The role of Michael Batanin's monoidal globular categories[1]

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This is a report[2] on recent work of Michael Batanin [3] which replaces and extends [2]. The goal of his work is to provide an environment for defining the concepts associated with weak ω-categories and for developing the ensuing theory. The approach is "globular".

WHAT ARE MONOIDAL GLOBULAR CATEGORIES ?

Let $\mathbb{G}$ denote the subcategory of **Set** consisting of the sets $\{0,n\}$, $n \geq 0$, and, apart from identity functions, only the constant functions $\{0,n\} \dashrightarrow \{0,m\}$ with $n > m$. A *globular object* X of a category $\mathcal{X}$ is a functor $X : \mathbb{G} \dashrightarrow \mathcal{X}$. We write X_n for $X\{0,n\}$ and $s_m, t_m : X_n \dashrightarrow X_m$ for the image under X of the functions $\{0,n\} \dashrightarrow \{0,m\}$ constant at 0, m, respectively.

$$X_3 \underset{t_3}{\overset{s_3}{\rightrightarrows}} X_2 \underset{t_2}{\overset{s_2}{\rightrightarrows}} X_1 \underset{t_1}{\overset{s_1}{\rightrightarrows}} X_0 \underset{t_0}{\overset{s_0}{\rightrightarrows}}$$

The globular object X is called *n-skeletal* when $X_m = X_n$ and s_m, t_m are identities for all $m > n$. The globular object X is called *reflexive* when it is equipped with arrows $i : X_n \dashrightarrow X_{n+1}$ for all $n \geq 0$ such that $s_n \circ i = t_n \circ i = 1_{X_n}$. We also write i or $i^k : X_n \dashrightarrow X_{n+k}$ for the arrow obtained by composing k consecutive i arrows.

Suppose the category $\mathcal{X}$ has pullbacks. For $m < n$, define the object $X_n \underset{m}{\times} X_n$ to be the following pullback.

$$\begin{array}{ccc} X_n \underset{m}{\times} X_n & \xrightarrow{p_2} & X_n \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle t_m} \\ X_n & \xrightarrow[s_m]{} & X_m \end{array}$$

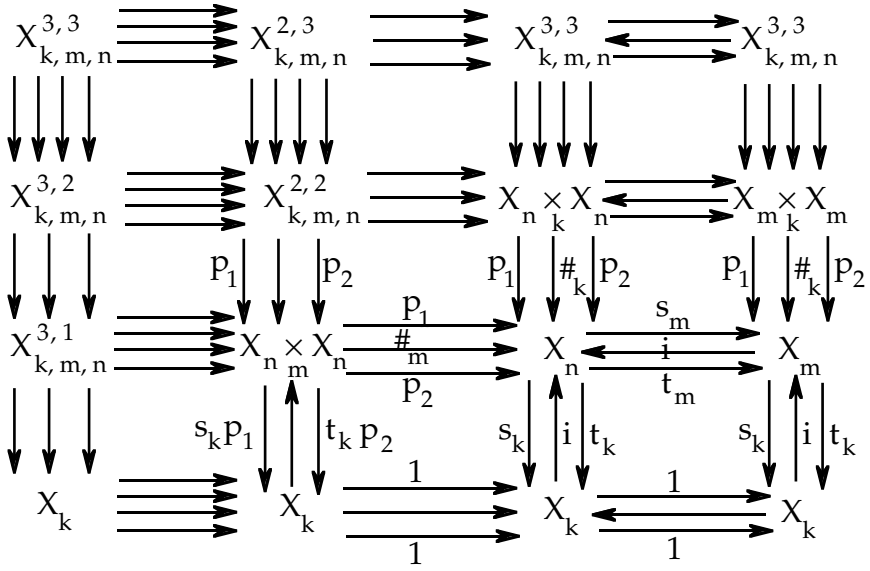
Recall that an *ω-category in* $\mathcal{X}$ is a reflexive globular object X together with arrows

$$\#_m : X_n \underset{m}{\times} X_n \dashrightarrow X_n \quad \text{for all } m < n$$

such that, for all $k < m < n$, the following induced 4×4 diagram is a truncated double simplicial object where each $X^{r,s}_{k,m,n}$ is the pullback of the previous outside pair of arrows.

[1] These are notes of two invited lectures at the Conference on Higher Category Theory and Mathematical Physics, Northwestern University (Evanston, Illinois; 28-30 March 1997).

[2] All that is correct is Batanin's; all errors are mine.



When $\mathcal{X}$ = **Set**, a globular object is called a *globular set* (or "ω-graph") and an ω-category in $\mathcal{X}$ is called an *ω-category.* We are also interested in the case $\mathcal{X}$ = **Cat** when a globular object is called a *globular category;* these form a 2-category **GC**. However, we are interested in something weaker than ω-categories in **Cat** – Batanin calls them *monoidal globular categories.* These are reflexive globular categories $\mathcal{C}$ together with functors $\#_m : \mathcal{C}_n \times_m \mathcal{C}_n \dashrightarrow \mathcal{C}_n$ satisfying

$s_m(A \#_m B) = s_m A, \quad t_m(A \#_m B) = t_m A, \quad s_m(A \#_k B) = s_m A \#_k s_m B, \quad t_m(A \#_k B) = t_m A \#_k t_m B,$

and globular natural isomorphisms

$$\alpha_m : (A \#_m B) \#_m C \dashrightarrow A \#_m (B \#_m C) \quad \textit{(associativity constraint)}$$

$$\lambda_m : i\, s_m A \#_m A \dashrightarrow A \quad \textit{(left unit constraint)}$$

$$\rho_m : A \#_m i\, s_m A \dashrightarrow A \quad \textit{(right unit constraint)}$$

$$\eta_{k,m} : (A \#_k B) \#_m (C \#_k D) \dashrightarrow (A \#_m C) \#_k (B \#_m D) \quad \textit{(interchange constraint)},$$

for all $k < m < n$ and all $A, B, C, D \in \mathcal{C}_n$ for which the expressions make sense, such that the obvious induced $4 \times 4 \times 4$ pseudo-diagram is a truncated triple pseudo-simplicial category. There is 2-category **MGC** of monoidal globular categories; the arrows are globular functors $F : \mathcal{C} \dashrightarrow \mathcal{D}$ equipped with natural isomorphisms $\phi_2 : FA \#_m FB \dashrightarrow F(A \#_m B)$, $\phi_0 : iFA \dashrightarrow FiA$ satisfying appropriate coherence conditions.

HOW ABOUT SOME EXAMPLES ?

1. An ω-category is precisely a monoidal globular category $\mathcal{C}$ for which each $\mathcal{C}_n$ is a discrete category.

2. An ω-category in **Cat** is precisely a strict monoidal globular category (that is, all the constraints are identities).

3. A bicategory is precisely a 1-skeletal monoidal globular category $\mathcal{C}$ for which $\mathcal{C}_0$ is a discrete category.

4. A double category is a 1-skeletal strict monoidal globular category.

5. A braided monoidal category is precisely a 2-skeletal monoidal globular category $\mathcal{C}$ for which $\mathcal{C}_0$ and $\mathcal{C}_1$ are equal to the terminal category $\mathbb{1}$.

6. For each $n > 2$, a symmetric monoidal category is precisely an n-skeletal monoidal globular category $\mathcal{C}$ for which $\mathcal{C}_m = \mathbb{1}$ for all $m < n$.

7. Given a globular category $\mathcal{C}$, we can define its *suspension* to be the globular category $\mathcal{C}^+$ with $\mathcal{C}^+_0 = \mathbb{1}$, $\mathcal{C}^+_n = \mathcal{C}_{n-1}$ for $n > 0$ with s_m, t_m translated by 1 for $m > 0$, and with s_0, t_0 the only functors possible. A monoidal structure on $\mathcal{C}^+$ gives a monoidal structure on $\mathcal{C}$ by translating by 1; but it also gives extra functors $\otimes = \#_{-1} : \mathcal{C}_n \times \mathcal{C}_n \dashrightarrow \mathcal{C}_n$ and objects $I_n \in \mathcal{C}_n$. We call the monoidal globular set $\mathcal{C}$ *augmented* when it is equipped with this extra structure making $\mathcal{C}^+$ monoidal. There is an obvious 2-category **AMGC** of augmented monoidal globular categories.

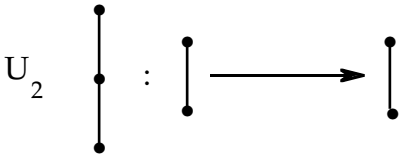
THE ALGEBRA OF TREES

The general operation, called *pasting*, available in an ω-category has received considerable attention. In order to discuss general pasting I introduced a very general concept of higher dimensional graph, called *computad*. The definition of n-computad is recursive: *a priori* the construction of the free (n–1)-category on an (n–1)-computad is required. It seems clear that some form of higher pasting should be involved in the definition of weak n-category. For example, as the main axiom for a tricategory looks like the fifth oriental or associahedron, it seems likely that we will need computads more general than ω-graphs. A useful kind of computad which can be defined directly, without recursion, is *parity complex* (or "pasting scheme"). These are globular objects in the category whose objects are finite sets and whose arrows are relations: but there are axioms which imply a lack of circuits ("loop freeness").

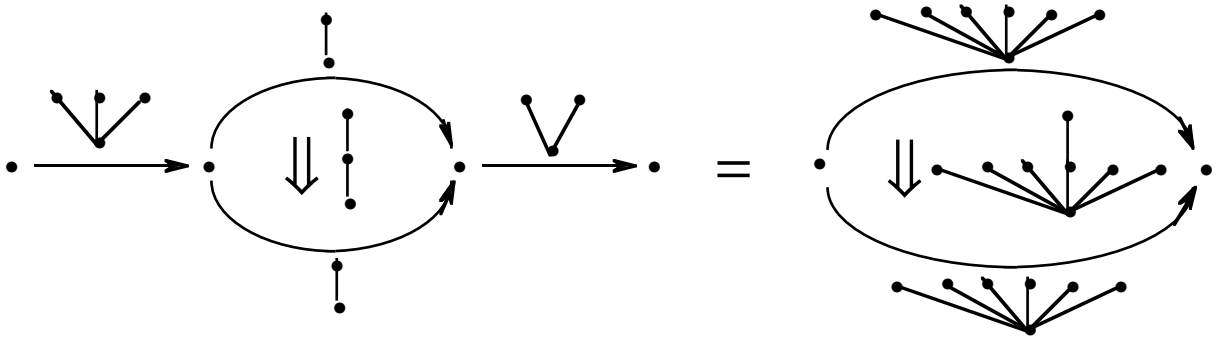
A major insight of Batanin was that a definition of weak ω-category could exist using the pasting generated by ω-graphs (= globular sets). This seems very restrictive since, in a 2-category, this would mean pasting only diagrams of 2-cells whose sources and targets were single arrows, not paths. As in many similar situations, this kind of pasting is controlled by how it behaves on the *terminal globular set*. That is, we must first understand the free ω-category **Tr** on the terminal globular set **1**: the globular set with exactly one element U_n in each dimension. This is as non loop free as one could get: *all* cells are endomorphisms. So we cannot resort to the construction of the free ω-category on a parity complex.

Clearly **Tr** has one 0-cell $U_0 = \bullet$ since there are no operations creating new 0-cells from old. The underlying 1-category of **Tr** is the free category on the terminal graph: it has one object U_0 and one arrow $U_1 \#_0 U_1 \#_0 \ldots \#_0 U_1 : U_0 \dashrightarrow U_0$ for each natural number n; we denote this by what Trimble calls the *n-sprout:*

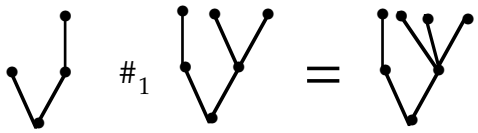
Then we have the 2-cell $U_2 : U_1 \Rightarrow U_1 : \bullet \dashrightarrow \bullet$ which we denote by the tree



and can then whisker on the right by an n-sprout and on the left by an m-sprout: the result is an (m+1+n)-sprout with an extra edge going up from the (m+1)-st top vertex of the sprout.



The composition $\#_1$ works as follows.



This leads us to the following model of the ω-category **Tr**.

We write $\mathbb{\Delta}$ for the category whose objects are finite ordinals $[n] = \{0,1,\ldots,n\}$, including the empty ordinal $[-1]$, and whose arrows are order-preserving functions. An *n-stage tree* is a functor $T : [n]^{op} \dashrightarrow \mathbb{\Delta}$ such that $T(0) = [0]$. Elements of $T(i)$ are called *vertices of* T *of height* i. We obtain a planar graph from T by defining there to be an edge e from $v' \in T(i+1)$ to $v \in T(i)$ when v is the value of the function $T(i < i+1) : T(i+1) \dashrightarrow T(i)$ at v′. Such an edge e is called an *input edge at* v. Vertices with no input edges are called *input vertices of* T or *leaves of* T. Write $\mathcal{T}r_n$ for the functor category $[[n]^{op}, \mathbb{\Delta}]$ of n-stage trees[3]. Restriction along the inclusion $[m] \dashrightarrow [n]$ for $m < n$ gives a functor $\partial_m : \mathcal{T}r_n \dashrightarrow \mathcal{T}r_m$; and we make a globular category $\mathcal{T}r$ by putting $s_m = t_m = \partial_m$. We make $\mathcal{T}r$ reflexive by defining $i : \mathcal{T}r_n \dashrightarrow \mathcal{T}r_{n+1}$ as follows:

[3] Notice that the only invertible arrows in this category are identities. Consequently, limits which exist are unique (not just up to isomorphism) and the skeleton of the category is the set of objects as a discrete category.

$$(iT)(m) = T(m) \text{ for } m \leq n, \qquad (iT)(n+1) = [-1].$$

Indeed, $\mathcal{T}r$ becomes a strict monoidal globular category by defining $\#_m : \mathcal{T}r_n \times_m \mathcal{T}r_n \dashrightarrow \mathcal{T}r_n$ via the following pushout for all n-stage trees S, T with $\partial_m S = \partial_m T$.

$$\begin{array}{ccc} i^m\partial_m S & \longrightarrow & T \\ \downarrow & & \downarrow \\ S & \longrightarrow & S \#_m T \end{array}$$

The ω-category **Tr** is defined by taking the skeleta of the categories $\mathcal{T}r_n$ with the ω-category structure induced from $\mathcal{T}r$.

We write U_n for the n-stage tree with one vertex of each height $m \leq n$. The fundamental role played by **Tr** is explained by the next result which we have indicated above.

Theorem *The ω-category* **Tr** *is freely generated by its cells* U_n, $n \geq 0$.

FREE MONOIDAL GLOBULAR CATEGORIES

It is fairly clear that the forgetful functor ω-Cat(**Cat**) ---> Glob(**Cat**) from strict monoidal globular categories to globular categories is monadic. In particular, it has a left adjoint

$$D_s : \text{Glob}(\mathbf{Cat}) \dashrightarrow \omega\text{-Cat}(\mathbf{Cat}).$$

The explicit description of $D_s(\mathcal{C})$ is a crucial aspect of this work. If $\mathcal{C}$ is terminal in Glob(**Cat**), the Theorem of the last Section tells us that $D_s(\mathcal{C}) = \mathbf{Tr}$. So we expect trees to be involved in the description of $D_s(\mathcal{C})$ for a general globular category $\mathcal{C}$. Just as the free strict monoidal category on a category $\mathcal{X}$ has as objects the families of objects of $\mathcal{X}$ indexed by the free strict monoidal category $\mathbb{N}$ on the terminal category, the objects of $D_s(\mathcal{C})$ are "families of objects of $\mathcal{C}$ indexed by trees" or "$\mathcal{C}$-labelled trees".

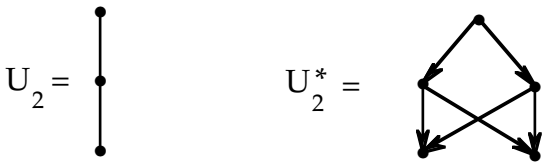
A *$\mathcal{C}$-labelling for a tree* $T \in \mathcal{T}r_n$ is a globular functor $A : T^* \dashrightarrow \mathcal{C}$ where T^* is a globular set determined by T as follows. For vertices u, v of T, write $u \vee v$ for the highest vertex for which there exist directed paths $p : u \dashrightarrow u \vee v$, $q : v \dashrightarrow u \vee v$ of edges in T. The globular set T^* is generated by the leaves of T in dimensions equal to their height, subject only to the relations

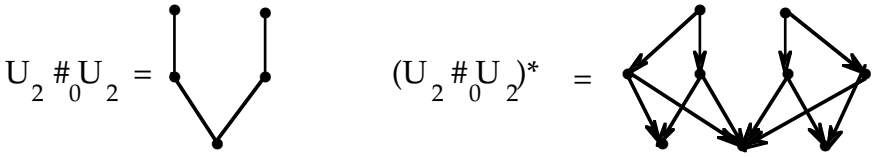
$$t_m(u) = s_m(v)$$

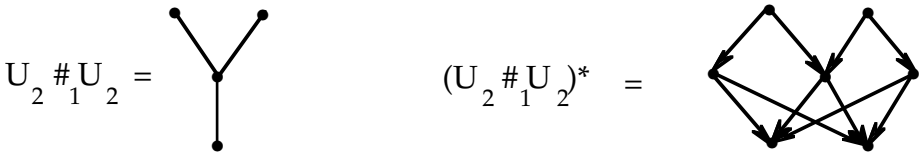
for leaves u, v having $u \vee v \in T(m)$ and either

u, v are consecutive vertices of the same height, or

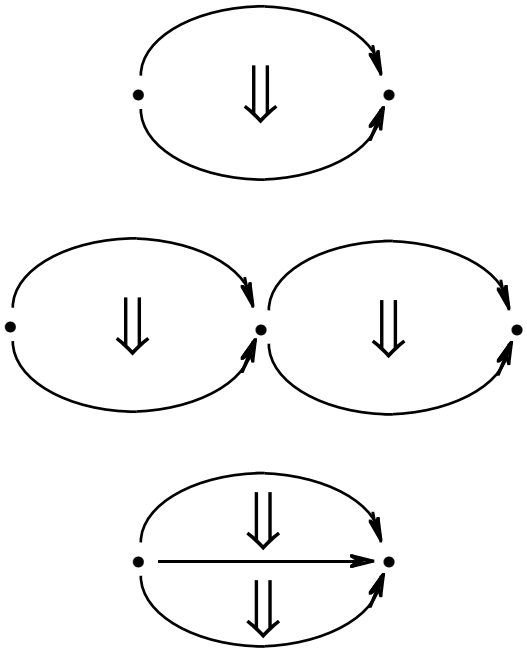
the paths p, q go through only maximal, minimal (respectively) edges of T at every height. It is in fact possible to reconstruct the tree T from the globular set T^*. Some examples of T^* are displayed below.







Respectively, in these examples, globular functors $A : T^* \dashrightarrow \mathcal{C}$ amount to globular pasting diagrams in $\mathcal{C}$ of the following shapes.



We define an *n-stage diagram in* $\mathcal{C}$ to be a pair (T,A) where T is an n-stage tree and A is a $\mathcal{C}$-labelling for T. (Later we shall see how to assign a *value* [A] to such a diagram when $\mathcal{C}$ is monoidal.)

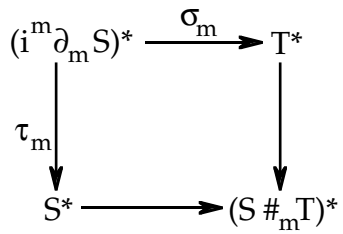
The category $D_s(\mathcal{C})_n$ is the category whose objects are n-stage diagrams in $\mathcal{C}$ and whose arrows are pointwise globular natural transformations. The obvious inclusions

$$\sigma_m, \tau_m : (\partial_m U_n)^* = U_m^* \dashrightarrow U_n^*$$

generate globular functions

$$\sigma_m, \tau_m : (\partial_m T)^* \dashrightarrow T^*$$

for every n-stage tree T, and restriction along these gives the required globular category structure on $D_s(\mathcal{C})$. Since every n-stage diagram (T,A) can be regarded as an (n+1)-stage diagram (iT,A), we have our reflexive structure. For the monoidal structure we use the following pushouts of globular sets which exist for m-composable trees S, T.



Since diagrams (S,A), (T,B) are m-composable when $A\tau_m = B\sigma_m$, the pushout yields a globular functor $A\#_m B : (S\#_m T)^* \dashrightarrow \mathcal{C}$; so we put $(S,A)\#_m(T,B) = (S\#_m T, A\#_m B)$. There is a globular functor $\upsilon : \mathcal{C} \dashrightarrow D_s(\mathcal{C})$ which takes $C \in \mathcal{C}_n$ to the diagram (U_n, C) where C is identified with the globular functor $U_n^* \dashrightarrow \mathcal{C}$ whose value at the unique highest node is C.

Theorem $D_s(\mathcal{C})$ *is the free strict monoidal globular category on a globular category* $\mathcal{C}$.

COHERENCE FOR MONOIDAL GLOBULAR CATEGORIES

By a familiar-style recursive construction, it is possible to describe the free monoidal globular category $D(\mathcal{C})$ on a globular category $\mathcal{C}$. The universal property of the inclusion $\mathcal{C} \dashrightarrow D(\mathcal{C})$ of generators is that, for each monoidal globular category $\mathcal{D}$ and each globular functor $F : \mathcal{C} \dashrightarrow \mathcal{D}$, there exists a unique strict monoidal globular functor $F' : D(\mathcal{C}) \dashrightarrow \mathcal{D}$ whose restriction to $\mathcal{C}$ is F. In particular, there is a strict monoidal globular functor

$$\Gamma : D(\mathcal{C}) \dashrightarrow D_s(\mathcal{C})$$

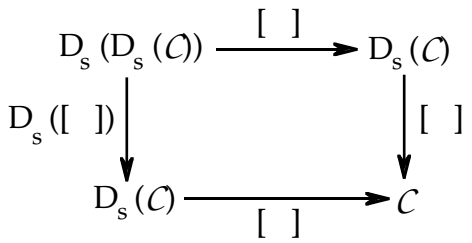
whose restriction to $\mathcal{C}$ is $\upsilon : \mathcal{C} \dashrightarrow D_s(\mathcal{C})$.

Theorem *For all globular categories* $\mathcal{C}$, *the monoidal globular functor* $\Gamma : D(\mathcal{C}) \dashrightarrow D_s(\mathcal{C})$ *is an equivalence. Every monoidal globular functor* $D(\mathcal{C}) \dashrightarrow \mathcal{D}$ *is isomorphic to a strict monoidal globular functor* $D(\mathcal{C}) \dashrightarrow \mathcal{D}$.

For any <u>monoidal</u> globular category $\mathcal{C}$, let $\mu : D(\mathcal{C}) \dashrightarrow \mathcal{C}$ be the unique strict monoidal globular functor whose restriction to $\mathcal{C}$ is the identity. The composite of μ with an inverse equivalence for Γ gives a monoidal globular functor

$$[\ ] : D_s(\mathcal{C}) \dashrightarrow \mathcal{C}$$

whose composite with $\upsilon : \mathcal{C} \dashrightarrow D_s(\mathcal{C})$ is isomorphic to the identity of $\mathcal{C}$. This gives a *value* $[A] \in \mathcal{C}_n$ to each n-stage diagram (T,A) in $\mathcal{C}$. The following square commutes up to canonical isomorphism.



Corollary *Every globular monoidal category* $\mathcal{C}$ *is equivalent to a strict one* $\mathbf{st}(\mathcal{C})$ *which is the full image of* $[\ ]: D_s(\mathcal{C}) \dashrightarrow \mathcal{C}$.

By applying these results to $\mathcal{C}^+$ we get coherence results for the augmented case.

THE ALGEBRA OF HIGHER SPANS

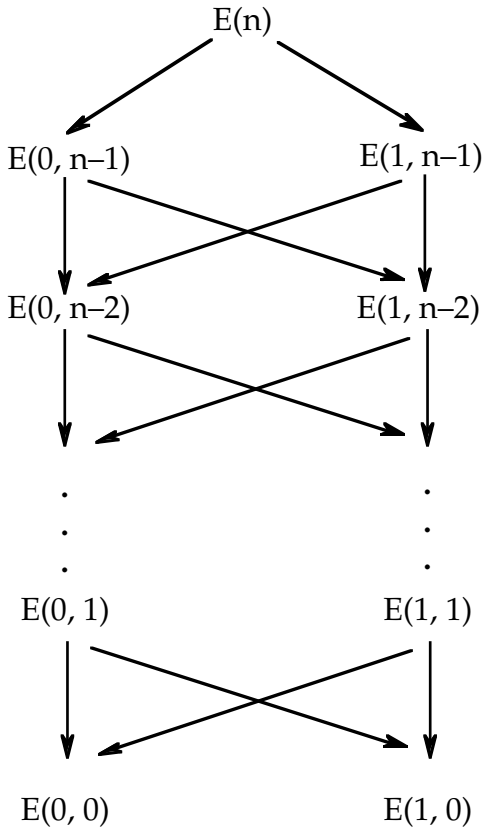
There is an augmented monoidal globular category **Span** in which all monoidal globular categories are "enriched". For each natural number n, let $\mathbf{Oct}_n$ be the poset

$$(\{0, 1\} \times \{0, 1, \ldots, n-1\}) \cup \{n\}$$

where n is the maximum element and $(\alpha,k) < (\beta,m)$ iff $k < m$. Define functions

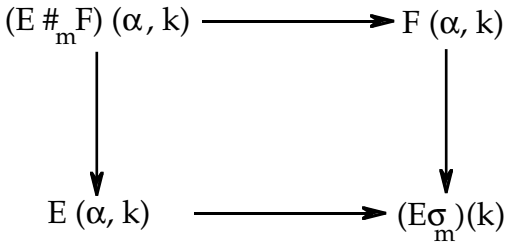
$$\sigma_m, \tau_m : \mathbf{Oct}_m \dashrightarrow \mathbf{Oct}_n$$

by $\sigma_m(\alpha,k) = \tau_m(\alpha,k) = (\alpha,k)$ for $k < m$, $\sigma_m(m) = (0,m)$, $\tau_m(m) = (1,m)$. An *n-span*[4] in a finitely complete category $\mathcal{X}$ is a functor $E : \mathbf{Oct}_n{}^{op} \dashrightarrow \mathcal{X}$ and a morphism of n-spans is a natural transformation; this gives a functor category $\mathbf{Span}(\mathcal{X})_n$.



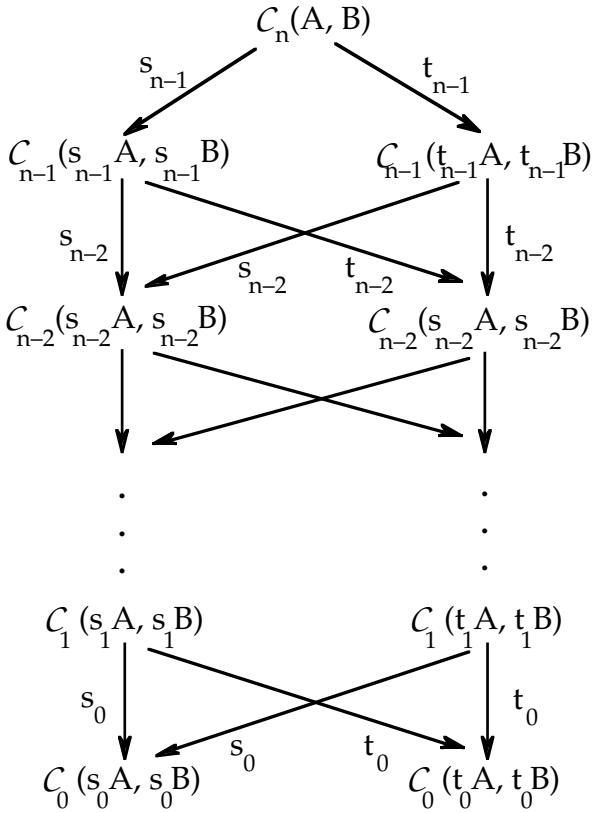
[4] In Paris, 1994, P. Cartier gave a lecture on 'multicategories' where n-spans were called 'telescopes of height n'. Stephen Lack kept notes.

Restriction along σ_m, τ_m gives the globular category structure s_m, t_m on the sequence **Span**($\mathcal{X}$) of categories **Span**($\mathcal{X}$)$_n$, $n \geq 0$. For each n-span E, the (n+1)-span iE is given by iE(α,k) = E(α,k) for $k < n$ and iE(α,n) = iE(n+1) = E(n); this makes **Span**($\mathcal{X}$) reflexive. If E, F are m-composable n-spans then the composite $E \#_m F$ is given by the pullbacks



and with the m-source, m-target of $E \#_m F$ equal to $s_m E$, $t_m F$, respectively. With the canonical constraints coming from the universal property of pullback, **Span**($\mathcal{X}$) becomes a monoidal globular category. Taking $\#_{-1}$ to be binary categorical product, we see that **Span**($\mathcal{X}$) becomes augmented. (In fact, this augmenting process can be iterated: **Span**($\mathcal{X}$)$^+$ becomes augmented by again taking $\#_{-1}$ to be product.)

Merely write **Span** for the augmented monoidal globular category **Span**(**Set**). Suppose $\mathcal{C}$ is any globular category. For all objects A, B of $\mathcal{C}_n$, we obtain the n-span $HOM_{\mathcal{C}}(A,B)$ as follows.



For a category $\mathcal{X}$ with finite colimits, there is the augmented globular category **Cospan**($\mathcal{X}$) = **Span**($\mathcal{X}^{op}$)^{op}.

LAX-MONOIDAL CATEGORIES

Batanin's notion of operad is a monoid in his category of "collections". Yet, the category of collections is not monoidal in the usual sense. Fortunately, it is monoidal enough to admit the definition of internal monoids and to allow the construction of free such monoids.

A *lax-monoidal category* consists of a category $\mathcal{V}$ together with n-ary functorial operations

$$\underset{n}{\otimes} : \mathcal{V} \times \ldots \times \mathcal{V} \dashrightarrow \mathcal{V}$$

for all $n \geq 0$ (where we put $\underset{n}{\otimes}(A_1, \ldots, A_n) = A_1 \otimes - \ldots - \otimes A_n$ for $n > 1$, require $\underset{1}{\otimes}$ to be the identity functor, and identify $\underset{0}{\otimes}$ with an object of $\mathcal{V}$ denoted by I) and natural transformations

$$\alpha_{k,m} : A_1 \otimes - \ldots - \otimes A_n \dashrightarrow \dashrightarrow {}_1 \otimes - \ldots - \otimes (A_k \otimes - \ldots - \otimes A_{k+m}) \otimes - \ldots - \otimes A_n$$

for all $1 \leq k$, $-1 \leq m$, $k + m \leq n$, satisfying various axioms.

In fact, there is a natural **Cat**-operad h (that Batanin has used before [1]) whose algebras are lax-monoidal categories. The objects of the ordered category h_n are bracketed words w in two symbols 0, 1 where n is the number of 0s; the order is defined by declaring $w \leq w'$ if and only if w can be obtained from w' by removal of some brackets or deletion of some 1s. For example, we have

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in h_3. There is a functor

$$\gamma : h_{j_1} \times \ldots \times h_{j_2} \times h_n \longrightarrow h_{j_1 + \ldots + j_n}$$

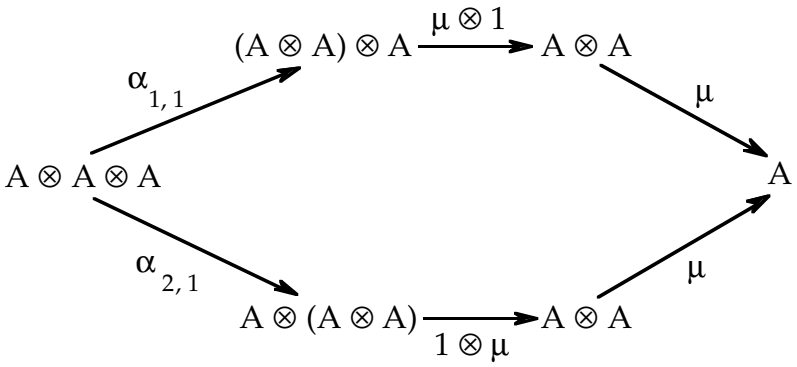
which substitutes the object of h_{j_i} with brackets around it in the place of the i-th 0 in h_n.

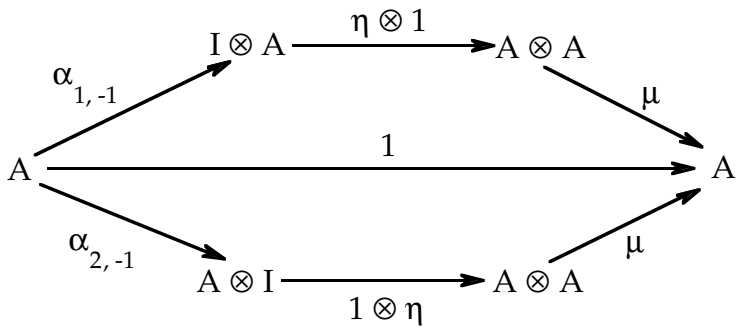
Coherence Theorem *Lax-monoidal structures on a category* $\mathcal{V}$ *are in natural bijection with* h*-algebra structures on* $\mathcal{V}$.

A *monoid* in a lax-monoidal category $\mathcal{V}$ is an object A together with arrows

$$\mu : A \otimes A \dashrightarrow A, \quad \eta : I \dashrightarrow A$$

such that the following diagrams commute.





Proposition *Free monoids exist in any lax-monoidal category* $\mathcal{V}$ *that admits coproducts, preserved in each variable by n-fold tensor product, and has each* $\alpha_{k,\,m}$ *a coprojection into a coproduct.*

Proof The usual "geometric series formula" needs modifying. Let X be an object of $\mathcal{V}$. For each object w of h_n, we let X^w denote the object of $\mathcal{V}$ obtained by replacing each 0 in w by X, replacing each 1 in w by I, leaving the brackets, and inserting ⊗s in a meaningful way. This gives a functor h_n ---> $\mathcal{V}$ taking w to X^w: the colimit of this functor (which exists because of the hypotheses on $\mathcal{V}$) is denoted by $X^{\otimes n}$. The free monoid on X is then given by

$$F(X) = \sum_{n \geq 0} X^{\otimes n} .$$

COLLECTIONS

A *collection* in a globular category category $\mathcal{C}$ is a globular functor A : **Tr** ---> $\mathcal{C}$. Write **Coll**($\mathcal{C}$) = **GC**(**Tr**,$\mathcal{C}$) for the category of collections in $\mathcal{C}$.

If $\mathcal{C}$ is merely $\mathcal{V}^+$ for a monoidal category $\mathcal{V}$ then a collection in $\mathcal{C}$ is just a sequence of objects A_n, n ≥ 0, of $\mathcal{V}$. These are the correct objects to support the usual structure of (non-permutative) operads on $\mathcal{V}$.

To support Batanin's higher operads he requires collections in spans. At this point notice that the category of collections in **Span** is isomorphic to the category of **Set**-valued functors on the category whose objects are trees $T \in \mathbf{Tr}_n$ and whose arrows are generated by formal symbols s, t : T ---> ∂T subject to the usual source-target relations for successive pairs.

We take the liberty of stating the following result rather imprecisely to show where we are headed. The construction required will be explained afterwards.

Theorem *If* $\mathcal{C} \in$ **AMGC** *admits coproducts which behave well under the augmented monoidal structure (such as* $\mathcal{C}$ = **Span***) then* **Coll**($\mathcal{C}$) *is a lax-monoidal category.*

For each globular category $\mathcal{G}$, put $D_s^n \mathcal{G} = D_s D_s \ldots D_s \mathcal{G}$ (n terms). Let

$$\delta_n : D_s^n \mathcal{G} \dashrightarrow \mathbf{Tr}$$

be the strict monoidal globular functor induced by the unique globular functor from $D_s^{n-1}\mathcal{G}$ to the terminal globular category; explicitly, $\delta_n(T,\sigma : T^* \dashrightarrow D_s^{n-1}\mathcal{G}) = T$. For $\mathcal{G} = \mathbf{Tr}$, we also have $\delta_0 = 1 : \mathbf{Tr} \dashrightarrow \mathbf{Tr}$.

Consider the biadjunction

$$\mathbf{MGC} \underset{\text{forgetful}}{\overset{D_s}{\underset{\longrightarrow}{\longleftarrow}}} \mathbf{GC} \qquad (\perp)$$

which restricts to a 2-adjunction on strict monoidal globular categories. It generates a pseudo-comonad on the 2-category **MGC**; that is, a pseudo-comonoid in the pseudo-functor category Hom(**MGC**,**MGC**). This gives a strict monoidal pseudo-functor

$$D_s^* : \mathbb{\Delta}^{op} \dashrightarrow^{\mathbb{A}} \mathrm{Hom}(\mathbf{MGC},\mathbf{MGC})$$

(that is, an augmented pseudo-simplicial object in Hom(**MGC**,**MGC**)) where

$$(D_s^*[n])\mathcal{M} = D_s^{n+1}\mathcal{M}$$

for any monoidal globular category $\mathcal{M}$ and $n \geq -1$. Write

$$[\ ]^n : D_s^{m+n}\mathcal{M} \dashrightarrow D_s^m\mathcal{M}$$

for the monoidal globular functor which is the value of $(D_s^*-)\mathcal{M}$ at the inclusion $[m-1] \dashrightarrow [m+n-1]$ in $\mathbb{\Delta}$; it is an n-fold composite of monoidal globular functors of the form $[\ ] : D_s^{k+1}\mathcal{M} \dashrightarrow D_s^k\mathcal{M}$. By composition, each globular functor $H : \mathcal{G} \dashrightarrow D_s^{m+n}\mathcal{M}$ gives a globular functor $[H]^n : \mathcal{G} \dashrightarrow D_s^m\mathcal{M}$. If $F : \mathcal{G} \dashrightarrow \mathcal{M}$ is a globular functor, we put

$$[F]_n = [D_s^n F]^n : D_s^n\mathcal{G} \dashrightarrow \mathcal{M}.$$

If $\mathcal{C} \in \mathbf{AMGC}$ and $A, B : \mathbf{Tr} \dashrightarrow \mathcal{C}$ are collections then there is a canonical isomorphism

$$[A \otimes B]_n \cong [A]_n \otimes [B]_n .$$

We can now define the n-fold tensor product required by the above Theorem. If $\mathcal{C} \in \mathbf{AMGC}$ admits good coproducts and $A_1, \ldots, A_n : \mathbf{Tr} \dashrightarrow \mathcal{C}$ are collections then

$$(A_1 \otimes \ldots \otimes A_n)(T) = \sum_{\substack{\tau \in D_s^{n-1}(\mathbf{Tr}) \\ [\tau]^{n-1} = T}} ([A_1]_{n-1} \otimes [A_2\,\delta_1]_{n-2} \otimes \ldots \otimes [A_n\,\delta_{n-1}]_0)(\tau) .$$

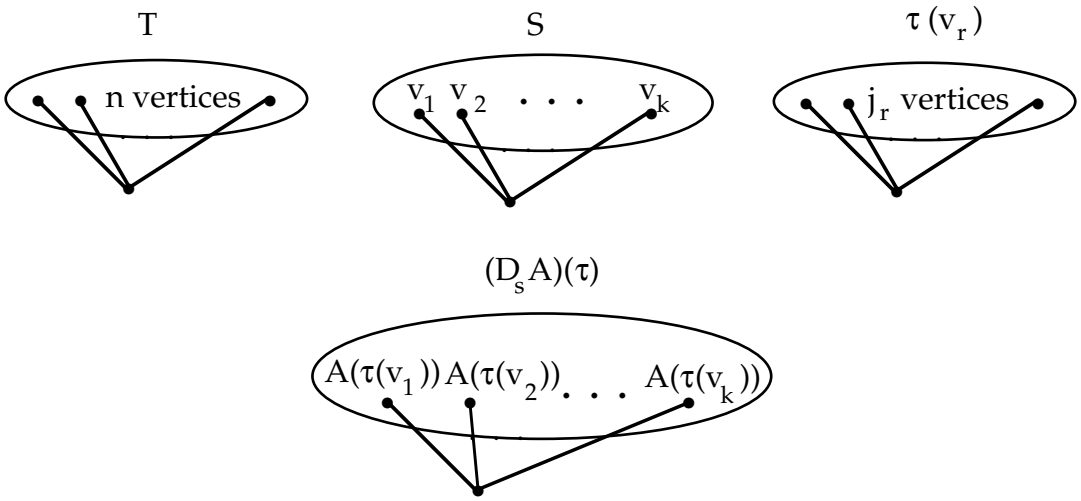
The definition of the lax constraints $\alpha_{k,m}$ is even more technical. We omit this, preferring instead to look a little at the binary tensor product

$$(A \otimes B)(T) = \sum_{\substack{\tau \in D_s(\mathbf{Tr}) \\ [\tau] = T}} ([A]_1 \otimes [B\delta_1]_0)(\tau) = \sum_{\substack{\tau \in D_s(\mathbf{Tr}) \\ [\tau] = T}} ([D_s A] \otimes B\delta_1)(\tau) .$$

when $\mathcal{C} = \mathcal{V}^{\mathbb{A}}$ for a cocomplete closed braided monoidal category $\mathcal{V}$. The interest centres on a 1-stage tree T and a 1-stage diagram $\tau : S^* \dashrightarrow \mathbf{Tr}$ of trees with $[\tau] = T$. The 1-stage tree T can be

identified with the number n of vertices of T of height 1. Then τ can be identified with a partition

$$n = j_1 + \ldots + j_k .$$



We can see that $[(D_s A)](\tau) = A(\tau(v_1)) \otimes A(\tau(v_2)) \otimes \ldots \otimes A(\tau(v_k))$ and $(B\delta_1)(\tau) = B(S)$. Identifying collections $A : \mathbf{Tr} \dashrightarrow \mathcal{V}$ with sequences $(A_n)_{n \geq 0}$ of objects of $\mathcal{V}$ via the specification

$$A_n = A(T),$$

we recapture the formula for the tensor product needed for the usual (non-permutative) operads, namely,

$$(A \otimes B)_n = \sum_{\substack{k \geq 0 \\ j_1 + \ldots + j_k = n}} A_{j_1} \otimes \ldots \otimes A_{j_k} \otimes B_k .$$

HIGHER OPERADS AND THEIR ALGEBRAS

Let $\mathcal{C}$ be an appropriately cocomplete augmented globular category.

An *operad* in $\mathcal{C}$ is a monoid in the lax-monoidal category $\mathbf{Coll}(\mathcal{C})$. Write $\mathbf{Oper}(\mathcal{C})$ for the category of operads in $\mathcal{C}$. We are particularly interested in

$$\mathbf{Coll} = \mathbf{Coll}(\mathbf{Span}) \quad \text{and} \quad \mathbf{Oper} = \mathbf{Oper}(\mathbf{Span}).$$

Theorem *With reasonable conditions on* $\mathcal{C}$ *(satisfied in the case* $\mathcal{C} = \mathbf{Span}$*), the forgetful functor* $\mathbf{Oper}(\mathcal{C}) \dashrightarrow \mathbf{Coll}(\mathcal{C})$ *is monadic and* $\mathbf{Oper}(\mathcal{C})$ *is cocomplete.*

The monad generated by $\mathbf{Oper}(\mathcal{C}) \dashrightarrow \mathbf{Coll}(\mathcal{C})$ and its left adjoint is denoted by

$$\mathcal{F} : \mathbf{Coll}(\mathcal{C}) \dashrightarrow \mathbf{Coll}(\mathcal{C}).$$

A *pointed* collection is a collection A together with a morphism $e : I \dashrightarrow A$. Write $\mathbf{Coll}_p$ for the category of pointed collections. It is an easy consequence of the Theorem that the forgetful functor $\mathbf{Oper} \dashrightarrow \mathbf{Coll}_p$ is monadic; the monad generated here is denoted by

$$\mathcal{F}_p : \mathbf{Coll}_p \dashrightarrow \mathbf{Coll}_p.$$

A *globular object* W of a globular category $\mathcal{G}$ is a globular functor $W : \mathbf{1} \dashrightarrow \mathcal{G}$ from the terminal globular category **1**.

By the freeness of **Tr**, every globular object $W : \mathbf{1} \dashrightarrow \mathcal{C}$ in a monoidal globular category $\mathcal{C}$ extends to a monoidal globular functor $[W] : \mathbf{Tr} \dashrightarrow \mathcal{C}$. Globular objects in monoidal globular categories are exponentiated by trees $T \in \mathbf{Tr}$ according to the formula:

$$W^T = [W](T).$$

For example, if $\mathcal{C} = \mathcal{V}^+$ where $\mathcal{V}$ is a monoidal category, then a globular object W in $\mathcal{C}$ is just an object W of $\mathcal{V}$ and exponentiation gives the tensor power

$$W^T = W \otimes \ldots \otimes W \quad \text{(n terms)}$$

where T is a 1-stage tree with n leaves.

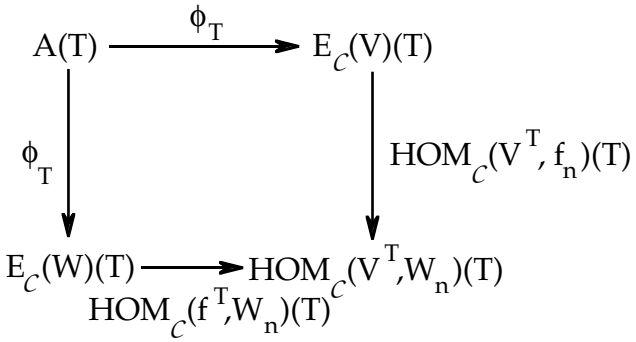
Proposition *For every globular object* W *in* $\mathcal{C}$*, there is a canonical structure of operad in* **Span** *on the collection* $E_{\mathcal{C}}(W)$ *given by*

$$E_{\mathcal{C}}(W)(T) = HOM_{\mathcal{C}}(W^T, W_n) \quad \textit{for} \quad T \in \mathbf{Tr}_n.$$

Suppose A is an operad in **Span** and $\mathcal{C}$ is a monoidal globular category. An A-*algebra in* $\mathcal{C}$ is a globular object W of $\mathcal{C}$ together with an operad morphism

$$\phi : A \dashrightarrow E_{\mathcal{C}}(W).$$

An A-*algebra morphism* $f : V \dashrightarrow W$ is a globular object morphism such that the following square commutes for all $T \in \mathbf{Tr}_n$.



It is possible to modify these definitions in a fairly straightforward way to yield the notion of *n-operad:* it is a monad in the lax-monoidal category of *n-collections.*

Examples 1) A 0-operad is a monoid in the monoidal category $\mathcal{C}_0$ with $\#_0$ as tensor product.

2) If $\mathcal{V}$ is a braided monoidal category then 1-operads in $\mathcal{V}^+ \in \mathbf{AMGC}$ are precisely non-permutative operads in $\mathcal{V}$.

3) The terminal collection $\mathcal{M}$ is naturally an operad. The category of $\mathcal{M}$-algebras is isomorphic to **ω-Cat**.

4) There is an easily described 2-operad in **Span** whose algebras are derivation schemes.

5) There is a fairly easily described 2-operad in **Span** whose algebras are sesquicategories.

CONTRACTIBILITY

A globular set X is called *contractible in dimension* $n \geq 0$[5] when, for all $x, y \in X_n$ with

$$s_{n-1}(x) = s_{n-1}(y), \quad t_{n-1}(x) = t_{n-1}(y),$$

there exists $z \in X_{n+1}$ with

$$s_n(z) = x, \quad t_n(z) = y.$$

A choice function $(x,y) \overset{jA}{\longmapsto} z$ is called a *contraction.*

A collection $A : \mathbf{Tr} \dashrightarrow \mathbf{Span}$ is called *contractible* when, for all $T \in \mathbf{Tr}_n$, the globular set

$$G_T(A) : \quad \cdots \rightrightarrows A(iT)(n+1) \rightrightarrows A(T)(n) \rightrightarrows \cdots \rightrightarrows A(\partial_0 T)(0)$$

is contractible in all dimensions $m \geq n$. An operad is *contractible* when its underlying collection is contractible.

An operad A is *normalized* when $A(U_0) = I_0$.

We need to distinguish a particular collection $P : \mathbf{Tr} \dashrightarrow \mathbf{Span}$. Write 0^k_n for the n-span made up of all singleton sets 1 for $k = n$ and, for $k < n$, the description of 0^k_n is

$$0^k_n(\alpha, m) = \begin{cases} 1 & \text{for} \quad m \leq k \\ 0 & \text{for} \quad k < m < n \end{cases}$$

$$0^k_n(n) = 0 .$$

Now define P by $P(T) = 0^{n-h}_n$ if there exists a $1 \leq k \leq h$ such that $\partial_h T = U_h \#_k U_h$ but there exists no $1 \leq k \leq h$ such that $\partial_{h+1} T = U_{h+1} \#_k U_{h+1}$. Let $\rho : I \dashrightarrow P$ be the canonical morphism.

A *system of compositions* for an operad A is a collection morphism $m : P \dashrightarrow A$ such that $m\,\rho = \varepsilon$. The idea is that such an operad A is equipped with the "compositions" required of a weak ω-category but not necessarily with any of the "constraints".

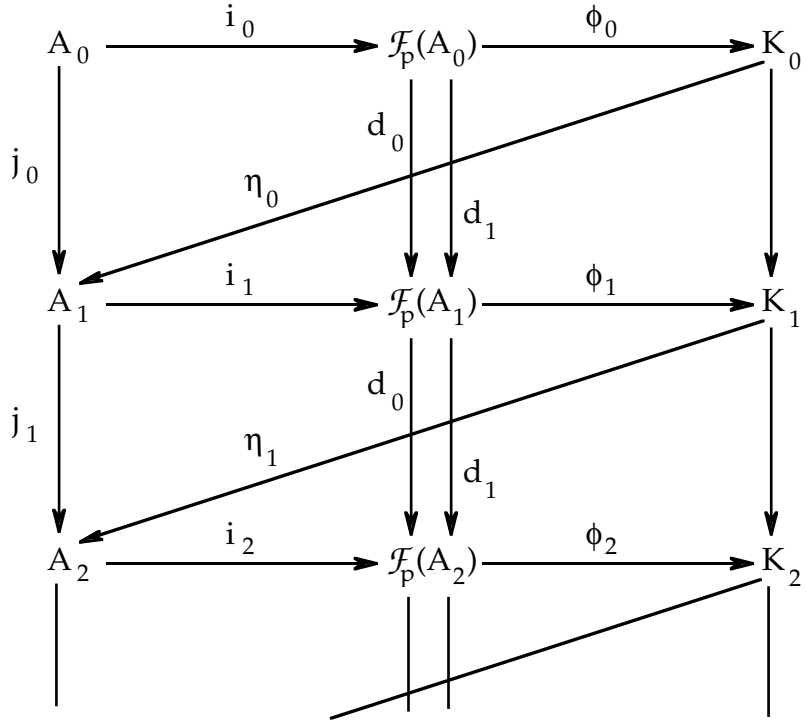
Lemma *There exists a pointed endofunctor* $C : \mathbf{Coll} \dashrightarrow \mathbf{Coll}$, $\eta : 1 \dashrightarrow C$, *such that* $C(X)$ *is contractible for all* $X \in \mathbf{Coll}$, *and, for every* $f : X \dashrightarrow Y$ *in* $\mathbf{Coll}$ *with* Y *contractible, there exists* $f_c : C(X) \dashrightarrow Y$ *with* $f = f_c\,\eta$.

Theorem *The category of contractible operads equipped with a system of compositions has a weak initial object* K.

Construction Put $A_0 = P$ and $K_0 = \mathcal{F}_p(A_0)$ (with identity morphism called ϕ_0). We have the canonical inclusion $i_0 : A_0 \dashrightarrow K_0$. We certainly now have an operad K_0 with a system of compositions, but it is not contractible. Enter the Lemma: we have $\eta_0 : K_0 \dashrightarrow C(K_0) = A_1$. Let $i_1 : A_1 \dashrightarrow \mathcal{F}_p(A_1)$ be the canonical inclusion. Put $d_0 = i_1\,\eta_0$ and $d_1 = \mathcal{F}_p(j_0) : \mathcal{F}_p(A_0) \dashrightarrow \mathcal{F}_p(A_1)$. Notice that d_1 is an operad morphism while d_0 is not. Define $\mathcal{F}_p(A_1) \dashrightarrow K_1$ to be the

[5] By convention, X_{-1} is taken to be a singleton.

coequalizer (in the category of operads) of the operad-morphism extensions $\mathcal{F}_p(\mathcal{F}_p(A_0))$ ---> $\mathcal{F}_p(A_1)$ of d_0, d_1. Clearly we have an operad morphism K_0 ---> K_1. This process iterates.



Put $K = \text{colim } K_n$ in the category of operads; it is clearly contractible with a system of compositions. An easy argument shows it is weakly initial among such operads.

PUNCH LINE

A *weak ω-category* is a K-algebra in **Span**.

STOP PRESS *19 March 1997*

Last evening Michael Batanin and Clemens Berger proved that each operad A in **Span** determines a monad $\hat{A}$ on the category $[\mathbb{G},\mathbf{Set}]$ of globular sets, where

$$\hat{A}(W) = \sum_T A(T) \times W^T,$$

and the category of A-algebras is isomorphic to the category of Eilenberg-Moore $\hat{A}$-algebras. A typical consequence of this is the cocompleteness of the category of weak ω-categories.

REFERENCES

[1] M.A. Batanin, Homotopy coherent homotopy theory and A_∞-structures in monoidal categories, *Journal of Pure & Applied Algebra* (to appear).

[2] M.A. Batanin, On the definition of weak ω-category, *Macquarie Mathematics Report* 96/207 (1996) 24pp.

[3] M.A. Batanin, Monoidal globular categories as a natural environment for the theory of weak n-categories (submitted).