LOGNORMAL APPROXIMATION TO PRODUCTS AND QUOTIENTS

BY S. R. BROADBENT

The British Coal Utilization Research Association

SUMMARY. Measurements have been made which are subject to error, and we are required to give
limits to a combination of the values measured. The combinations considered here are products and
quotients. Some exact results are available in simple cases, but otherwise an approximation is required.
The lognormal distribution, which is asymptotically exact, is shown to give useful approximations when
fitted by moments to the combination. This method of fitting is nearly optimum in a defined metric.
Tables are given which make its application simple.

1. INTRODUCTION

The problem we shall consider is assessing the precision of certain combinations of measure-
ments which have been made with a known distribution of error. Suppose the unbiased
measurements \( x_i \) \((i = 1, \ldots, n)\) are made, that \( E[x_i] = \mu_i \), and that we are interested in a
combination of the \( x_i \) of the form

\[
q = \frac{x_1 x_2 \ldots x_j}{x_{j+1} \ldots x_n} \quad (1 < j < n)
\]

(more generally, the observations \( x_i \) may be raised to any power).

We may be required to set fiducial limits to

\[
(\mu_1 \mu_2 \ldots \mu_j)/(\mu_{j+1} \ldots \mu_n),
\]

i.e. to combine the fiducial distributions of the \( \mu_i \) which we have attempted to measure, or
to set probability limits to \( q \), i.e. to combine the probability distributions of the \( x_i \). In the
first case we assume the distributions of \( x_i \) known, except for the means \( \mu_i \), and that the
observations \( x_i \) are given; in the second we assume the distributions of the \( x_i \) completely
known. The two problems are formally identical, and we shall henceforward speak only
of setting limits to \( q \).

To fix the ideas with a particular example of the first case, consider the efficiency \( E \) of
a steam boiler determined by a single trial in which the heat supplied to and the heat
obtained from the boiler are obtained from sample measurements. The efficiency may be
calculated from

\[
E = \frac{HW(1-C)(Qh(1-C'))}{},
\]

where \( H \) is the heat in 1 lb. of steam, \( W \) the weight of water evaporated during the trial,
\( C \) the air-dried moisture content of the coal, \( Q \) the weight of coal fired, \( h \) its air-dried calorific
value and \( C' \) its moisture content as fired. All these measurements are subject to errors whose
distribution is known; the effect of these errors on the precision of \( E \) is required.

In general, the error of a measurement has two possible sources: sampling technique and
instrument. In this paper we shall for simplicity suppose the errors to be either normal or
rectangular, of known coefficients of variation or half-range, and with small coefficients of
variation, although the restriction to these distributions is not necessary. In many applica-
tions, the distributions of the errors are known to be of one of these two forms, and their
standard deviations or half-ranges can usually be found by simple investigations. In other
applications the errors can only be guessed, and then the roughest approximations are
appropriate.

* We assume that fiducial distributions can be combined in the same way as probability distributions,
but see the discussion in Creasy (1954) on this point.
It is impossible to list exhaustively all such combinations of errors. Some of the simpler combinations can be discussed individually; in the event of a large number of errors being combined we can use with confidence the asymptotic distribution. It is in the intermediate cases that approximations must be critically considered. The choice of a suitable family of approximating distributions will always be more of an art than a science.

It is possible in a particular case to refine approximation to any required degree. Gram–Charlier or similar series may be used; the saddlepoint method given by Daniels (1954) also generates approximating functions. However, in this paper we are concerned with a working or first-order approximation only: all that some data and applications merit.

It is well known that, even when \( n \) is small and the component distributions are far from normal, \( x_1 + \ldots + x_n \) has a distribution close to normal. When the component variates are not added but multiplied and divided, the fundamental approximating distribution is the lognormal, as was pointed out, for example, by Shellard (1952). We consider the questions, how is this distribution best fitted and when may we use it with confidence?

2. Known results

2-1. We now suppose the errors of measurement are independent, and we denote a normal variate by \( N \) and a rectangular variate by \( R \). The quotient of the standard deviation and the mean of \( N \), and the quotient of the half-range and mean of \( R \) are both denoted by \( \alpha_1, \alpha_2 \), etc., referring to the variates \( x_1, x_2, \ldots \) in this order. Thus \( R/N \) \((\alpha_1 = 0.1, \alpha_2 = 0.05)\) denotes the quotient of a rectangular variate whose mean is ten times its half-range and an independent normal variate whose coefficient of variation is 5%.

For \( q = N/N \) Geary’s approximation (1930) is used in all practical cases. This states that if \( Q \) is the quotient of the means of the numerator and denominator,

\[
(q - Q)/(\alpha_1 q^2 + \alpha_2 Q^2)\]

is approximately normally distributed with zero mean and unit variance. This approximation is very good up to values of \( \alpha_1 \) and \( \alpha_2 \) as large as 0.25 (see Creasy, 1954). The exact percentage points of this distribution may also be calculated using existing tables as was shown by Fieller (1932). The product \( N \times N \) was discussed by Craig (1936) and Aroian (1947). Percentage points of \( R/N \) have been given by the author (1954). He has given also points of the quotient of a triangular and a normal variate.

The distributions of \( R \times R, R/R \) and so on are not difficult to calculate exactly.

Distributions of products, quotients and powers of variates have been extensively studied, but not many results have been obtained that can be applied to practical problems.

3. Lognormal approximations

3-1. The distribution of

\[
q = (x_1 x_2 \ldots x_j)/(x_{j+1} \ldots x_n)
\]

tends to the lognormal as \( n \to \infty \) under very general conditions. The lognormal distribution has been discussed by Finney (1941), Gaddum (1943) and Johnson (1949).

The most general lognormal approximation to \( q \) is a variate \( z \) such that \( \log(z - \xi) \) is normally distributed with mean \( \mu \) and variance \( \sigma^2 \). Here for simplicity we consider the choice of \( \mu \) and \( \sigma \) only, \( \xi \) being always taken as zero. A variate with this distribution is necessarily positive, while \( q \) may have negative values. When \( \alpha_1, \alpha_2, \ldots \) are small, the approximation by this lognormal distribution may nevertheless be satisfactory, since the probability attached to such negative values is very small.
3-2. The first method of choosing $\mu$ and $\sigma$ is to calculate the moments of $\log q$ and to set $\mu$ and $\sigma^2$ equal to the first and second corrected moments. We call this the method of fitting by moments to $\log q$, or the fit to $\log q$; this method is used for estimation by Finney (1941). It is also intuitively attractive to choose that lognormal approximation whose mean and variance are equal to the mean and variance of $q$ (see Wicksell, 1917). We call this the method of fitting by moments to $q$, or the fit to $q$. The two fits are in general different, although the difference tends to zero as $n$ increases. The difference throws a doubt on the intuitive appeal of fitting by moments, and raises the question we consider below, by what criterion are we to choose our approximation?

Let a lognormal distribution have mean $m$ and variance $s^2$, and let the distribution of the logarithm of a variate from the lognormal have mean $n$ and variance $\sigma^2$. Then it is known that

$$
m = \exp(\mu + \sigma^2/2),
$$
$$s^2 = \left\{ \exp(2\mu + \sigma^2) \right\} \left\{ \exp(\sigma^2) - 1 \right\},
$$
$$\mu = \log m - \frac{1}{2} \log (1 + s^2/m^2),
$$
$$\sigma^2 = \log (1 + s^2/m^2).
$$

For the fit to $\log q$, we must find the mean and variance of $\log q$, and set $\mu$ and $\sigma^2$ equal to them. The $t\%$ point of this fit is

$$\exp(\mu + \sigma_1 \sigma), \quad (2)$$

where the probability is $1/t\%$ that a standardized normal variate is less than $\mu_1$.

For the fit to $q$, we must find the mean and variance of $q$, and set $m$ and $s^2$ equal to them. The $t\%$ point of this fit is

$$(m \exp\left[ \frac{\mu_1 \log (1 + y^2)}{1 + \sigma^2} \right])/(1 + \sigma^2), \quad (3)$$

where $100v = 100s/m$, the coefficient of variation of $q$. This $t\%$ point is tabulated in Table 1 for $m = 1$, $100v = 0(0.01)15$ and $t = 1, 5, 95$ and 99. To use this table it is necessary to find the mean and coefficient of variation of $q$, to enter the table with the appropriate $v$, and multiply the value in the table by $m$. The table was calculated from a series expansion of (3).

3-3. To find the moments of $q$ or of $\log q$ we require the moments of various powers (positive and negative) of $x$ or of $\log x$, when $x$ is normally and when $x$ is rectangularly distributed. We write $x = \mu(1 + ay)$, where $y$ is either a standardized normal variate or is uniformly distributed between $-1$ and 1, and $a$ is small (less than 0.15). We begin by finding the moments of $\log (1 + ay)$, not when $y$ is normally distributed, but (since $Pr \{1 + ay < 0\} = e$ is less than $10^{-10}$) when $y$ is from the truncated distribution

$$\exp\left( -\frac{1}{2} y^2 \right) dy/[1 - e \sqrt{(2\pi)}] \quad (y > -1/\alpha), \quad (4)$$

which is for practical purposes indistinguishable from the normal. We avoid mathematical difficulties by taking this distribution as the parent. Alternatively, the treatment of convergence given by Derksen (1939) could be used.

The moment-generating function of $\log (1 + ay)$ is $E[(1 + ay)^t]$, and this may be written

$$1/\sqrt{(2\pi)} \int_{-1/\alpha}^{1/\alpha} (1 + ity + \frac{1}{2} it(it - 1) \alpha^2 y^2 + \ldots) \exp\left( -\frac{1}{2} y^2 \right) dy + K,$$

where $|K| \leq \epsilon/(1 - \epsilon)$.

Let

$$J(\alpha, r) = \frac{1}{\sqrt{(2\pi)}} \int_{1/\alpha}^{\infty} y^r \exp\left( -\frac{1}{2} y^2 \right) dy$$

$$= \left[ 1 - I[1/(2\alpha^2), \frac{1}{2}(r - 1)] \right] 2^{1-r} - \Gamma[\frac{1}{2}(r + 1)] / \sqrt{\pi}. \quad (5)$$
Here \( I(y, p) \) is the Incomplete Gamma ratio \( \int_0^y v^p e^{-v} dv / \Gamma(p + 1) \), and is nearly one when \( y \) is large and \( p \) small; we deduce from Pearson's table (1922) that for \( \alpha \leq 0.15 \) and \( r \leq 6 \), \( J(\alpha, r) \leq 10^{-8} \), and we therefore neglect it in this region.

The series above is uniformly convergent within the range of integration, so that term-by-term integration is permissible. We may replace the limits of integration by \(( -\infty, \infty)\) in the first four non-zero terms when \( \alpha \approx 0.15 \), since the error so introduced is a factor of \( \{1 - 2J(\alpha, r)/(1 - e)\} \) for \( r = 0, 2, 4 \) and 6. As \( K \) is negligible we obtain to sufficient accuracy the first four terms of the convergent series for the moment-generating function

\[ 1 + \alpha^2 i(t - 1)/2 + \alpha^4 i(t - 1)(i - 2)/(i - 3)/8 + \alpha^6 i \ldots (i - 5)/48, \]

and hence the cumulants of \( \log (1 + ay) \),

\[
\begin{align*}
\mu_1 &= \kappa_1 = -\alpha^2/2 - 3\alpha^4/4 - 5\alpha^6/2 - \ldots \\
\mu_2 &= \kappa_2 = \alpha^2 + 5\alpha^4/2 + 32\alpha^6/3 + \ldots \\
\kappa_3 &= -3\alpha^4 - 22\alpha^6 - \ldots \\
\kappa_4 &= 20\alpha^6 + \ldots 
\end{align*}
\]

The cumulants of \( \log (1 + ay) \) when \( y \) is rectangularly distributed in \((-1, 1)\) are obtained in a similar way. We have

\[
\begin{align*}
\mu_1 &= \kappa_1 = -\alpha^2/6 - \alpha^4/20 - \alpha^6/42 - \ldots \\
\mu_2 &= \kappa_2 = \alpha^2/3 + 7\alpha^4/45 + 29\alpha^6/315 + \ldots \\
\kappa_3 &= -2\alpha^4/15 - 1513\alpha^6/3780 - \ldots \\
\kappa_4 &= -2\alpha^4/15 - 16\alpha^6/315 - \ldots 
\end{align*}
\]

We next require the expectation of various powers of \((1 + ay)\) when \( y \) is normally distributed and when \( y \) is rectangularly distributed. These have been calculated for the truncated normal distribution (4) and the rectangular distribution; the arguments are similar to those given above and are not repeated here. Finally, we have arranged the results commonly required in Table 2. These formulae, with (2) or Table 1, enable the percentage points of the two lognormal fits to \( q \) to be calculated.

The power series above have been formed by identifying coefficients in the expansion of the moment-generating function. We have given only the first four terms of this expansion, for moderate \( \alpha \). For later terms or larger \( \alpha \), important corrections would have to be applied to the coefficients obtained by uncritically continuing the series. Indeed, Wicksell (1921) pointed out that some of the series in (5) and (6) may diverge if the later coefficients are uncritically formed. The simple rules for obtaining these coefficients apply only under the conditions stated.

The results given may be extended to cases in which the variates are not independent. It is necessary only to find the moments for correlated variates in the same way as above; some moments are given by Haldane (1942).

As an example, consider the lognormal approximations to \( q = N/N \); by convention the coefficients of variation of numerator and denominator are respectively \( 100\alpha_1 \) and \( 100\alpha_g \).

Using Table 2 we obtain

\[ E[q] = m = (1) \left(1 + \alpha_1^2 + 3\alpha_1^4 + \ldots\right), \]

and

\[ E[q^2] = (1 + \alpha_1^2) \left(1 + 3\alpha_1^2 + 15\alpha_1^4 + \ldots\right), \]

i.e.

\[ Var[q] = s^2 = \alpha_1^2 + \alpha_1^4 + 3\alpha_1^6 \alpha_2^4 + 8\alpha_1^8 + \ldots. \]
Lognormal approximation to products and quotients

Table 1. Standardized lognormal percentage points

Given the mean \( m \) and standard deviation \( s \), let \( v = \frac{100s}{m} \). The percentage point of the lognormal distribution with this mean and standard deviation is the entry in the table, multiplied by \( m \).

Lower 1% points \((t = 1)\)

<table>
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<tr>
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<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
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<th>( \Delta )</th>
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<td>0.7805</td>
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Lower 5% points \((t = 5)\)

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S. R. Broadbent

Table 1 (cont.)
Upper 5% points ($t = 95$)

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<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
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Upper 1% points ($t = 99$)

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</table>
Lognormal approximation to products and quotients

Similarly,

\[ E[\log q] = \mu = -(\frac{1}{2}a_1^2 + \frac{1}{3}a_1^3 + \ldots) + \left(\frac{1}{2}a_2^2 + \frac{1}{3}a_2^3 + \ldots\right), \]
and

\[ V[\log q] = \sigma^2 = (a_1^2 + \frac{1}{3}a_1^3 + \ldots) + (a_2^2 + \frac{1}{3}a_2^3 + \ldots). \]

Using these values of \( \mu \) and \( \sigma^2 \) we obtain for the first two moments of the lognormal fitted to \( \log q \),

\begin{align*}
\text{mean:} & \quad 1 + a_2^2 + \frac{1}{3}a_2^3 + \ldots, \\
\text{variance:} & \quad a_1^2 + a_2^2 + 3a_1^4 + 3a_1^2a_2^2 + 5a_1^2 + \ldots.
\end{align*}

Since these agree with \( m \) and \( s^2 \) to \( O(a_1^2, a_2^2) \), and to higher order when \( a_1 = a_2 \), the difference between the two fits will be small.

Table 2

<table>
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<tr>
<th>( r )</th>
<th>( y ) is distributed normally with mean zero and variance one</th>
<th>( y ) is distributed rectangularly in the interval ((-1, 1))</th>
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<td>( \frac{1}{2} )</td>
<td>( 1 - a^2/8 - 15a^4/128 - \ldots )</td>
<td>( 1 - a^2/24 - a^4/128 - \ldots )</td>
</tr>
<tr>
<td>1</td>
<td>( 1 )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 1 + a^2 )</td>
<td>( 1 + a^2/3 )</td>
</tr>
<tr>
<td>4</td>
<td>( 1 + 6a^2 + 3a^4 )</td>
<td>( 1 + 2a^2 + a^4/5 )</td>
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<tr>
<td>( -\frac{1}{2} )</td>
<td>( 1 + 3a^2/8 + 105a^4/128 + \ldots )</td>
<td>( 1 + a^2/8 + 7a^4/128 + \ldots )</td>
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<tr>
<td>( -1 )</td>
<td>( 1 + a^2 + 3a^4 + \ldots )</td>
<td>( 1 + a^2/3 + a^4/5 + \ldots )</td>
</tr>
<tr>
<td>( -2 )</td>
<td>( 1 + 3a^2 + 15a^4 + \ldots )</td>
<td>( 1/(1 - a^2) )</td>
</tr>
<tr>
<td>( -4 )</td>
<td>( 1 + 10a^2 + 105a^4 + \ldots )</td>
<td>( 1 + 10a^2/3 + 7a^4/45 + \ldots )</td>
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</table>

It is not possible to state how different the two will be in general. We compare below the percentage points given by the two fits with some exact values (Table 3). The comparisons are made for quite large coefficients of variation, and for distributions rather different from lognormal. The agreement is surprisingly good (it would be worse at more extreme percentage points). The two fits generally give points closer to each other than to the exact value, i.e. choice between the two methods does not appear to be important.

3.4. We turn now to the normal approximation. Cramér (1951) has shown that if \( H \) is a function of the central moments of a multi-dimensional sample of size \( S \), such that \( H \) and its first two derivatives with respect to these moments are continuous, then \( H \) is asymptotically normal. If \( H(m_1, \ldots) \) is this function, its mean and variance are asymptotically \( H(\mu_1, \ldots) \), and

\[ \mu_\mu(m_\mu) \left( \frac{\partial H}{\partial \mu_\mu} \right)^2 + \ldots + 2\mu_{r1}(m_\mu m_{r1}) \left( \frac{\partial H}{\partial \mu_{r1}} \right) \left( \frac{\partial H}{\partial \mu_\mu} \right) + \ldots \]

Now \( q = (x_1x_2 \ldots x_j)/(x_{j+1} \ldots x_n) \) is in the form required by this theorem, with \( S = 1 \).

It is sometimes said that \( q \) is approximately normally distributed, with mean

\[ (\mu_1 \mu_2 \ldots \mu_j)/(\mu_{j+1} \ldots \mu_n), \]

and variance

\[ \sigma_q^2 \left( \frac{\partial q}{\partial x_1} \right)^2 + \ldots + 2\rho_{r1} \sigma_{r1} \sigma_q \left( \frac{\partial q}{\partial x_1} \right) \left( \frac{\partial q}{\partial x_r} \right) + \ldots, \]

that is, with coefficient of variation

\[ 100(\sigma_1^2 + \ldots + \alpha_{j+1}^2 + \ldots + 2\rho_{1j+1} \alpha_1 \alpha_2 + \ldots - 2\rho_{1j+1j+1} \alpha_1 \alpha_{j+1} - \ldots)^4. \]
The mean and variance of $q$ agree with these formulae to $O(\alpha^2)$ and $O(\alpha^4)$ respectively when the $x_i$ are normally distributed, and so do not differ greatly for small $\alpha$. Now the normal and lognormal distributions do not differ greatly in their 1, 5, 95 and 99% points for small coefficients of variation. If the lognormal approximation to $q$ is good, the normal approximation will also be good for small $\alpha$. This appears to be a better justification for the normal approximation in such cases than reliance on the application of Cramér's asymptotic theorem to a sample of size one. Brunt (1931) has derived this approximation by a Taylor expansion of $q$ and the assumption of normality.

4. General Use of the Approximation

4·1. It is necessary to know how useful the lognormal approximation is in practice. Since the exact distributions for which we require approximate percentage points are not usually known, it is impossible to give exact results. Equally, in many problems it is as reasonable to suppose the variates are lognormally distributed as to consider them normally distributed, and then the question is trivial.

In Table 3 some typical comparisons with simple distributions are given. It is intuitively clear that the approximation will improve as the $\alpha$ decrease or as the number of component variates increases. If we are satisfied by the agreement indicated in Table 3, we may use the method with confidence in more complicated situations with smaller $\alpha$.

But as the number of variates increases, larger coefficients of variation than those given in Table 3 may be allowed without impairing the approximation. It is interesting to know how much larger the coefficients of variation may be. Some quantitative conclusions may be drawn from the cumulants of $\log q$, where $q$ is the product of $n$ independent normal variates:

$$q = x_1 x_2 \ldots x_n.$$  

Suppose the constant $\mu_1 \mu_2 \ldots \mu_n$ has already been removed, so that $x_i$ has mean one and variance $\alpha_i^2$: suppose also that the lognormal approximation to $q$ has been deemed satisfactory. We now wish to approximate to $q' = qx_{n+1}$, where $x_{n+1}$ has mean one and variance $\beta^2$. The lognormal approximation to $q'$ will remain satisfactory for small $\beta$; for large $\beta$ the distribution of $q'$ will be unduly affected by $x_{n+1}$ and the lognormal approximation will no longer satisfy us. The problem is to determine conditions on $\beta$, in relation to the $\alpha_i$, which allow us to use with confidence a new lognormal approximation. These conditions cannot ensure that the new approximation is from every point of view as good as the old, for example, the exact and approximate cumulative distributions will not generally coincide at the points where they coincided before.

It follows from (5) that the fit to $\log q$ supposes that

$$u = (\log q - \kappa_1)/\kappa_1^2$$

is approximately normally distributed with zero mean and unit variance, where

$$\kappa_1 = -\lambda/2 - 3\mu/4 - 5\nu/2,$$

and

$$\kappa_2 = \lambda + 5\mu/2 + 32\nu/3.$$

Here

$$\lambda = \sum_{i=1}^n \alpha_i^2, \quad \mu = \sum_{i=1}^n \alpha_i^3 \quad \text{and} \quad \nu = \sum_{i=1}^n \alpha_i^4.$$

The third and fourth cumulants of $u$ are approximately

$$\kappa_3 = -3\mu/\lambda^3 \quad \text{and} \quad \kappa_4 = 20\nu/\lambda^2,$$

and are of the order of $3\alpha/n^3$ and $20\alpha^2/n$ respectively.
Lognormal approximation to products and quotients

For $q'$ similar relations hold, with

$$\lambda' = \lambda + \beta^2, \quad \mu' = \mu + \beta^4 \quad \text{and} \quad \nu' = \nu + \beta^4.$$

If the new third cumulant is less than or equal to $\kappa_3$, in absolute value, and the new fourth cumulant less than or equal to $\kappa_4$, we have grounds for believing that the normal approximation to $\log q'$ and the lognormal approximation to $q'$ will be at least as satisfactory as the approximations to $q$. The condition on the third cumulant is approximately

$$\lambda' \leq \mu' \leq \kappa_3.$$

Table 3. Exact and lognormal approximation percentage points

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<td>$N$</td>
<td>$R/N$</td>
<td>$N/N$</td>
<td>$N \times N$</td>
<td>$N/N$</td>
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<td>10</td>
<td>4</td>
<td>5</td>
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<tr>
<td>100$\times_2$</td>
<td>---</td>
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<td>4</td>
<td>5</td>
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</table>

* Exact points communicated by Prof. L. A. Aroian.

We write $b = \beta^2/\lambda$ and $\gamma = \lambda^2/\mu$; then the condition on $b$ is

$$b^2\gamma^2 - b^2 + (2\gamma - 3)b - 3 \leq 0.$$

It follows from Cauchy's inequality that $1 \leq \gamma \leq n$. The condition is satisfied at $b = 0$ and until

$$\gamma = \{(b + 1)(1 - 1)/b^2\}.$$

Similarly, the condition on the fourth cumulant is approximately

$$\lambda^2(\nu + \beta^4) \leq (\lambda + \beta^2)^2 \nu.$$

We write $\delta = \lambda^2/\nu$; it follows from Cauchy's inequality that $1 \leq \delta \leq n^2$. The condition becomes

$$\delta b^2 - b - 2 \leq 0,$$

and is satisfied at $b = 0$ and until

$$b = \{(1 + 8\delta)(1 + 1)/2\delta\}.$$
Similar regions may be defined when the \( x_i \) are raised to positive or negative powers, and for rectangular variates. When \( g \) consists of the product of independent rectangular variates we see from (6) that both \( \kappa_3 \) and \( \kappa_4 \) are \( O(x^3) \), and hence that the conditions on both these cumulants result in an acceptable region similar to the first of the two defined above.

4·2. Some practical conclusions are now drawn.

All coefficients of variation equal. Suppose \( x_1 = \ldots = x_n = \alpha \) and \( x_{n+1} \) has coefficient of variation \( \beta \). Then the lognormal fit to the new product will be as good as the fit to the old if \( \beta \) is less than the value given in Table 4, \( n = 1 \) (1) 6; the asymptotic criterion is also shown.

| Table 4. Critical values of \( \beta \) for \( x_1 = \ldots = x_n = \alpha \) |
|-----------------|-----------------|-----------------|
| \( n \)        | \( \kappa_3 \)   | \( \kappa_4 \)   |
| 1              | 1.46\( \alpha \) | 1.41\( \alpha \) |
| 2              | 1.38\( \alpha \) | 1.30\( \alpha \) |
| 3              | 1.32\( \alpha \) | 1.26\( \alpha \) |
| 4              | 1.28\( \alpha \) | 1.24\( \alpha \) |
| 5              | 1.26\( \alpha \) | 1.24\( \alpha \) |
| 6              | 1.24\( \alpha \) | 1.24\( \alpha \) |
| \( \infty \)   | 1.22\( \alpha \) | 1.22\( \alpha \) |

Coefficients of variation unequal. For various values of \( \alpha_2 \) and \( \alpha_3 \), Table 5 gives the critical values of \( \beta \) in terms of \( \alpha_1 = \alpha \).

| Table 5. Critical values of \( \beta \), \( \alpha_i \) unequal |
|-----------------|-----------------|-----------------|-----------------|
| \( n \)        | \( \alpha_2 \)   | \( \alpha_3 \)   | \( \kappa_3 \)   | \( \kappa_4 \)   |
| 2              | \( \alpha/2 \)   | —               | 1.28\( \alpha \) | 1.29\( \alpha \) |
| 2              | \( \alpha/5 \)   | —               | 1.40\( \alpha \) | 1.39\( \alpha \) |
| 3              | \( \alpha/2 \)   | \( \alpha/2 \)   | 1.27\( \alpha \) | 1.25\( \alpha \) |
| 3              | \( \alpha/5 \)   | \( \alpha/5 \)   | 1.21\( \alpha \) | 1.36\( \alpha \) |

It is clear from Tables 4 and 5 that when a new component is added to \( q \) and it is known that the lognormal approximation to \( q \) is satisfactory, the coefficient of variation of the new variate can be of the order of 1·25 times the largest coefficient of variation already present. When this is the case, we have some confidence that the new \( q' \) will also have a satisfactory lognormal approximation. In this way we may extend the results of Table 3. We may, for example, conclude that it is likely that the lognormal approximations are satisfactory for \( RN/N, \alpha_1 < 0.1, \alpha_2 < 0.06 \) and \( \alpha_3 < 0.05 \), and for \( RN/NN, \alpha_1 < 0.1, \alpha_2 < 0.06, \alpha_3 < 0.07 \) and \( \alpha_4 < 0.05 \).

5. THE DISTANCE BETWEEN DISTRIBUTIONS

5·1. We have already pointed out that there are several methods of choosing a lognormal approximation to \( q \), and we have given examples to show that the fit by moments to \( q \) gives good practical results. It is possible to give this method some theoretical justification, which of course rests on the criterion adopted for judging an approximation.
Suppose that we are given the cumulative distribution function $F(x)$, for which the first and second moments $\mu$ and $\sigma^2$ exist. Suppose also the cumulative distribution $G(x)$ has moments $m$ and $s^2$, and that we are to choose from some family that $G(x)$ which most resembles $F(x)$. For example, $G(x)$ may be a well-tabulated distribution, and we are at liberty to choose the parameters $m$ and $s^2$. The manner in which $G(x)$ is to resemble $F(x)$ is of course critical, and the number of possible criteria is unbounded. We develop first two almost trivial criteria and then discuss two approaches to the distance between $F(x)$ and $G(x)$.

5-2. In the first place, suppose we really require two percentage points of $F(x)$; this is the type of problem we stated initially. Suppose two parameters of $G(x)$ are available, and for particular values of these parameters the two percentage points of $G(x)$ coincide with those of $F(x)$. These are the values we should choose. For example, in fitting a normal distribution to the 5 and 95% of $R/N$, $x_1 = 0.1$, $x_2 = 0.03$, we have to solve

\[ m - 1.645s = 0.882, \quad m + 1.645s = 1.133, \]

to obtain a unique solution. This method cannot be usefully applied, since if the points were known we would not require to fit $G(x)$.

Now suppose it is known that a good fit will be obtained by selecting a set of known points for agreement. For example, the author (1954) gives the 1 and 5% points of $R/N$; suppose the 2% point is required. The lognormal distribution is a reasonable fit to this distribution; we therefore fit a normal distribution to the logarithms of the 1 and 5% points $p_1$ and $p_2$. We solve

\[ m - 2.326s = \log p_1, \quad m - 1.645s = \log p_2. \]

The 2% point, $p_{2\%}$, is then given by

\[ m - 1.960s = 0.462 \log p_1 + 0.538 \log p_2 = \log p_{2\%}. \]

Calculations show that this interpolation is within the accuracy of the tables given. The method is also exact when applied to interpolation or extrapolation in Table 1 of this paper.

5-3. The distance between two cumulative distribution functions at the value $x$ is naturally defined as some monotonic increasing function, $M(z)$, of $|F(x) - G(x)|$. The distance between the two functions may then be defined by either

\[ \int_{-\infty}^{\infty} M(|F(x) - G(x)|) \psi(F(x)) \, dF(x), \]

or

\[ \sup_x [M(|F(x) - G(x)|) \psi(F(x))], \]

where $\psi(z)$ is some weighting function. The first definition is associated with Cramér (1928), von Mises (1931) and Šmírnov (1936); the second with Kolmogorov (1933).

We are concerned with distances along the x-axis rather than perpendicular to it. That is, we aim to fit approximate percentage points which are close to the true points rather than to fix points at which $F(x)$ is close to the desired value.

Suppose that $F(x) = p$ and $G(x) = p$ have respectively the unique inverses $x = \phi(p)$ and $x = \gamma(p)$ for almost all $p$. We are concerned with a distance between the functions at $p$ of $M(|\phi(p) - \gamma(p)|)$, and we require definitions of distance between the distributions in the form

\[ \int_0^1 M(|\phi(p) - \gamma(p)|) \psi(p) \, dp, \]

or

\[ \sup_p [M(|\phi(p) - \gamma(p)|) \psi(p)]. \]
We do not here consider the second definition, which gives an unbounded distance for certain very simple cases, e.g. if $F(x)$ and $G(x)$ are normal distributions differing only in variance, $M(z) = z$ and $\psi(p) = 1$.

A natural form of the first definition for our purpose is given by $M(z) = z^2$, and $\psi(p)$ the characteristic function of some set $E$ contained in the interval $(0, 1)$ in which we are specially interested. For example, $E$ might be the interval $(0, 1)$ itself, or the union of the intervals $(0.001, 0.05)$ and $(0.95, 0.995)$.

We therefore define the distance $\theta$ between $F(x)$ and $G(x)$ by

$$\theta = \int_E (\psi(p) - \gamma(p))^2 dp / |E|.$$

The standardized form of $x$ corresponding to $F(x)$ is $(x - \mu) / \sigma$; let $\xi(p)$ be the inverse of the distribution function corresponding to the standardized form and $\eta(p)$ the similar function derived from $G(x)$:

$$\phi(p) = \mu + \sigma\xi(p), \quad \gamma(p) = m + s\eta(p).$$

It follows that

$$|E| \theta = (\mu - m)^2 + 2\sigma(\mu - m)\int_E \xi(p) dp - 2\sigma(\mu - m)\int_E \eta(p) dp + \int_E (\sigma^2\xi(p) - \sigma^2\eta(p))^2 dp.$$

Consider the particular case $|E| = 1$, that is, $E$ consists of all points in $(0, 1)$ except possibly for a set of measure zero. For instance, we would exclude values of $p$ for which $\xi(p)$ and $\eta(p)$ are not unique. Here

$$\theta = (\mu - m)^2 + \sigma^2 - 2\sigma^2 I + \delta^2,$$

where

$$I = \int_0^1 \xi(p) \eta(p) dp.$$

Now if $G(x)$ differs from $F(x)$ in mean and variance at most, $I = 1$, and

$$\theta = (\mu - m)^2 + (\sigma - s)^2.$$

This is reduced to zero by choosing $m = \mu$ and $s = \sigma$.

In all other cases $I < 1$. To minimize $\theta$ we should choose $m = \mu$ and $s = I\sigma$, when $\theta$ takes the value $\sigma^2(1 - I^2)$. If more than the mean and variance can be chosen we should make our choice to maximize $I$.

By this criterion the method of fitting $G(x)$ by moments is correct in that it sets $m = \mu$, but is not optimum in taking $s = \sigma$. The method gives $\theta = 2\sigma^2(1 - I)$. The optimum procedure is to calculate $I$ and then to choose $s$ less than $\sigma$ accordingly. However, the method of moments will not be far from optimum if $I$ is near one, i.e. if the standardized forms of $F(x)$ and $G(x)$ are not too different. Suppose $I = 1 - \epsilon$, where $\epsilon$ is small. By the optimum procedure $\theta = 2\sigma^2\epsilon - \sigma^2 \epsilon^3$, and by the method of moments $\theta$ is only slightly greater: $\theta = 2\sigma^2 \epsilon$.

It appears that $\epsilon$ is often small, and in this sense the method of fitting by moments is nearly optimum. $I$ cannot usually be calculated in the cases for which we require approximations. But in fitting a normal distribution to $\chi^2 / \chi^2$, where the $\chi^2$-distribution has 30 d.f., we obtain $I = 0.995$. In fitting a normal distribution even to $R/R$, $\alpha_1 = 0.1, \alpha_2 = 0.05$, we obtain $I$ as large as 0.92.
Lognormal approximation to products and quotients

The more general case, $|E| < 1$, will not be discussed here except to point out that the procedure of minimizing $\theta$ is very similar. If we write (8) as

$$|E| \theta = (\mu - m)^2 + 2(\mu - m)(U\sigma - Vs) + \sigma^2X - 2\sigma Y + s^2Z,$$

we see we must choose

$$m = \mu + (V Y - U Z)\sigma/(V^2 - Z),$$
$$s = (U Y - Y)\sigma/(V^2 - Z),$$

provided $G(x)$ is non-singular.

If, for example, $F(x), G(x)$ and $E$ are symmetrical about their means, (9) reduces to

$$m = \mu \quad \text{and} \quad s = Y\sigma/Z.$$

5.4. Although the preceding section may be regarded as a refinement and in part a justification of the method of fitting by moments, it cannot be applied as it stands to the lognormal approximation to $q$. For the lognormal, and for a wide class of distributions, the standard form $\eta(p)$ of § 5-3 does not exist. It exists only if the family $G(x)$ is linear. The product and quotient of standard forms are not a linear family, nor can they be transformed into a linear family. This is evident from the fact that such a product, say $(a + bx)(c + dy)$, or quotient has three parameters $(ac, b/c$ and $d/c)$, while a linear family has only two.

Although we cannot fit a best lognormal to a general product or quotient of variates, the argument above still shows that the method of fitting by moments to $q$ is nearly optimum.

The definitions of $\xi(p)$ and $\eta(p)$ must now follow from (7).

The method of fitting to log $q$ is nearly optimum in another metric, in which the distance between $F(x)$ and $G(x)$ is

$$\int_0^1 [\log(\phi(p)/\gamma(p))]^2 dp.$$

6. Conclusions

We return to the problem of setting limits to

$$q = (x_1 x_2 \ldots x_j)/(x_{j+1} \ldots x_n).$$

There is no single answer to the question, what approximation should we use to calculate good limits? We must use judgement in selecting the known distribution used in the calculation: normal, lognormal, or one of those described in § 2. When the family is chosen, the method of fitting by moments to $q$ is recommended. It is recommended only because the metric in which it is nearly optimum is the more natural; in practice there is little difference between the two approximations. Given Tables 1 and 2, the fit to $q$ is easily calculated; without Table 1, the fit to log $q$ is simpler.

If the number of component variates is larger than two, the lognormal approximation will give satisfactory results when the coefficients of variation of the components are small and not too different. It is therefore a suitable approximation for general use. If the coefficient of variation of $q$ is sufficiently small, the normal approximation gives very similar results.

To calculate the first two moments of $q$ it is necessary to know the coefficient of variation, or quotient of half-range and mean, of each component, and then to combine the values given in Table 2. The percentage points are then given in Table 1, interpolation and extrapolation for other percentage points being discussed in § 5-2. Finally, these points are to be multiplied by

$$(\mu_1 \mu_2 \ldots \mu_j)/(\mu_{j+1} \ldots \mu_n).$$
The example we gave in § 1 may now be completed. Suppose, for example, \( E = NR^2/N^2 \) (i.e. \( W \) is measured with rectangular error, the other components with normal error) and \( \alpha_1 = 0.02, \alpha_2 = 0.01, \alpha_3 = 0.005, \alpha_4 = 0.015, \alpha_5 = 0.005 \) and \( \alpha_6 = 0.005 \). We find the mean and coefficient of variation of \( E \) are 1.000 275 and 2.57 %.

The 1 and 99 % limits to \( E \) are 0.941 and 1.061 times its calculated value, i.e. the efficiency has been determined plus or minus about 6 %. The normal approximation gives very similar limits: 0.937 and 1.063.

If, however, \( \alpha_2 \) were large in relation to \( \alpha_1, \alpha_3 \), etc., we might prefer to use \( R/N \) rather than the lognormal as an approximation, and to obtain percentage points from Broadbent (1954).

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