ORDINAL OPERATIONS ON SURREAL NUMBERS

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ABSTRACT

An open problem posed by John H. Conway in [2] was whether one could, on his system of numbers and games, '...define operations of addition and multiplication which will restrict on the ordinals to give their usual operations'. Such a definition for addition was later given in [4], and this paper will show that a definition also exists for multiplication. An ordinal exponentiation operation is also considered.

1. Introduction

An open problem posed by John H. Conway in [2] was whether one could, on his system of numbers and games, '...define operations of addition and multiplication which will restrict on the ordinals to give their usual operations'. Such a definition for addition was later given in [4], and this paper will show that a definition also exists for multiplication. An ordinal exponentiation operation is also considered.

In [2, 3, 4], Conway developed a system of numbers and games that, at the same time, formalised classical game theory and unified two traditional constructions of numbers, the real numbers of Dedekind and the ordinal numbers of Cantor. Adapting terminology due to Knuth [6], we use the term surreal numbers to denote Conway's numbers and games.

The operations Conway defined for his numbers, when restricted to the ordinals, were not the usual ordinal operations of Cantor, but rather the so-called natural (or maximal) operations of Hessenberg [7, p. 367]. Conway later [4] gave a definition of addition that generalised the Cantor sum. A definition of multiplication has been proposed by Simon P. Norton of Cambridge University. The main result given here shows that Norton's definition works, and is a generalisation of Cantor's multiplication.

The rest of this paper is as follows. Section 2 introduces the surreal numbers, Section 3 the generalised Cantor ordinal sum, and Section 4 the new definition of multiplication, and finally Section 5 will consider whether or not a generalised ordinal exponentiation exists.

2. Numbers and games

The definitions in this section can all be found in [4].
2.1 The basic inductive definitions. Conway defined a game to be a pair \( \{L \mid R\} \), where \( L \) and \( R \) are sets of previously created games. The process starts with \( L = R = \emptyset \), and the first game is \( \{\emptyset \mid \emptyset\} \), usually simplified to \( \{|\} \), and called 0 or the zero game.

If \( G = \{L \mid R\} \) is a game, then we write \( G^L \) for a typical member of \( L \), \( G^R \) for a typical member of \( R \), and refer to these (respectively) as the Left (Right) options of \( G \). We also write \( G = \{G^L \mid G^R\} \).

Order between two games \( G \) and \( H \) is defined inductively by the following.

\[
G \geq H \iff \text{(no } G^R \leq H \text{ and } G \leq \text{ no } H^L), \quad G \nRightarrow H \iff G \geq H \text{ does not hold,}
\]

\[
G > H \iff G \geq H \text{ and } H \nRightarrow G, \quad G = H \iff G \geq H \text{ and } H \geq G.
\]

A particular game \( \{L \mid R\} \) is called a number if each member of \( L \cup R \) is already a number, and no member of \( L \) is \( \geq \) any member of \( R \). Numbers are usually written in lower-case letters, that is, \( x = \{x^L \mid x^R\} \).

2.2 Value and form of a game. As equality is a defined relation, a distinction has to be made between the form of a game and its value. Two equal games are said to have the same value, even though their forms \( \{L \mid R\} \) may be different. Two games \( G \) and \( H \) are identical, written \( G \equiv H \), if every \( GL \) is identical to some \( HL \), every \( GR \) is identical to some \( HR \), and vice versa.

2.3 Operations on games. An operation \( * \) over games is introduced in two stages. First, for arbitrary games \( G \) and \( H \), \( G * H \) is defined. Secondly, it is verified that if \( G_1 = G_2 \) and \( H_1 = H_2 \), then \( G_1 * H_1 = G_2 * H_2 \). This second stage, we shall say, shows that \( * \) extends to values. An operation is said to be well defined on a class of games \( \mathcal{G} \) if it is closed on \( \mathcal{G} \) and extends to values over \( \mathcal{G} \).

2.4 Reversible and dominated options. If \( G^L_0 \) and \( G^L_1 \) are two Left options of \( G \), and \( G^L_0 \leq G^L_1 \), then we say that \( G^L_0 \) is dominated by \( G^L_1 \). If \( G^L_0 \) has a Right option \( G^R_0 \) such that \( G^L_0 R_0 \leq G \), then we say that \( G^L_0 \) is a reversible option through \( G^L_0 R_0 \). Similar definitions exist for the Right options of \( G \).

The value of a game \( G \) is left unaltered if [4, Theorem 68, p. 110]:

(i) all dominated options are deleted;

(ii) a reversible Left option \( G^L_0 \) through \( G^L_0 R_0 \) is replaced by all Left options of \( G^L_0 R_0 \)—similarly for the reversible Right options.

Rule (ii) can be applied repeatedly to a game until eventually no reversible options exist [4, p. 210]. A game \( G \) is said to be free of reversible options if, first, every option of \( G \) is free of reversible options, and secondly, \( G \) has no reversible options.

3. The ordinal sum

Conway's inductive definition for a generalised ordinal sum [4, pp. 192, 210] is

\[
X + Y = \{X^L, X + Y^L \mid X^R, X + Y^R\}.
\]

This satisfies the following, easily proved, identities: (i) \( X + 0 \equiv 0 + X \equiv X \), (ii) \( X + (Y + Z) \equiv (X + Y) + Z \), (iii) \( -(X + Y) \equiv (-X) + (-Y) \).

Conway used the colon symbol to denote the ordinal sum, but the third identity allows us to define ordinal subtraction by \( X - Y = X + (-Y) \).
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Over what class of games is the ordinal sum well defined? Theorem 87 of [4, p. 189] (also called the Colon principle in [1, p. 184]) shows that

\[ X + Y \geq X + Z \iff Y \geq Z. \]  

(1)

Hence a corollary is that the value of \( X + Y \) is not affected by the form of \( Y \). However, the form of \( X \) does affect the value of \( X + Y \). For example, if \( X_1 = \{ \} \) and \( X_2 = \{-1\} \), then \( X_1 + 1 = \{X_1 + 0\} = \{0 + 0\} = 1 \), but \( X_2 + 1 = \{X_2^L, X_2 + 0\} = \{-1, 0\} = \frac{1}{2} \).

It was stated in [4, pp. 211–212] that the ordinal sum is well defined over the class of games free of reversible options. However, there are some contrived cases where this is not so, the case \( X + 1 \), where \( X = \{0, \{\|\} | 0\}(= *) \), being an example. This slight oversight is easily resolved, and the details can be found in [5].

We finish this section by noting that if \( x, y \) are numbers, so is \( x + y \). The proof is simply inductive and uses (1). Lastly, as Conway noted [4, p. 31], 'The sign-expansion of \( x + y \) is obtained by following that of \( x \) by that of \( y \). This is enough to show that Conway's ordinal sum is a generalisation of that of Cantor. Sign-expansions are another way Conway used to represent his numbers, this time in terms of sequences of pluses and minuses.

4. An ordinal product

4.1 Definition and identities. S. P. Norton’s proposed generalisation of Cantor’s ordinal multiplication is as follows.

**DEFINITION.** \( XY = (XY^L) + X^L, (XY^R) - X^R | (XY^L) + X^R, (XY^R) - X^L \).

Is this a generalisation of Cantor’s definition? Arguing informally, this would be so if for ordinals \( \alpha = \{\alpha^- \} \) and \( \beta = \{\beta^- \} \), \( \alpha \beta \) was just the repeated ordinal addition of \( \alpha \) \( \beta \) times—that is, \( \alpha \beta = \alpha + \alpha + \ldots (\beta \text{ times}) \). By the definition, \( \alpha \beta = \{\alpha \beta^- + \alpha^- \} \).

Now assuming the simpler case of \( \alpha \beta^L \) to be the repeated ordinal addition of \( \alpha \), \( \beta \) times, the options of \( \alpha \beta \) are just of the form \( \alpha + \alpha + \ldots (\beta \text{ times}) + \alpha^- \), which are indeed typical options of \( \alpha + \alpha + \ldots (\beta \text{ times}) \).

The rest of the ordinal product definition comes from trying to extend the definition to ‘negative’ ordinals in the simplest way.

The ordinal product definition satisfies the following identities.

**THEOREM 4.1.** For all games \( X, Y, Z \) we have

- (i) \( X0 \equiv 0X \equiv 0 \)
- (ii) \( X1 \equiv 1X \equiv X \)
- (iii) \( X(-Y) \equiv (-X)Y \equiv -(XY) \)
- (iv) \( (-X)(-Y) \equiv XY \)
- (v) \( X(Y + Z) \equiv XY + XZ \)
- (vi) \( X(YZ) \equiv (XY)Z \).

Each proof is of the type called ‘1-line proofs’ [4, p. 17], simply using the definition and induction, and maybe some properties of the ordinal sum. For example: (ii) \( X1 \equiv \{X0^L \} \) \( X0^L \equiv X \equiv \{1X^L + 0 \} \) \( X0^L \equiv 1X \). Further details can be found in [5].

Over what class of games is this definition of ordinal multiplication well defined? It is not for all games, as \( \ast \cdot 2 = \ast \cdot \{1\} = \{\ast \cdot 1 + 0, \ast \cdot 1 + 0\} = \{\ast \cdot \ast \} = 0 \), but \( \ast \cdot \{0, 1\} = \{\ast \cdot 0 + 0, \ast \cdot 0 + 0, \ast \cdot 1 + 0 \} = \{0, \ast \cdot 0, \ast \} \neq 0 \).

So the ordinal product for games does not extend to values, even for games free of reversible options.
The next case to consider is the class of all numbers. At least the ordinal sum of two numbers is always a number, whatever the form of the numbers involved. However, this is not so for the ordinal product, as the following example shows. The definition first gives $\frac{1}{2} \cdot 2 = \frac{3}{2}$, and then if $y$ is the number $\{0,1|2\}(=\frac{1}{2})$,

$$\frac{1}{2} \cdot y = \{\frac{1}{2} \cdot 0 + 0, \frac{1}{2} \cdot 1 - 1, \frac{1}{2} \cdot 2 - 1 | \frac{1}{2} \cdot 0 + 1, \frac{1}{2} \cdot 1 - 0, \frac{1}{2} \cdot 2 - 0\}$$

$$= \{0, \frac{1}{2} - 1, \frac{3}{2} - 1 | 1, \frac{3}{2}, \frac{5}{2}\}$$

$$= \{0, \frac{1}{2} \parallel \frac{3}{2} | \frac{3}{2} \parallel \frac{1}{2}\}$$

and this is not a number.

So the ordinal product is not even closed over all numbers. The next possibility is the class of numbers free of reversible options. Conway called his class of numbers $\text{No}$, so we shall call the subclass of numbers which are free of reversible options $\text{Nor}$.

We shall eventually show that over $\text{Nor}$ the ordinal product is well defined. First we consider some properties enjoyed by the class $\text{Nor}$.

4.2 Numbers free of reversible options. The first result can be found in [4, p. 210].

**Lemma 4.2.** Suppose $X$ and $Y$ are free of reversible options. Then $X = Y$ if and only if each option of either is dominated by a corresponding option of the other.

**Lemma 4.3.** If $x \in \text{Nor}$, then (i) each $x^L \geq$ some $x^R$, and (ii) each $x^R \leq$ some $x^L$.

**Proof.** (i) As $x \in \text{Nor}$, each $x^L \leq x$. So either $x \leq$ some $(x^L)^L$, or some $x^R \leq x^L$. The former case gives $x \leq (x^L)^{RL} < x^L$, as $x^L \in \text{Nor}$, but $x < x^L$ is impossible. (ii) is shown similarly.

**Definition.** An *iterated Left option* of $x$, denoted by $x^L \cdots x_L$, is the position after a consecutive (but finite) number of left moves. Similarly define $x^R \cdots x_R$.

The following lemmas are proved using arguments similar to those in the proof of Lemma 4.3. Further details can be found in [5].

**Lemma 4.4.** Suppose $x \in \text{Nor}$.

(i) If two Left options of $x$ satisfy $x^L \leq x^L$, then $x^L$ is equal to an iterated Left option of $x^L$.

(ii) If two Right options of $x$ satisfy $x^R \leq x^R$, then $x^R$ is equal to an iterated Right option of $x^R$.

(iii) For any pair $(x^L, x^R)$, either $x^R$ equals an iterated Right option of $x^L$, or $x^L$ equals an iterated Left option of $x^R$.

**Lemma 4.5.** Suppose $x, y \in \text{Nor}$ and $x = y$. Then each $x^L = \text{some iterated Left option of } y$ and each $x^R = \text{some iterated Right option of } y$.

**Lemma 4.6.** If $x \in \text{Nor}$ and $x \geq 0$, then all options of $x$ are non-negative.

4.3 Main results. As a reminder, before they are used in the following proofs, we list the following properties of the ordinal sum.
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1. \( x + y \geq x + z \) if and only if \( y \geq z \).
2. The ordinal sum over \( \text{Nor} \) is closed.
3. If \( x, y \in \text{Nor} \) and \( x = y \), then \( x + z = y + z \).
4. All Left (Right) options of \( x \) are Left (Right) options of \( x + y \).

We also note that all \( xy^{y^*} + x^L \) and \( xy^R - x^R \) are iterated Left options of \( xy \), and all \( xy^{y^*} + x^R \) and \( xy^R - x^L \) are iterated Right options of \( xy \).

**Theorem 4.7.** For all \( x, y, z \in \text{Nor} \) we have (1) \( xy \) is a number, and then (2a) \( xy \in \text{Nor} \), (2b) \( y = z \) implies \( xy = xz \).

**Proof.** The proof will be inductive on all parts of the theorem simultaneously.

(1) To verify that \( xy \) is a number, we must show first that each of the options of \( xy \) is a number, and secondly, that each \( (xy)^L \notin \) each \( (xy)^R \).

For each \( y^L \), \( y^R \), induction gives that \( xy^L \), \( xy^R \) are numbers. As the ordinal sum of two numbers is a number, it follows that all options of \( xy \) are numbers. Finally, we are left to check that the following four sets of inequalities hold.

\[(i) \ xy^{y^L} + x^L \notin xy^{y^R} + x^R \quad (ii) \ xy^{y^R} - x^R \notin xy^{y^L} - x^L \]
\[(iii) \ xy^L + x^L \notin xy^R - x^L \quad (iv) \ xy^R - x^R \notin xy^L + x^R \]

(i) Three cases arise, depending on how \( y^L \) and \( y^R \) compare.

Case (a). If \( y^L = y^R \), then induction gives \( xy^{y^L} = xy^{y^R} \) with both in \( \text{Nor} \). So \( xy^{y^L} + x^L = xy^{y^R} + x^L \), as \( x^L \notin x^R \).

Case (b). If \( y^L < y^R \), then by Lemma 4.4, \( y^L = \text{some} (y^L)^L \ldots L \), and so inductively \( xy^{y^L} = x(y^L)^L \ldots L \) with both in \( \text{Nor} \). Hence \( xy^{y^L} + x^L = x(y^L)^L \ldots L + x^L \), but \( x(y^L)^L \ldots L + x^L \) is an iterated Left option of \( xy^{y^L} \), and so also of \( xy^{y^L} + x^R \). Therefore \( xy^{y^L} + x^L \notin xy^{y^L} + x^R \).

Case (c). If \( y^L > y^R \), then by Lemma 4.4, \( y^L = \text{some} (y^L)^R \ldots R \), and so inductively \( xy^{y^L} = x(y^L)^L \ldots L \) with both in \( \text{Nor} \). Hence \( xy^{y^L} + x^L = x(y^L)^L \ldots L + x^L \), but the latter is an iterated Right option of \( xy^{y^L} \), and so also of \( xy^{y^L} + x^R \). Therefore \( xy^{y^L} + x^R > xy^{y^L} + x^L \).

(ii) Use arguments similar to those in (i).

(iii) Lemma 4.4 shows there are two cases, depending on whether \( y^L = \text{some} (y^R)^L \ldots L \), or \( y^R = \text{some} (y^L)^R \ldots R \).

Case (a). If \( y^L = (y^R)^L \ldots L \), then induction gives \( xy^L = x(y^R)^L \ldots L \) with both in \( \text{Nor} \), and so \( xy^L + x^L = x(y^R)^L \ldots L + x^L \). As \( x(y^R)^L \ldots L + x^L \) is an iterated Left option of \( xy^R \), and so also of \( xy^R - x^L \), we must have \( xy^L + x^L < xy^R - x^L \).

Case (b). If \( y^R = (y^L)^R \ldots R \), then induction gives \( xy^R = x(y^L)^R \ldots R \) with both in \( \text{Nor} \), and so \( xy^R - x^R = x(y^L)^R \ldots R - x^R \). As \( x(y^L)^R \ldots R - x^L \) is an iterated Right option of \( xy^L \), and so also of \( xy^R - x^L \), we must have \( xy^R - x^L > xy^L + x^R \).

(iv) Use arguments similar to those in (iii).

(2a) It is next shown that, assuming \( xy \) is a number, \( xy \) is free of reversible options. So we already know \( (xy)^L < xy < (xy)^R \).

By induction, all \( xy^L \) and \( xy^R \) are in \( \text{Nor} \), and hence it follows that each option of \( xy \) is in \( \text{Nor} \). Finally, we have to check that each \( (xy)^L \notin \) each \( (xy)^R \). The Right options of the Left options of \( xy \) are given by

\[
(xy)^L = ((xy)^L)^R = (xy^L + x^L)^R, (xy^R - x^R)^R \]
\[
= \{xy^{LL} + x^L, xy^{LR} - x^L, xy^{RL} + x^R, xy^{RR} - x^L, xy^L + x^LR, xy^R - x^RL\}.
\]
First consider the pair \( xy_{LL} + xR \) and \( xy_{LR} - xL \). Both are Right options of \( xy_{L} \), and so also of \( xy^{+} + x^{R} \), which itself is a Right option of \( xy \). Hence, as \( xy \) is a number, both \( xy_{LL} + xR \) and \( xy_{LR} - xL \) are strictly greater than \( xy \).

Similarly, both \( xy_{RL} + xR \) and \( xy_{RR} - xL \) are Right options of \( xy_{R} \), and so also of \( xy^{+R} - xL \), which itself is a Right option of \( xy \).

Hence, as \( xy \) is a number, both \( xy_{LL} + xR \) and \( xy_{LR} - xL \) are strictly greater than \( xy \).

Similarly, both \( xy_{RL} + xR \) and \( xy_{RR} - xL \) are Right options of \( xy_{R} \), and so also of \( xy^{+R} - xL \), which itself is a Right option of \( xy \).

Now for \( xy^{+} + x^{R} \), Lemma 4.3 gives that \( x^{+} \geq \) some \( x^{L} \). Hence \( x^{+} + x^{R} \geq x^{L} + x^{R} > xy \). Similarly, each \( x^{+L} \leq \) some \( x^{L} \) by Lemma 4.3, and hence \( x^{+} + x^{L} \geq x^{+} - x^{L} > xy \).

Therefore in all cases we have \( (xy)^{R} \leq xy \), with similar arguments also showing each \( (xy)^{R} \leq xy \), and so \( xy \) is free of reversible options.

(2b) Now assume \( z = y \), and we ask: is \( xy = xz \)? Yes, if inequalities like \( xy \leq (xz)^{L} \), for example, can be established. By Lemma 4.5, each \( z^{L} = \) some \( y^{L} \cdot \cdot \cdot L \), and so by induction \( xz^{L} = xy^{L} \cdot \cdot \cdot L \). Also, by induction, both \( xz \) and \( xy \cdot \cdot \cdot L \) are in \( Nor \), and hence \( xz^{L} + x^{L} = xy^{L} \cdot \cdot \cdot L + x^{L} \). As \( xy^{L} \cdot \cdot \cdot L + x^{L} \) is an iterated Left option of \( xy \) and \( xy \) is assumed to be a number, \( xz^{L} + x^{L} < xy \). Similarly, each \( z^{R} = \) some \( y^{R} \cdot \cdot \cdot R \), by Lemma 4.5, and so \( xz^{R} = xy^{R} \cdot \cdot \cdot R \) by induction, with both in \( Nor \), and hence \( xz^{R} - x^{R} = xy^{R} \cdot \cdot \cdot R - x^{R} < xy \). So \( xy \leq \) each \( (xz)^{L} \), and all other inequalities are shown similarly.

This then completes the proof of the theorem.

In particular, Theorem 4.7 says that over \( Nor \) the value of \( xy \) is independent of the form of \( y \). The next result shows that this is also true for \( x \).

**Theorem 4.8.** If \( x_{1}, x_{2}, y \in Nor \) and \( x_{1} = x_{2} \), then \( x_{1}y = x_{2}y \).

**Proof.** By Theorem 4.7, both \( x_{1}y \) and \( x_{2}y \) belong to \( Nor \). Therefore Lemma 4.2 gives that \( x_{1}y = x_{2}y \) if and only if each option of either is dominated by a corresponding option of the other. Consider the two typical Left options of \( x_{1}y \).

(i) \( x_{1}y^{L} + x_{1}^{L} \). By induction, \( x_{1}y^{L} = x_{2}y^{L} \). Also, as \( x_{1} = x_{2} \), we can apply Lemma 4.2 to give that each \( x_{1}^{L} \leq \) some \( x_{2}^{L} \). Hence \( x_{1}y^{L} + x_{1}^{L} = x_{2}y^{L} + x_{1}^{L} \leq x_{2}y^{L} + x_{2}^{L} \), and so \( x_{1}y^{L} + x_{1}^{L} \) is dominated by a Left option of \( x_{2}y \).

(ii) \( x_{1}y^{R} - x_{1}^{R} \). Again \( x_{1}y^{R} = x_{2}y^{R} \) by induction, and so \( x_{1}y^{R} - x_{1}^{R} = x_{2}y^{R} - x_{2}^{R} \leq x_{2}y^{R} - x_{2}^{R} \), as some \( x_{2}^{L} \leq x_{1}^{L} \) by Lemma 4.2.

Similarly, the Right options of \( x_{1}y \) and the Left and Right options of \( x_{2}y \) are dominated by corresponding options. Hence \( x_{2}y = x_{1}y \).

Theorems 4.7 and 4.8 combine to give the following.

**Corollary 4.9. The ordinal product is well defined over Nor.**

The next result is the ordinal product equivalent of Theorem 87 of [4]. It shows that over \( Nor \), the ordinal product is what Conway would call an example of a strictly order-preserving (SOP) function.

**Theorem 4.10.** If \( x, y, z \in Nor \) and \( x > 0 \), then \( xy \geq xz \) if and only if \( y \geq z \).

**Proof.** If \( xy \geq xz \), then each \((xy)^{R} \leq xz \) and \( xy \leq \) each \((xz)^{L} \). Specifically, \( xy^{R} \geq xz \cdot \cdot \cdot z + x^{L} \) for all \( y^{R} \), as each \( x^{L} > 0 \) by Lemma 4.6. Also, \( xy \leq xz^{L} + x^{L} \geq xz^{L} \), for all \( z^{L} \). Hence, by induction, each \( y^{R} \leq z \) and \( y \leq \) each \( z^{L} \), and so \( y \geq z \).
Now suppose that \( y \geq z \). If \( y = z \), then \( xy = xz \) by Theorem 4.7, so consider when \( y > z \). That is, \( z \not< y \), and so either some \( z^R \leq y \) or \( z \leq y^L \). Hence, by induction, either \( xz^R \leq xy \) or \( xz \leq xy^L \), and so either \( xz^R - x^L \leq xz^R \leq xy \) or \( xz \leq xy^L \leq xy^L + x^L \). Therefore either some \( (xz)^R \leq xy \) or \( xz \leq some (xy)^L \), and hence \( xz \not> xy \). Also, as \( xy \) and \( xz \) are numbers, \( xy > xz \).

**Corollary 4.11.** If \( x, y \in \text{Nor} \) and \( x > 0, y > 0 \), then \( xy > 0 \).

**Proof.** By Theorem 4.10, \( xy > x0 = 0 \).

5. Ordinal exponentiation

Cantor also defined an exponentiation operation. If \( \alpha = \{\alpha^L | \} \) and \( \beta = \{\beta^L | \} \) are two ordinals, then \( \alpha^\beta \) is the repeated ordinal product of \( \alpha \), \( \beta \) times; that is, informally, \( \alpha^\beta = \alpha \cdot \alpha \ldots (\beta \text{ times}) \). Now the right-hand side of this has Left options \( (\alpha \cdot \alpha \ldots (\beta^L \text{ times}) \cdot \alpha)^L \), that is, \( (\alpha^\beta \cdot \alpha)^L \). Hence we can inductively define \( \alpha^\beta \) to be

\[
\alpha^\beta = \{0, (\alpha^\beta \cdot \alpha)^L \} = \{0, \alpha^\beta \cdot \alpha^L + (\alpha^\beta)^L \}.
\]

This can be extended to give a definition of \( X^\beta \), for positive numbers \( X \). Now again \( X^\beta \) should be \( X \cdot X \ldots (\beta \text{ times}) \), which has Left options \( (X \cdot X \ldots (\beta^L \text{ times}) \cdot X)^L \) and \( (X \cdot X \ldots (\beta^L \text{ times}) \cdot X)^R \) for Right options. Hence for ordinal \( \beta \) and \( X > 0 \),

\[
X^\beta = \{0, (X^\beta \cdot X)^L | (X^\beta \cdot X)^R \}
= \{0, X^{\beta^L} \cdot X^L + (X^\beta)^L, X^{\beta^L} \cdot X^R - (X^\beta)^R | X^{\beta^L} \cdot X^L + (X^\beta)^R, X^{\beta^L} \cdot X^R - (X^\beta)^L \}.
\]

This definition satisfies all the expected properties of an ordinal exponential definition, as the following theorem shows. Each proof is again of the type Conway calls a ‘1-line proof’.

**Theorem 5.1.** \( X^\beta \), as defined by (3), satisfies (i) \( X^0 = 1 \), (ii) \( X^1 = X \), (iii) \( 1^\beta = 1 \), (iv) \( X^{\alpha \beta} = X^{\alpha} \cdot X^\beta \), (v) \( (X^\beta)^\beta = X^{\alpha \beta} \).

Now, can definition (3) be extended to give a full definition of \( X^Y \) which satisfies the same five properties as Theorem 5.1? A number of possible definitions have been tried, but none seems to satisfy the fifth property, \( (X^Y)^Z = X^{YZ} \).

Another approach is to suspend the search for a definition of \( X^Y \), and instead assume that a definition already exists satisfying all five properties. Any \( Y \) can be written as \( Y = (\pm 1) + (\pm 1) + \ldots \), and so \( X^Y = X^{(\pm 1)+(\pm 1)+\ldots} = X^{\pm 1 \cdot X \pm 1 \ldots} \). As \( X^1 = X \), we therefore only need to know \( X^{-1} \) in order to compute \( X^Y \). Also, \( (X^{-1})^{-1} = X^{-1\cdot-1} = X^1 = X \). A plausible definition for \( X^{-1} \) is

\[
X^{-1} = \{0, (X^R)^{-1} | (X^L)^{-1} \}.
\]

This definition gives the sign-expansion of \( X^{-1} \) to be the reverse of that of \( X \), except that the first sign is unaltered. However, this definition still does not lead to a definition of \( X^Y \) which satisfies \( (X^Y)^Z = X^{YZ} \).

So all attempts at a generalisation of ordinal exponentiation have partially failed. This is probably not surprising, as equalities like \( (-\alpha) + (-\beta) = -(\alpha + \beta) \), \( (-\alpha) \beta = \alpha(-\beta) = - (\alpha \beta) \), for example, give some motivation to why the ordinal sum and product can be extended to all numbers. However, no such motivation exists for exponentiation; for instance, what should \( \alpha^{-1} \) be?
References


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