Λ
I_n	$2^n \cdot n!$	1	$\frac{1}{4}$	$\frac{n}{4}$	$[0] = (0^n)$
A_n	$(n+1)! \cdot 2$ $(n \ge 2)$	$n+1$	$\frac{1}{2}$	$\frac{a(n+1-a)}{n+1}$, $a = \left[\frac{n+1}{2} \right]$	$[i] = \left(\left(\frac{j}{n+1} \right)^i , \left(\frac{-i}{n+1} \right)^j \right)$ $0 \le i \le n$, $i + j = n+1$
A_n^*		$\frac{1}{n+1}$	$\frac{n}{4n+4}$	$\frac{n(n+2)}{12(n+1)}$	
D_n	$192 \cdot 3!$ $(n=4)$ $2^{n-1} n! \cdot 2$ $(n \ge 5)$	4	$\frac{1}{2}$	$\frac{n}{4}$ $(n \ge 4)$	$[0] = (0^n)$, $[1] = \left(\left(\frac{1}{2} \right)^n \right)$ $[2] = (0^{n-1}, 1)$, $[3] = \left(\left(\frac{1}{2} \right)^{n-1}, -\frac{1}{2} \right)$
D_n^*		$\frac{1}{4}$	$1 (n \ge 4)$	$\frac{n}{8}$ (n even ≥ 4) $\frac{2n-1}{16}$ (n odd ≥ 5)	
E_6	$2^7 3^4 5 \cdot 2$	3	$\frac{1}{2}$	$\frac{4}{3}$	$[0] = (0^8)$, $[2] = -[1]$, $[1] = \left(0; \left(-\frac{2}{3} \right)^2, \left(\frac{1}{3} \right)^4; 0 \right)$
E_6^*		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	
E_7	$2^{10} 3^4 5\ 7 \cdot 1$	2	$\frac{1}{2}$	$\frac{3}{2}$	$[0] = (0^8)$, $[1] = \left(\left(\frac{3}{4} \right)^2, \left(-\frac{1}{4} \right)^6 \right)$
E_7^*		$\frac{1}{2}$	$\frac{3}{8}$	$\frac{7}{8}$	
E_8	$2^{14} 3^5 5^2 7 \cdot 1$	1	$\frac{1}{2}$	1	$[0] = (0^8)$

as well as the number of minimal vectors M_μ (the kissing number, i.e. the number of vertices of the contact polytope).

One of the chief reasons for our interest in these lattices is that they provide the best answers presently known (and often the best possible answers) to many common questions about lattices in low dimensions. To illustrate this Table III gives, in dimensions $n \le 9$, the lattices which provide the densest packings, thinnest coverings, and have the smallest covering multiplicities. The question marks indicate that these entries are only conjectured to be optimal. The results in the first row of Table III are primarily due to Blichfeldt [see [4], [41]), and in the second row to Ryskov and Baranovskii [37]. Λ_9 (in the first row) is a *laminated lattice* – see Chapter 6 of [13]. For the third row of Table III see [14]. The lattices in this row are not unique, since small perturbations usually do not change the covering multiplicity. For information about packings and coverings in higher dimensions, as well as what happens when nonlattice arrangements

Table II. Theta series.

$$I_n: 1 + 2nq + 2^n \binom{n}{2} q^2 + 2^3 \binom{n}{3} q^3 + \left\{ 2^4 \binom{n}{4} + 2n \right\} q^4 + \cdots$$

$$A_n: 1 + 2\binom{n+1}{2} q^2 + \binom{n+1}{4}\binom{4}{2} q^2 + \left\{ \binom{n+1}{6}\binom{6}{3} + 6\binom{n+1}{3} \right\} q^6 + \cdots$$

$$A_n^*: 1 + \sum_{i=1}^{n} \binom{n+1}{i} q^{i(n+1-i)/(n+1)} + 2\binom{n+1}{2} q^2 + \cdots$$

$$D_n: 1 + 4\binom{n}{2} q^2 + \left\{ 2^4 \binom{n}{4} + 2n \right\} q^4 + \left\{ 2^6 \binom{n}{6} + 3 \cdot 2^3 \binom{n}{3} \right\} q^6 + \cdots$$

$$D_n^*: 1 + 2nq + 2^2 \binom{n}{2} q^2 + 2^3 \binom{n}{3} q^3 + \left\{ 2^4 \binom{n}{4} + 2n \right\} q^4 + \cdots$$
$$+ 2^n q^{n/4} + 2^n n q^{(n+8)/4} + 2^n \binom{n}{2} q^{(n+16)/4} + \cdots$$

$$E_6: 1 + 72q^2 + 270q^4 + 720q^6 + 936q^8 + \cdots$$

$$E_6^*: 1 + 54q^{4/3} + 72q^2 + 432q^{10/3} + 270q^4 + \cdots$$

$$E_7: 1 + 126q^2 + 756q^4 + 2072q^6 + 4158q^8 + \cdots$$

$$E_7^*: 1 + 56q^{3/2} + 126q^2 + 576q^{7/2} + 756q^4 + \cdots$$

$$E_8: 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + \cdots$$

of spheres are permitted, the reader is referred to [13].

Table III. The optimal lattices.

Dimension	1	2	3	4	5	6	7	8	9
Densest packing	I_1	A_2	A_3	D_4	D_5	E_6	E_7	E_8	Λ_9?
Thinnest covering	I_1	A_2	A_3^*	A_4^*	A_5^*	A_6^*?	A_7^*?	A_8^*?	A_9^*?
Smallest covering multiplicity and lattice	1 I_1	2 I_2, A_2	3 A_3^*	4 A_4^*, D_4	5 A_5^*	6 A_6^*, E_6^*	7 A_7^*	8 A_8^*	11? A_9^*?

In later sections we shall encounter a number of different polytopes. To help distinguish them we have decided to give ‘‘English names’’ to many more polytopes than usual. What we call the Schläfli, Hesse and Gosset polytopes probably all first appeared (as polytopes) in Gosset [26]. However, we have used the names Schläfli and Hesse to recall well-known tactical configurations described by those authors (see Sections 9, 10). We use n-polytope to mean an n-dimensional polytope. The $(n-1)$-dimensional faces of an n-polytope are called its *cells*.

Table IV summarizes the most important polytopes encountered.

3. Diagrams and graphs

In the 1930’s Coxeter [15], [16] classified all irreducible discrete groups generated by reflections; the list is given on page 196 of [20]. (For the history of this theorem see [5, p. 237], [20, p. 209], [21, p. 122], [34].) The groups are classified by diagrams introduced by Coxeter, now generally called Coxeter-Dynkin diagrams, which provide (among many other things) a set of defining relations for the groups.

For our application to the study of lattices we need only consider those groups on this list in which the angles between the reflecting hyperplanes are always 60° or 90°. The others either produce the same lattices or (the ‘‘non-crystallographic’’ groups) do not correspond to lattices at all. Thus we need only

Table IV. Summary of polytopes.

Number of vertices	Dim.	Symbol	Name
$n+1$	n	n	simplex
$2n$	n	$(n-3)_{1,1}$	orthoplex
2^{n-1}	n	$1_{n-3,1}$	hemicube
16	7	–	diplo-simplex = Delaunay polytope for E_7^*
27	6	2_{21}	Schlafli polytope = Delaunay polytope for E_6
54	6	–	diplo-Schlafli polytope = Voronoi polytope for E_6
56	7	3_{21}	Hesse polytope = contact polytope of E_7^*
72	6	1_{22}	contact 6-polytope
126	7	2_{31}	contact 7-polytope
240	8	4_{21}	Gosset polytope = contact 8-polytope
576	7	1_{32}	Voronoi polytope of E_7^*
720	6	–	ambo contact 6-polytope = Voronoi polytope of E_6^*
2160	8	2_{41}	‘‘deep hole polytope’’ of E_8
17280	8	1_{42}	‘‘shallow hole polytope’’ of E_8

consider the groups defined by the diagrams(3) $a_n (n \ge 1)$, $A_n (n \ge 2)$, $d_n (n \ge 3)$, $D_n (n \ge 4)$, $e_n (n = 6, 7, 8)$ and $E_n (n = 6, 7, 8)$. These are shown in Figure 3.

The rules for constructing the reflection group from the Coxeter-Dynkin diagram can be found in [5], [11], [13], [20], [21], [28], etc., and we do not reproduce them here. The groups defined by the diagrams A_n, D_n, E_n are infinite and are called the *affine* (or *Euclidean*) *Weyl groups* $W(A_n)$, $W(D_n)$, $W(E_n)$. The a_n, d_n and e_n diagrams definite the *finite* (or *spherical*) *Weyl groups* $W(a_n)$, $W(d_n)$, $W(e_n)$, of orders $(n+1)!$ (for a_n), $2^{n-1} n!$ (for d_n), $2^7 3^4 5$ (for e_6), $2^{10} 3^4 5\ 7$ (for e_7) and $2^{14} 3^5 5^3 7$ (for e_8).

The same Coxeter-Dynkin diagrams also represent simplexes (in Euclidean or spherical space) which are fundamental regions for these groups – see the references above, especially Chapters 4 and 21 of [13]. For example, Figures 5 and 13 below give fundamental simplexes for the groups $W(A_n)$ and $W(D_n)$.

By circling certain nodes, the diagrams are also used to represent polytopes (if the group is finite), or honeycombs and lattices (if the group is infinite). In this notation, used by Coxeter and others ([18], [20, p. 196], [44]) a diagram with a single circled node represents the set of images of the corresponding vertex of the fundamental simplex under the group. A diagram containing several circles represents the orbit of a suitably chosen point in the interior of the convex hull of the vertices represented by the individual circles. (In this paper we shall meet only a few instances when there is more than one circle.)

In particular, *the root lattices A_n, D_n, E_n are the vertices of the honeycombs obtained by circling* (*or starring*) *a single node in the A_n, D_n, E_n diagrams*, as shown in Figures 6, 14(M), 17(M), 19(M) and 23(M). The finite Weyl groups $W(a_n)$, $W(d_n)$, $W(e_n)$, when extended by the automorphism groups of the corresponding diagrams a_n, d_n, e_n, yield the point groups of the root lattices A_n, D_n, E_n, respectively (see column 2 of Table I).

We follow Coxeter in using the symbol n_{ij} to denote the polytope or honeycomb specified by the diagram with a single circle shown in Figure 4. Then the root lattices E_6, E_7 and E_8 are the vertices of the honeycombs 5_{21}, 3_{31} and 2_{22} respectively.

(3) The diagrams that we call a_n, d_n, e_n, A_n, D_n, E_n, are more usually called A_n, D_n, E_n, $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_n$, respectively. Since our chief interest is in the honeycombs we have preferred to adopt this slightly unusual terminology (used also in Chapter 23 of [13].)

Figure 3. Coxeter-Dynkin diagrams.

Figure 4. The polytope or honeycomb n_{ij}.

4. The lattice $\mathbf{A_n}$ $(\mathbf{n \geq 2})$

The root lattice A_n $(n \geq 2)$ consists of all vectors $(u_0, u_1, \ldots, u_n)$ for which the u_i are integers satisfying $u_0 + u_1 + \cdots + u_n = 0$.[(4)] (Thus we are using $n+1$ coordinates to define an n-dimensional lattice.) Coordinates for the vertices of a fundamental simplex are given in Figure 5.

Figure 5. Fundamental simplex for A_n. Also $[i]$ denotes the particular vector $\left(\left(\frac{j}{n+1} \right)^{j}, \left(\frac{-i}{n+1} \right)^{j} \right)$ where $i+j = n+1$.

The lattice points are the vertices of the honeycomb represented in the circle notation by the diagram in Figure 6. That is, the lattice points consist of the set of images of the circled node of the fundamental simplex under the affine Weyl group $W(A_n)$.

By Coxeter's rule [26], [3, p. 197], the holes in this lattice, as well as the associated Delaunay polytopes, can be found as follows. The holes are the images under $W(A_n)$ of all the nodes of Figure 6 whose removal (together with their adjacent edges) does not disconnect any node from the circled node. All nodes except the circled one satisfy this condition, so the holes in A_n are the images under $W(A_n)$ of the vertices $[i]$, $1 \leq i \leq n$, of the fundamental simplex.

We describe the set of all these holes using a modification of the circle notation, shown in Figure 7. In this notation a diagram with a single star has the same meaning as that with a single circled node, and

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(4) Although this definition is also valid for $n=0$ and 1 (in particular $A_1 = \sqrt{2}\, I_1$), the diagrams would need special treatment in these cases.

Figure 6. Diagram for the root lattice A_n.

Figure 7. Holes in the root lattice A_n.

henceforth we shall prefer this notation. For example the points of A_n are now indicated as in Figure 8. A diagram with several stars denotes the union of the sets of points indicated by the individual stars.

Figure 8. Star diagram for the root lattice A_n.

The Delaunay polytope corresponding to a particular hole is obtained by deleting the corresponding node from Figure 6. So for A_n the Delaunay polytopes are as shown in Figure 9.

The vertices of a typical Delaunay polytope consist of the lattice vectors nearest to a starred vertex $[i]$ $(1 \leq i \leq n)$ in Figure 7. Equivalently, these are the points of the coset $A_n - [i]$ closest to the origin, and consist of all $\binom{n+1}{i}$ permutations of

$$\left(\left(\frac{-j}{n+1} \right)^i , \left(\frac{i}{n+1} \right)^j \right) , \quad j = n+1-i \, .$$

These are the midpoints of the $(i-1)$-dimensional faces of a regular simplex, so we call this Delaunay

Figure 9. Delaunay polytopes of A_n.

polytope an $(i-1)^{\text{st}}$-*order ambo-simplex*.(5) The 0^{th}-order ambo-simplex is a simplex (whose vertices are the images of the vector [1] under $W(a_n)$). By *the* ambo-simplex we mean the first-order ambo-simplex, whose vertices are the midpoints of edges of a simplex (for example the vector [2] and its images). The second-order ambo-simplex is the convex hull of the midpoints of the two-dimensional faces of a simplex (for example [3] and its images), and so on. The *ambo-polytope* and *higher-order ambo-polytopes* for any sufficiently regular polytope are defined similarly. For example an ambo-tetrahedron is an octahedron and an ambo-cube is a cuboctahedron. The typical ambo-polytope of arbitrary order for a regular polytope is an intersection of suitably scaled versions of both that polytope and its dual.

Of course the holes (indicated in Figure 7) are also the vertices of the Voronoi tessellation corresponding to A_n. The vertices of the Voronoi cell $V(0)$ around the origin are the images of the starred nodes in Figure 7 under the finite Weyl group $W(a_n)$, consisting of all permutations of the $n+1$ coordinates. There are $\binom{n+1}{1} + \binom{n+1}{2} + \cdots + \binom{n+1}{n} = 2^{n+1} - 2$ vertices. Thus $V(0)$ is represented by the diagram shown in Figure 10. It is the convex hull of the union of ambo-simplexes of all orders.

Figure 10. Voronoi cell $V(0)$ for A_n.

Examples. The Delaunay polygons for the hexagonal lattice A_2 are triangles and ambo-triangles (which are inverted triangles) – see Figure 2b. The Voronoi polygon, a regular hexagon, is the convex hull of the union of two triangles – see Figure 2a. For A_3, the face-centered cubic lattice, the Delaunay polyhedra are tetrahedra, octahedra (ambo-tetrahedra), and inverted tetrahedra (second-order ambo-tetrahedra). The Voronoi polyhedron is the convex hull of the union of three such polyhedra, namely a rhombic dodecahedron with $4+6+4 = 14$ vertices.

(5) The prefix ambo- can mean either ''edge'' (or ''rim''), as in the Greek ''αμβων'', or ''both'', from the Latin ''ambo''.

The contact polytope of A_n is the convex hull of all points of the form $(1, -1, 0^{n-1})$. (This is an ambo-diplo-simplex in the notation introduced later.) This somewhat exceptional polytope can be represented by the diagram obtained by circling two ends of the a_n diagram (see Figure 11 (C)).

The Cell Schematic Diagram. We summarize the analysis in this section in what we shall call the *Cell Schematic* (or CS) *diagram* for A_n, shown in Figure 11. (Later sections will be largely in the form of CS diagrams.) Such a schematic consists of a diagram for the vertices of the mother honeycomb (M), followed by a diagram (H) describing the holes and then diagrams (D) for the Delaunay polytopes corresponding to the various holes. In some cases we also give diagrams for the contact polytope (C) and the Voronoi polytope $V(0)$ (labeled V). We shall sometimes give cell schematics for certain polytopes, in which case the diagram (H) describes the points of the cells furthest from the vertices, and the diagrams labeled (D) describe the cells.

Figure 11. Cell Schematic diagram for A_n.

Usually H is obtained from M by starring the nodes furthest in the graph from those starred in M. When this is the case the diagrams D are obtained from M by individually omitting the vertices that were starred in H. Again, C and V are usually obtained from M and H by omitting a vertex that was starred in M. However, these rules are only guidelines, not universal truths, and we occasionally see exceptions such as Figure 11 (C). The true solution to any particular problem of this type involves finding the points of the fundamental simplex furthest from some given ones. Often these desired points will themselves be vertices of the fundamental simplex, but not always, as we shall see in the next section. We have found the computer programs MINOS [35] and AMPL [25] useful for solving particular instances of these problems (see [14]).

Summary for $\mathbf{A_n}$. *The Delaunay polytopes are ambo-simplexes of all orders, the Voronoi polytope $V(0)$ is the convex hull of the union of ambo-simplexes of all orders, and the contact polytope is an ambo-diplo-simplex. The squared packing and covering radii are respectively ½ and $N([a]) = a(n+1-a)/(n+1)$, where $a = [(n+1)/2]$.*

5. The lattice $\mathbf{A_n^*}$, $\mathbf{n \geq 2}$

The dual lattice to A_n is the weight lattice A_n^*, consisting of all vectors $(u_0, u_1, \ldots, u_n)$ where $u_0 + u_1 + \cdots + u_n = 0$, $u_i \in \frac{1}{n+1} \mathbb{Z}$, and $u_0 \equiv u_1 \equiv \cdots \equiv u_n$ (mod 1). A_n^* is the union of $n+1$ cosets of A_n, represented by the vectors $[i]$ shown in Figure 5. Since each of these cosets is one orbit under $W(A_n)$, A_n^* is represented by the first figure (M) in the CS diagram shown in Figure 12.

Figure 12. Cell Schematic diagram for A_n^*.

In other words, A_n^* consists of the images of all vertices of the fundamental simplex for A_n under $W(A_n)$. Obviously the centroid of the fundamental simplex,

$$P = \frac{1}{n+1} \left(-\frac{n}{2}, 1-\frac{n}{2}, 2-\frac{n}{2}, \ldots, \frac{n}{2}-2, \frac{n}{2}-1, \frac{n}{2} \right)$$

is at the maximal distance from its vertices, and so this point and its images under $W(A_n)$ are the holes in A_n^*. In the circle notation this is represented by Figure 12 (H).

The vertices of the Voronoi polytope $V(0)$ are the images of P under the finite group $W(a_n)$. Since this group consists of all permutations of the $n+1$ coordinates, this Voronoi polytope is sometimes called a *permutohedron* [13, p. 472].

The Delaunay polytope centered at P *is* the fundamental simplex. We adopt the convention that an uncircled and unstarred diagram (D) simply represents that simplex.

The minimal vectors of A_n^* are the $2n+2$ points of the form

$$\pm \left(\frac{n}{n+1}, \left(\frac{-1}{n+1} \right)^n \right)$$

The contact polytope (Figure 12 (C)) is the convex hull of these points. We call this a *diplo-simplex*,(6) since its vertices are the vertices of two oppositely oriented regular simplexes. The two-dimensional diplo-simplex is a regular hexagon, and the three-dimensional diplo-simplex is a cube. Both these figures tessellate space. We find it remarkable that, as we shall see in Section 9, the seven-dimensional diplo-simplex also tessellates space.

Summary for $\mathbf{A_n^*}$. *The Delaunay polytope is the fundamental simplex, the Voronoi polytope $V(0)$ is a permutohedron, and the contact polytope is a diplo-simplex. The squared packing and covering radii are respectively* $\frac{1}{4} N([1]) = n/(4n+4)$ *and* $N(P) = n(n+2)/12(n+1)$.

6. The lattice $\mathbf{D_n}$, $\mathbf{n \geq 4}$

The root lattice D_n, $n \geq 4$ consists of all vectors $(u_1, \ldots, u_n)$ for which the u_i are integers with an even sum.(7) A fundamental simplex is shown in Figure 13 and the CS diagram in Figure 14.

Figure 13. Fundamental simplex for D_n.

Two of the Delaunay polytopes are *hemicubes*, consisting of alternate vertices of a cube. For example the vertices of the Delaunay polytope containing $[1] = \left(\frac{1}{2}^n \right)$ are the 2^{n-1} points of the form $(1^{2k}, 0^{n-2k})$, $0 \leq k \leq n/2$.

The vertices of the Delaunay polytope containing $[2] = (1, 0^{n-1})$ are all the points $[2] + v$, when v is one of the $2n$ vectors $(\pm 1, 0^{n-1})$. This is a regular polytope which we shall call an n-dimensional

(6) The prefix diplo- means double. In general the vertices of the *diplo-polytope* of Π are the vertices of Π and its opposite polytope $-\Pi$.

(7) Although this definition is also valid for $n < 4$ (in particular $D_1 = 2I_1$, $D_2 \cong \sqrt{2}\, I_2$, $D_3 \cong A_3$, where $\cong$ indicates congruent lattices, differing only by a rotation), the diagrams would need special treatment in these cases.

orthoplex(8) (abbreviating *orthant-complex*), since it has one cell for each orthant of n-dimensional space. It is represented by the diagram $(n-3)_{11}$, in the notation of Section 3. It is remarkable that the four-dimensional orthoplex is the same polytope as the four-dimensional hemicube.

The Voronoi polytope $V(0)$ has vertices of the form $\left(\left(\pm \frac{1}{2} \right)^n \right)$ and $(\pm 1,\ 0^{n-1})$. This is a *pyramidal cube*, obtained from a hypercube by attaching a pyramid to each face. The outer vertex of the pyramid is obtained by reflecting the origin in the face. In three dimensions this construction produces a rhombic dodecahedron, the Voronoi polytope for $A_3 \cong D_3$ (Figure 15). In four dimensions it produces the regular polytope $\{3,4,3\}$.

The contact polytope for D_n has $2n(n-1)$ vertices $(\pm 1^2,\ 0^{n-2})$, and is an ambo-orthoplex. It is remarkable that the four-dimensional orthoplex is also a copy of $\{3,4,3\}$.

Summary for $\mathbf{D_n}$, n ≥ 4. *The Delaunay polytopes are hemicubes and orthoplexes, the Voronoi polytope is a pyramidal cube, and the contact polytope is an ambo-orthoplex. The squared packing and covering radii are respectively* ½ *and* $N([1]) = n/4$.

7. The lattice $\mathbf{D_n^*}$, n ≥ 5

The dual to the root lattice D_n is the weight lattice D_n^*, which is the union of the four cosets $D_n + [i]$, $0 \le i \le 3$. Since D_4^* is geometrically similar to D_4, we can suppose $n \ge 5$, and so avoid certain modifications that would be required for $n \le 4$. The CS diagram is given in Figure 16.

The holes in D_n^* are the images under $W(D_n)$ of that point P in the fundamental simplex which is at the maximal distance from the four vertices $[0], \ldots, [3]$. If $n = 2t$ is even, $P = \left(\frac{1}{2}^t,\ 0^t \right)$ is itself a vertex corresponding to the central node of the diagram. But if $n = 2t+1$ is odd, $P = \left(\frac{1}{2}^t,\ \frac{1}{4},\ 0^t \right)$ is the point midway between the two vertices represented by the middle nodes of the diagram.

The vertices of the Delaunay polytope centered at P are the lattice points nearest P. If $n = 2t$ these form two t-dimensional hypercubes

$$P + \left(\pm \frac{1}{2}^t,\ \ 0^t \right),$$

$$P + \left(0^t,\ \pm \frac{1}{2}^t \right),$$

in complementary t-spaces, so this Delaunay polytope is what we call the *join* of two hypercubes, illustrated for $t = 4$ in Figure 16. If $n = 2t+1$ the vertices again form two t-dimensional hypercubes

$$P + \left(\pm \frac{1}{2}^t,\ 0,\ \ 0^t \right),$$

$$P + \left(0^t,\ \frac{1}{4},\ \pm \frac{1}{2}^t \right),$$

in orthogonal t-spaces, but now their centers are separated by the vector $\left(0^t,\ \frac{1}{4},\ 0^t \right)$ orthogonal to both these t-spaces. We call this Delaunay polytope the *separated join* of two hypercubes. This situation is illustrated for $t = 3$ in Figure 16 (D), using an *ad hoc* notation to represent the separated join.

The contact polytope (for $n \ge 5$) is an orthoplex with vertices $(\pm 1,\ 0^{n-1})$.

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(8) Other names are *generalized octahedron* or *cross-polytope.*

Figure 14. Cell Schematic diagram for D_n.

Summary for $\mathbf{D_n^*}$, n ≥ 4. *The Delaunay polytopes for D_n^* are the joins or separated joins of two $\left[\frac{n}{2}\right]$-dimensional hypercubes, according as n is even or odd. The Voronoi polytope is as described in Figure 16, and the contact polytope is an orthoplex. The squared packing and covering radii are respectively* $\frac{1}{4} N([2]) = 1$ *and* $N(P) = \frac{n}{8}$ *(n even) or* $\frac{2n-1}{16}$ *(n odd).*

8. The lattice $\mathbf{E_8}$ and the Gosset polytope $\mathbf{4_{21}}$

The 8-dimensional root lattice E_8 consists of all vectors $(u_0, \ldots, u_7)$ for which the u_i are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and are such that $u_0 + \cdots + u_7$ is even. E_8 is self-dual, so that the weight lattice of E_8 is the same lattice. To save space we shall not give coordinates for the fundamental simplexes for E_6, E_7 or E_8; they can be found on page 461 of [13].

The CS diagram is given in Figure 17. There are two types of holes in E_8, deep holes such as $(0^7, 1)$ and shallow holes such as $\left(\left(\frac{1}{3}\right)^7, -\frac{1}{3}\right)$

The vertices of the Delaunay polytope centered at the hole $P = (0^7, 1)$ are all the points $P + (\pm 1, 0^7)$, and form an orthoplex. The other type of Delaunay polytope is a simplex, which is more easily described in an alternative coordinate system for E_8 obtained by negating the final coordinate.[9] In these coordinates

Figure 15. The Voronoi polytope for $A_3 \cong D_3$, a rhombic dodecahedron, may be obtained by attaching a pyramid to each face of a cube (dashed lines).

the hole is $Q = \left(\frac{1}{3}^{8} \right)$ and the nearest lattice points are the origin and the eight vertices of shape $\left(-\frac{1}{2}, \left(\frac{1}{2}^{7} \right) \right)$ the convex hull of these nine points is a simplex.

Note that P is $\frac{1}{2}$ of a vector of norm 4 in E_8 and Q is $\frac{1}{3}$ of a primitive[10] vector of norm 8. The Weyl group $W(e_8)$ is transitive on these two types of vectors. Thus the vertices of the Voronoi polytope $V(0)$ are $\frac{1}{2} v_4$ and $\frac{1}{3} v_8$, where $v_4 \in E_8$ is any norm 4 vector and v_8 is any primitive norm 8 vector. The images of these vectors under $W(E_8)$ are all the vertices of the Voronoi tessellation, i.e. the holes in the lattice.

(9) This is the *odd coordinate system* for E_8 [13, p. 120].

(10) I.e. not the double of a norm 2 vector.

Figure 16. Cell Schematic diagram for D_n^*, $n \geq 5$. (Note that $D_4^* \cong D_4$.)

Figure 17. Cell Schematic diagram for E_8.

The contact 8-polytope (the contact polytope of E_8) is the vertex figure[(11)] 4_{21} of the honeycomb 5_{21} (see Figure 17 (C)). We call this the *Gosset polytope*, after Gosset [29]. Apart from a scale factor it is the convex hull of the 240 minimal vectors of E_8. Its CS diagram is given in Figure 18. The numbers of cells of each type are easily found by the method given in [3, § 11.8]. Further calculations of the same type (see [3, p. 204]) give the numbers of faces of every dimension; these are shown in Table V.

Table V. *d*-dimensional faces of Gosset polytope 4_{21}.

d	Face	Number
8	Gosset 4_{21}	1
7	7-orthoplex	2160
	7-simplex	17280
6	6-simplex	69120 + 138240
5	5-simplex	483840
4	4-simplex	483840
3	tetrahedron	241920
2	triangle	60480
1	edge	6720
0	vertex	240

Table VI shows how the 240 vertices are distributed among the sections, starting at a vertex, simplex or orthoplex. It is convenient to use different coordinate systems for the different sections. In this table $\left(\frac{1}{2}^{n+}\right)$ indicates a vector $\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$ with n components, the product of whose signs is positive.

(11) For definition see [3, p. 128].

Figure 18. Cell Schematic diagram for the 8-dimensional Gosset polytope 4_{21}.

Similarly for $\left(\frac{1}{2}^{n-} \right)$ only now the product of the signs must be negative.

Summary for $\mathbf{E_8}$. *The holes are the images of* $\frac{1}{2} v_4$ *and* $\frac{1}{3} v_8$ *under* $W(E_8)$*, where* $v_4 \in E_8$ *and* $v_8 \in E_8$ *are vectors of norm 4 and primitive vectors of norm 8 respectively. The vertices of the Voronoi polytope* $V(0)$ *are the images of* $\frac{1}{2} v_4$ *and* $\frac{1}{3} v_8$ *under* $W(e_8)$ *and form a polytope dual to the Gosset polytope. The corresponding Delaunay polytopes are an orthoplex and an 8-simplex. The contact 8-polytope is the Gosset polytope* 4_{21}*. The squared packing and covering radii are respectively ½ and 1.*

9. The lattices $\mathbf{E_7}$, $\mathbf{E_7^*}$ and the Hesse polytope $\mathbf{3_{21}}$

The 7-dimensional root lattice E_7 is best defined to consist of those vectors of E_8 that are perpendicular to a given minimal vector $v \in E_8$. According as we take v to be

$$\left(\frac{1}{2}^{8} \right), \quad (1, -1, 0^6), \quad \text{or} \quad (1, 1, 0^6) ,$$

we get three different coordinate systems for E_7: E_7 consists of those vectors $(u_0, \ldots, u_7) \in E_8$ that satisfy

$$\Sigma u_i = 0, \quad u_1 = u_2 , \quad \text{or} \quad u_1 + u_2 = 0 ,$$

respectively. The CS diagram is given in Figure 19.

There are two types of holes in E_7, deep holes such as $P = \left(\frac{3}{4}^{2}, -\frac{1}{4}^{6} \right)$ and shallow holes such as $Q = \left(\frac{7}{8}, -\frac{1}{8}^{7} \right)$ in the first coordinate system. The Delaunay polytope centered at Q has eight vertices, (0^8) and $(1; -1\ 0^6)$, and is a simplex.

The Delaunay polytope 3_{21} centered at P has 56 vertices, consisting of all points of the form

$$P \pm \left(\frac{3}{4}^{2}, -\frac{1}{4}^{6} \right)$$

We call this polytope 3_{21} (discussed by Gosset) the *Hesse polytope*, because its 28 diameters have the same symmetries as the configuration of 28 bitangents to the general plane quartic curve studied by Hesse [32] in 1855. Since Hesse and Cayley these bitangents have been labeled by unordered pairs from a set of eight objects, and the Hesse group ($Sp_6(2) \cong O_7(2)$) consists of all permutations of the 28 bitangents that

Table VI. Sections of Gosset polytope 4_{21}.

Coordinates	Number	Shape
$\frac{1}{2}^{8}$	1	zenith
$\frac{1}{2}^{6}\ -\frac{1}{2}^{2}$ and $1^2 0^6$	56	Hesse polytope
$\frac{1}{2}^{4}\ -\frac{1}{2}^{4}$ and $1\ -1\ 0^6$	126	contact 7-polytope
$\frac{1}{2}^{2}\ -\frac{1}{2}^{6}$ and $-1^2\ 0^6$	56	Hesse polytope
$-\frac{1}{2}^{8}$	1	nadir
$\frac{1}{2}^{7}\ -\frac{1}{2}$	8	simplex
$1^2\ 0^6$	28	ambo-simplex
$\frac{1}{2}^{5}\ -\frac{1}{2}^{3}$	56	2^{nd} ambo-simplex
$1\ -1\ 0^6$	56	ambo-diplo-simplex
$\frac{1}{2}^{3}\ -\frac{1}{2}^{5}$	56	2^{nd} ambo-simplex
$-1^2\ 0^6$	28	ambo-simplex
$\frac{1}{2}\ -\frac{1}{2}^{7}$	8	simplex
$1;\ \pm 1\ 0^6$	14	orthoplex
$\frac{1}{2};\ \frac{1}{2}^{7+}$	64	hemicube
$0;\ \pm 1^2\ 0^5$	84	ambo-orthoplex
$-\frac{1}{2};\ \frac{1}{2}^{7-}$	64	hemicube
$-1;\ \pm 1\ 0^6$	14	orthoplex

preserve the 315 quadruples such as $\{ab, bc, cd, da\}$ and $\{ab, cd, ef, gh\}$, where $a, \ldots, h$ are distinct. The bitangent ij in this notation corresponds to the diameter

$$\pm \left(\frac{3}{4}^{2},\ -\frac{1}{4}^{6} \right)$$

with the $\frac{3}{4}$ in the i^{th} and j^{th} coordinates. Since the 28 points of shape $\left(\frac{3}{4}^{2},\ -\frac{1}{4}^{6} \right)$ form an ambo-simplex, the Hesse polytope might also be called a diplo-ambo-simplex, but this would be misleading, since the polytope has much more symmetry than this name would suggest. The Hesse group is described for

Figure 19. Cell Schematic diagram for E_7.

example in [11, p. 46]. See also [2], [15], [26]. The CS diagram for the Hesse polytope is given in Figure 20, and the numbers of faces of each dimension are given in Table VII (after Coxeter [15, p. 7]).

Figure 20. Cell Schematic diagram for Hesse polytope 3_{21}.

Table VII. *d*-dimensional faces of Hesse polytope 3_{21}.

d	Face	Number
7	Hesse 3_{21}	1
6	6-orthoplex	126
	6-simplex	576
5	5-simplex	2016 + 4032
4	4-simplex	12096
3	tetrahedron	10080
2	triangle	4032
1	edge	756
0	vertex	56

The contact polytope for E_7 (i.e. the contact 7-polytope, see Figure 19 (C)) is the vertex figure 2_{31} of the honeycomb 3_{31}, and is described in Figure 21. Its 126 vertices are the centers of the orthoplex cells of the Hesse polytope.

Tables VIII and IX show the vertices of the Hesse and contact 7-polytope distributed by sections in the three appropriate directions.

Figure 21. Cell Schematic diagram for contact 7-polytope.

The weight lattice E_7^* is the union of the two cosets E_7 and $E_7 + [1]$, where (in the first coordinate system) $[1] = \left(\left(\frac{3}{4}\right)^{2}, \left(-\frac{1}{4}\right)^{6} \right)$ Its CS diagram is shown in Figure 22, and E_7^* itself is described in Figure 22 (M).

Worley [43] showed that the unique point R of the fundamental simplex most distant from the starred vertices in Figure 22 (M) is the starred vertex in Figure 22 (H), so that the honeycomb 1_{33} symbolized by this figure is actually the Voronoi tessellation for E_7^*. The Delaunay polytope, Figure 22 (D), is a diplo-simplex.

Summary for $\mathbf{E_7}$. *The holes are the images of vectors $v_{3/2}$ and $\frac{1}{2} v_{7/2}$ under $W(E_7)$, where $v_{3/2}$, $v_{7/2}$ are vectors of norms $3/2$, $7/2$ in E_7^*. The vertices of the Voronoi polytope $V(0)$ are the images of $v_{3/2}$ and $\frac{1}{2} v_{7/2}$ under $W(e_7)$. The corresponding Delaunay polytopes are the Hesse polytope 3_{21} and a 7-simplex. The contact 7-polytope is a 2_{31}. The squared packing and covering radii are respectively ½ and*

$N(P) = 3/2$.

Summary for $\mathbf{E_7^*}$. *The holes are the images of* $\frac{1}{2} v_{7/2}$ *under* $W(E_7)$, *and consist of the vertices of the honeycomb* 1_{33}. *The vertices of the Voronoi polytope* $V(0)$ *are the images of* $\frac{1}{2} v_{7/2}$ *under* $W(e_7)$ *and form the polytope* 1_{32}. *The Delaunay polytope is a diplo-simplex and the contact polytope for* E_7^* *is a Hesse polytope* 3_{21}. *The squared packing and covering radii are respectively* $\frac{1}{4} N([1]) = \frac{3}{8}$ *and* $N(R) = 7/8$.

Figure 22. Cell Schematic diagram for E_7^*.

Table VIII. Sections of Hesse polytope 3_{21}.

Coordinates	Number	Shape
$\frac{3}{4};\ \frac{3}{4}\ -\frac{1}{4}\ ^{6}$	7	simplex
$\frac{1}{4};\ -\frac{3}{4}\ ^{2}\ \ \frac{1}{4}\ ^{5}$	21	ambo-simplex
$-\frac{1}{4};\ \frac{3}{4}\ ^{2}\ -\frac{1}{4}\ ^{5}$	21	ambo-simplex
$\frac{3}{4};\ -\frac{3}{4}\ \ \frac{1}{4}\ ^{6}$	7	simplex
$\frac{1}{2}\ ^{2}\ ;\ \pm 1\ 0^{5}$	12	orthoplex
$0^{2};\ \frac{1}{2}\ ^{6-}$	32	hemicube
$-\frac{1}{2}\ ^{2}\ ;\ \pm 1\ 0^{5}$	12	orthoplex
$0^{2};\ \frac{1}{2}\ ^{6}$	1	zenith
$\frac{1}{2}\ ^{2+}\ ;\ +1\ 0^{5}$ and $0^{2};\ \frac{1}{2}\ ^{4}\ -\frac{1}{2}\ ^{2}$	27	Schlafli polytope
$\frac{1}{2}\ ^{2+}\ ;\ -1\ 0^{5}$ and $0^{2};\ -\frac{1}{2}\ ^{4}\ \ \frac{1}{2}\ ^{2}$	27	Schlafli polytope
$0^{2};\ -\frac{1}{2}\ ^{6}$	1	nadir

Table IX. Sections of contact 7-polytope.

Coordinates	Number	Shape
$1; -1\ 0^{6}$	7	simplex
$\frac{1}{2}; \left(\frac{1}{2}\right)^{3} \left(-\frac{1}{2}\right)^{4}$	35	2nd ambo-simplex
$0; 1\ -1\ 0^{5}$	42	ambo-diplo-simplex
$-\frac{1}{2}; \left(-\frac{1}{2}\right)^{3} \left(\frac{1}{2}\right)^{4}$	35	2nd ambo-simplex
$-1; 1\ 0^{6}$	7	simplex
$1^{2}; 0^{6}$	1	zenith
$\left(\frac{1}{2}\right)^{2}; \left(\frac{1}{2}\right)^{6+}$	32	hemicube
$0^{2}; \pm 1^{2}\ 0^{4}$	60	ambo-orthoplex
$\left(-\frac{1}{2}\right)^{2}; \left(\frac{1}{2}\right)^{6+}$	32	hemicube
$-1^{2}; 0^{6}$	1	nadir
$\left(\frac{1}{2}\right)^{2-}; \left(\frac{1}{2}\right)^{5} -\frac{1}{2}$ and $0^{2}; 1^{2}\ 0^{4}$	27	Schlafli polytope
$1^{2-}; 0^{6}$ and $\left(\frac{1}{2}\right)^{2-}; \left(\frac{1}{2}\right)^{3} \left(-\frac{1}{2}\right)^{3}$ and $0^{2}; 1\ -1\ 0^{4}$	72	contact 6-polytope
$\left(\frac{1}{2}\right)^{2-}; \left(-\frac{1}{2}\right)^{5} \frac{1}{2}$ and $0^{2}; -1^{2}\ 0^{4}$	27	Schlafli polytope

10. The lattices $\mathbf{E_6}$, $\mathbf{E_6^*}$ and the Schläfli polytope $\mathbf{2_{21}}$.

The 6-dimensional root lattice E_6 is best defined to consist of those vectors of E_8 that are perpendicular to a given A_2 sublattice, spanned by a pair of minimal vectors $v, w \in E_8$ with $v \cdot w = -1$. According as we take v and w to be

$$(1, 0^{6}, 1) \text{ and } \left(\left(\frac{1}{2}\right)^{8}\right) \quad \text{or} \quad (1, -1, 0^{6}) \text{ and } (0, 1, -1, 0^{5})$$

we get two different coordinate systems for E_6: E_6 consists of the vectors $(u_0, \ldots, u_7) \in E_8$ that satisfy

$$u_0 + u_7 = u_1 + \cdots + u_6 = 0\,, \quad \text{or} \quad u_0 = u_1 = u_2\,,$$

respectively. The CS diagram is given in Figure 23.

The weight lattice E_6^* is the union of the three cosets E_6, $E_6 + [1]$, $E_6 + [2]$, where (in the first coordinate system) $[1] = \left(0; \left(\frac{2}{3}\right)^{2}, \left(-\frac{1}{3}\right)^{4}; 0\right)$, $[2] = -[1]$. The CS diagram is given in Figure 24. The Delaunay polytope of E_6^* was first found by Worley [42]. As we see from Figure 24 (D), this is the join of three triangles in orthogonal planes. Again it is a surprise to us that this tessellates space.

Figure 23. Cell schematic diagram for E_6.

The Delaunay polytopes of E_6 (Figure 23 (D)) are 27-vertex polytopes 2_{21} (in two orientations), which we call *Schläfli polytopes*. Their centers are images of the minimal vectors $v_{4/3} \in E_6^*$ under the group $W(E_6)$.

If we take the second definition of E_6, so that E_6 consists of the vectors of E_8 lying in the subspace defined by $u_0 = u_1 = u_2$, then any vector of E_8 has the same inner product with these vectors as its projections onto this space. In particular, *the projections of* E_8 *vectors lie in* E_6^*.

In this way we obtain the 27 vectors on the right of the following display:

$$(1\ 1\ 0;\ 0^5) \rightarrow \left(\frac{2}{3}\ \frac{2}{3}\ \frac{2}{3};\ 0^5\right) \qquad (1)$$

$$\left(\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2};\ \frac{1}{2}^{5-}\right) \rightarrow \left(\frac{1}{6}\ \frac{1}{6}\ \frac{1}{6};\ \frac{1}{2}^{5-}\right) \qquad (16)$$

$$(-1\ 0\ 0;\ \pm 1\ 0^4) \rightarrow \left(-\frac{1}{3}\ -\frac{1}{3}\ -\frac{1}{3};\ \pm 1\ 0^4\right) \qquad (10)$$

These are all the shortest vectors from one coset of E_6 in E_6^*, and form a Schläfli polytope. The contact polytope for E_6^* consists of these vectors and their negatives, and so is a diplo-Schläfli polytope.

The reason for calling 2_{21} the Schläfli polytope is best seen using instead the first definition of E_6. The 27 vectors become six vectors

$$a_i = \left(\frac{1}{2};\ \frac{5}{6}, -\frac{1}{6}^5;\ -\frac{1}{2}\right),$$

with the $\frac{5}{6}$ in position i, $1 \le i \le 6$; six vectors

Figure 24. Cell schematic diagram for E_6^*.

$$b_j = \left(-\frac{1}{2} ; -\frac{1}{6}^{5} , \frac{5}{6} ; \frac{1}{2} \right)$$

with the $\frac{5}{6}$ in position j, $1 \le j \le 6$; and fifteen vectors

$$c_{ij} = \left(0 ; -\frac{2}{3} \ \frac{1}{3}^{4} \ -\frac{2}{3} ; 0 \right)$$

with the $-\frac{2}{3}$ in positions i and j, $1 \le i < j \le 6$.

There are 45 triples of these vectors that add to 0:

$$a_i + b_j + c_{ij} = 0, \quad c_{ij} + c_{kl} + c_{mn} = 0 ,$$

where $\{ i, j, k, l, m, n \} = \{ 1, 2, \ldots, 6 \}$, and the Weyl group $W(e_6)$ achieves all permutations of the 27 vectors that preserve this set of 45 triples. Now there are 27 lines in the general cubic surface in projective 3-space, that lie by 3's in 45 tritangent planes, for which Schläfli's notation is as above. It follows that $W(e_6)$ is isomorphic to the famous group of permutations of the 27 lines that preserve the 45 tritangent planes. (For additional information see [2], [3], [6, Appendix H], [11, p. 26], [15], [19], [31], [38], [43].)

The Schläfli polytope and the contact 6-polytope are further described in Figures 25, 26 and Tables X, XI.

Summary for $\mathbf{E_6}$. *The holes are the images of $v_{4/3} \in E_6^*$ under $W(E_6)$. The vertices of the Voronoi polytope $V(0)$ are the images of $v_{4/3}$ under $W(e_6)$, and form a diplo-Schläfli polytope. The Delaunay polytopes are Schläfli polytopes 2_{21}, and the contact 6-polytope is a 1_{22}. The squared packing and covering radii are respectively $\frac{1}{2}$ and $\frac{4}{3}$.*

Summary for $\mathbf{E_6^*}$. *The holes are the images of* $\frac{1}{3} v_6$, $v_6 \in E_6$, *under* $W(E_6)$. *The vertices of the Voronoi polytope* $V(0)$ *are the images of* $\frac{1}{3} v_6$ *under* $W(e_6)$ *and form an ambo*-1_{22}. *The Delaunay polytope is the join of three triangles in orthogonal planes, and the contact polytope for* E_6^* *is a diplo-Schläfli polytope. The squared packing and covering radii are respectively* $1/3$ *and* $2/3$.

Table X. *d*-dimensional faces of Schläfli polytope 2_{21}.

d	Face	Number
6	Schlafli 2_{21}	1
5	5-orthoplex	27
	5-simplex	72
4	4-simplex	216 + 432
3	tetrahedron	1080
2	triangle	720
1	edge	216
0	vertex	27

Figure 25. Cell Schematic diagram for Schläfli polytope 2_{21}.

Figure 26. Cell Schematic diagram for contact 6-polytope.

Table XI. Sections of Schläfli polytope 2_{21}.

Coordinates	Number	Shape
$\frac{2}{3}^{3}\ ;\ 0^5$	1	apex
$\frac{1}{6}^{3}\ ;\ \frac{1}{2}^{5-}$	16	hemicube
$-\frac{1}{3}^{3}\ ;\ \pm 1\ 0^4$	10	orthoplex
$a_i = \frac{1}{2};\ \frac{5}{6},\ -\frac{1}{6}^{5}\ ;\ -\frac{1}{2}$	6	simplex ($1 \le i \le 6$)
$c_{ij} = 0;\ -\frac{2}{3}^{2}\ \frac{1}{3}^{4}\ ;\ 0$	15	ambo-simplex ($1 \le i \le j \le 6$)
$b_j = -\frac{1}{2};\ \frac{5}{6},\ -\frac{1}{6}^{5}\ ;\ \frac{1}{2}$	6	simplex ($1 \le j \le 6$)

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