

Classical non mass preserving solutions of coagulation equations.

M. Escobedo¹, J. J. L. Velázquez²

Abstract: In this paper we construct classical solutions of a family of coagulation equations with homogeneous kernels that exhibit the behaviour known as gelation. This behaviour consists in the loss of mass due to the fact that some of the particles can become infinitely large in finite time.

1 INTRODUCTION

In this paper we prove existence of solutions of the classical coagulation equation for which the mass is not conserved in time. The coagulation equation reads as:

$$\frac{\partial f}{\partial t}(t, x) = Q[f](t, x) \quad , \quad x \geq 0 \quad , \quad t > 0 \quad (1.1)$$

$$Q[f] = \frac{1}{2} \int_0^x K(x-y, y) f(t, x-y) f(t, y) - \int_0^\infty K(x, y) f(t, x) f(t, y) dy \quad (1.2)$$

$$f(x, 0) = f_0(x) \quad , \quad x > 0 \quad (1.3)$$

where the kernel K whose specific form will be precised later, satisfies $K(x, y) = K(y, x) \geq 0$.

The solutions of (1.1)-(1.3) satisfy formally, assuming that Fubini's Theorem can be applied, the mass conservation property:

$$\frac{d}{dt} \left(\int_0^\infty x f(t, x) dx \right) = 0 \quad (1.4)$$

However, it is well known that for a large class of homogeneous kernels $K(x, y)$ solutions of (1.1)-(1.3) satisfying (1.4) cannot exist globally in time (cf. [2], [3], [10], [15]). More precisely, there exists solutions of (1.1)-(1.3) that preserve the total mass of the particles $\int_0^\infty x f(t, x) dx$ during a finite time interval $0 \leq t \leq T < \infty$, but the mass is not preserved for arbitrarily long times. This phenomenon is usually termed as gelation.

In this paper we will restrict our attention to the study of kernels with the form:

$$K(x, y) = (x y)^{\frac{\lambda}{2}} \quad , \quad 1 < \lambda < 2 \quad (1.5)$$

The range of exponents in (1.5) is the one in which changes of mass of order one can be expected in times of order one. Global weak solutions of (1.1) have been obtained in [12].

The main goal of this paper is to construct classical solutions of (1.1)-(1.3) exhibiting gelation. We will assume that the initial data behaves as a suitable power law for large values of x , and therefore that the loss of mass takes place since $t = 0$. In particular, in the classical solutions obtained in this paper, it will be possible to compute a detailed asymptotic behaviour of the solution $f(t, x)$ as $x \rightarrow \infty$, as well as the flux of mass escaping to infinity. The solutions obtained will be local in time, since we cannot avoid the possibility of discontinuities in the fluxes at infinity for positive times.

The results obtained in this paper rely heavily in the estimates obtained in the papers [6], [7], where some related linear coagulation models were studied. In particular we have obtained

¹Departamento de Matemáticas. Universidad del País Vasco. Apartado 644. E-48080 Bilbao, Spain. E-mail: miguel.escobedo@ehu.es

²ICMAT (CSIC-UAM-UC3M-UCM). Facultad de Matemáticas. Universidad Complutense. E-28040, Madrid, Spain. E-mail: JJ.Velazquez@mat.ucm.es

very detailed estimates for the fundamental solution of the linear coagulation equation that results linearizing (1.1)-(1.3) around the power law $f(x) = \frac{1}{x^{\frac{3+\lambda}{2}}}$ in [6]. On the other hand, we have introduced in [7] some natural functional spaces to study the linearized version of (1.1)-(1.3) that results considering small deviations of a bounded initial data $f_0(x)$ behaving asymptotically as $\frac{1}{x^{\frac{3+\lambda}{2}}}$ as $x \rightarrow \infty$. Both the fundamental solution in [6] and the functional framework introduced in [7] will be used extensively in this paper.

The coagulation equation is one among a large family of kinetic equations exhibiting particle fluxes for homogeneous solutions. Several examples can be found in [1]. A rigorous construction of solutions exhibiting loss of mass for small values of the energy for the so-called Uehling-Uhlenbeck equation (or quantum Boltzmann equation) has been obtained in [4], [5]. The type of methods used in those papers is closely related to the ones used in this paper, although there are some technical differences.

In both cases (coagulation and Uehling-Uhlenbeck) we can think that the obtained solutions are mass preserving measure valued solutions having a singular part at some distinguished point and a regular part that is described by the integro-differential equations. In the case of coagulation the singular part of the measure (or gel) would be supported at $x = \infty$, and in the case of Uehling-Uhlenbeck such atomic measure (or Bose-Einstein condensate) would correspond to a macroscopic fraction of particles with zero energy. A natural question that arises in both cases, and in general in the study of equations with particle fluxes is to understand the interaction between the singular measure and the regular part of the measure. For the solutions obtained in [4], [5] and in this paper we assume that the regular part of the measure is not affected by the singular part. However, it is well known that such interaction could be nontrivial. For instance, in the case of coagulation models, explicit examples for the kernel $K(x, y) = x \cdot y$ show that different solutions can be expected if there is interaction between the singular part and the regular part (cf. [9], [20]) or if such interaction does not exist. For more general kernels it is known that different dynamics can arise for different mass preserving regularizations of the kernel $K(x, y)$ after passing to the limit where gelation can occur (cf. [8]). In the case of Uehling-Uhlenbeck the computations and physical arguments in [11], [17], [18] suggest the existence of solutions of this equation exhibiting nontrivial interactions between the regular part of the particle distribution and the Bose-Einstein condensate. We also remark that in [13], [14] a construction of global mass preserving weak solutions for the Uehling-Uhlenbeck system has been given. Such a construction begins regularizing the collision kernel for small energies and pass to the limit in the cutoff parameter. It is not known if the solutions constructed in [13], [14] are the same as the ones in [4], [5]. In all these problems a detailed understanding of the physical regularizations yielding cutoff mechanisms plays a crucial role (cf. also [19] for a discussion about these problems).

The plan of this paper is the following. In Section 2 we describe the functional framework used to prove the main Theorem of this paper and state the main result. Section 3 gives a general sketch of the strategy of the proof. Section 4 summarizes some results that have been proved in [6], [7] that will be used in this paper. Section 5 contains some auxiliary technical results concerning the functional spaces as well as the fundamental solution $g(\tau, x; 1)$ studied in [6]. Section 6 provides some estimates for the nonlinear term. Section 7 describes the asymptotics of the solutions of some linear equations as $x \rightarrow \infty$ in a detailed manner. Finally Section 8 explains the fixed point argument that concludes the proof of the Theorem.

2 FUNCTIONAL FRAMEWORK AND MAIN RESULT.

In this paper we will choose the initial data in (1.3) satisfying $f_0 \in C^3(\mathbb{R}^+)$. We will assume also, as in [7], that the function f_0 is close to a power law for large x . To this end we define:

$$r = \frac{\lambda - 1}{2} \tag{2.1}$$

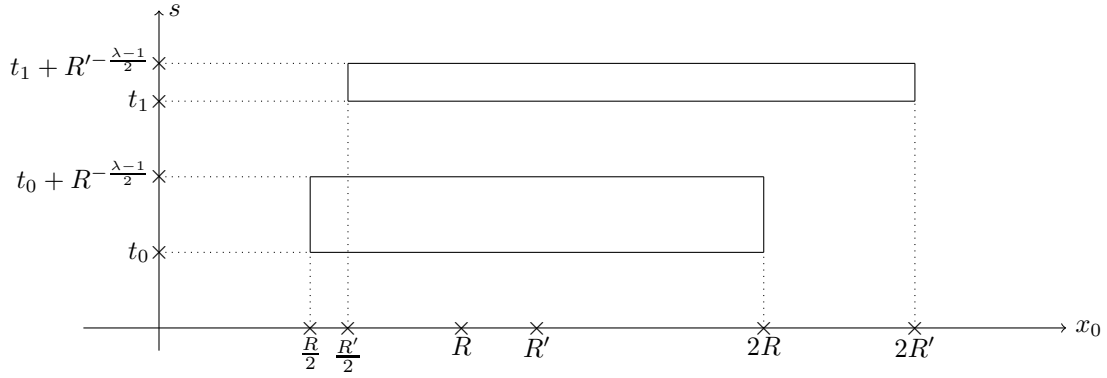


Figure 1: Two cubes of the kind appearing in the norms $N_{2,\sigma}$ and N_∞ defined below.

We fix also $\delta > 0$ satisfying $\delta < \min \{r, \frac{2-\lambda}{2}\}$. We will then assume that f_0 has the form:

$$f_0(x) = f_1(x) + f_2(x) + f_3(x), \quad f_1(x) = \frac{D_1 \xi(x)}{x^{\frac{3+\lambda}{2}}}, \quad f_2(x) = \frac{D_2 \xi(x)}{x^{\frac{3+\lambda}{2}+r}} \quad (2.2)$$

$$f_{1;2}(x) = f_1(x) + f_2(x) \quad (2.3)$$

where $D_1 > 0$, $D_2 \in \mathbb{R}$ and:

$$\xi \in C^\infty[0, \infty), \quad \xi(x) = 1 \text{ for } x \geq 1 \text{ and } \xi(x) = 0 \text{ if } 0 \leq x \leq 1/2, \quad \xi'(x) \geq 0 \quad (2.4)$$

$$|f_3^k(x)| \leq \frac{B}{(x+1)^{\frac{3+\lambda}{2}+r+k+\delta}}, \quad k = 0, 1, 2, 3, 4 \quad (2.5)$$

for some $B > 0$. The following auxiliary function will be used repeatedly:

$$h_0(x) = f_0(x) - f_1(x) = f_2(x) + f_3(x) \quad (2.6)$$

Notice that (2.2)-(2.5) imply:

$$\begin{aligned} & \left(1 + y^{\frac{3+\lambda}{2}+r}\right) |h_0(y)| + \left(1 + y^{\frac{3+\lambda}{2}+r+1}\right) |h_0'(y)| + \\ & + \left(1 + y^{\frac{3+\lambda}{2}+r+2}\right) |h_0''(y)| + \left(1 + y^{\frac{3+\lambda}{2}+r+3}\right) |h_0'''(y)| \leq CB. \end{aligned} \quad (2.7)$$

for some $C > 0$. We will assume in the rest of the paper that C is a generic constant that can change from line to line and that might depend only on D_1, D_2, B, λ and δ unless some additional dependence is written explicitly. Moreover, we will assume without loss of generality that $D_1 = 1$, since this parameter can be absorbed in a rescaling of t .

For any interval $I \subset (0, +\infty)$ we will denote as $L^2(I)$ the usual Lebesgue space of square integrable functions. For any $\sigma > 0$ we denote as $H^\sigma(I)$ the usual Sobolev space $W^{\sigma,2}(I)$. The corresponding norms will be denoted $\|\cdot\|_{L^2}$ and $\|\cdot\|_{H^\sigma}$. Dealing with functions depending on variables x and t we will write H_x^σ or L_t^2 in order to indicate the argument with respect to which the norm is taken.

In order to define suitable functional spaces we define, for any $T > 0$, $R > 0$:

$$N_{2;\sigma}(f; t_0, R) = \left(R^{\frac{\lambda-1}{2}+2\sigma-1} \int_{t_0}^{\min(t_0+R^{-(\lambda-1)/2}, T)} \|D_x^\sigma f(t)\|_{L^2(R/2, 2R)}^2 dt \right)^{1/2}, \quad \sigma \geq 0 \quad (2.8)$$

$$M_{2;\sigma}(f; R) = \left(R^{2\sigma-1} \int_0^T \|D_x^\sigma f(t)\|_{L^2(R/2, 2R)}^2 dt \right)^{1/2}, \quad \sigma \geq 0 \quad (2.9)$$

$$N_\infty(f; t_0, R) = \left(R^{\frac{\lambda-1}{2}} \int_{t_0}^{\min(t_0+R^{-(\lambda-1)/2}, T)} \|f(t)\|_{L^\infty(R/2, 2R)}^2 dt \right)^{1/2}$$

$$M_\infty(f; R) = \left(\int_0^T \|f(t)\|_{L^\infty(R/2, 2R)}^2 dt \right)^{1/2}$$

Then, for any $\sigma > 0$ we define the following norms:

$$\begin{aligned} \|f\|_{Y_{q,p}^\sigma(T)} &= \sup_{0 < R \leq 1} R^q M_{2;0}(f; R) + \sup_{0 < R \leq 1} R^q M_{2;\sigma}(f; R) + \\ &+ \sup_{0 \leq t_0 \leq T} \sup_{R \geq 1} R^p N_{2;0}(f; t_0, R) + \sup_{0 \leq t_0 \leq T} \sup_{R \geq 1} R^p N_{2;\sigma}(f; t_0, R) \end{aligned}$$

$$\|f\|_{X_{q,p}(T)} = \sup_{0 < R \leq 1} R^q M_\infty(f; R) + \sup_{0 \leq t_0 \leq T} \sup_{R \geq 1} R^p N_\infty(f; t_0, R)$$

$$\|f\|_{q,p} = \sup_{0 \leq x \leq 1} \{x^q |f(x)|\} + \sup_{x > 1} \{x^p |f(x)|\} \quad (2.10)$$

$$\|f\|_\sigma = \sup_{0 \leq t \leq T} \|f\|_{\frac{3}{2}, \frac{3+\lambda}{2}} + \|f\|_{Y_{\frac{3}{2}, \frac{3+\lambda}{2}}^\sigma(T)} \quad (2.11)$$

and the following spaces:

$$Y_{q,p}^\sigma(T) = \left\{ f : \|f\|_{Y_{q,p}^\sigma(T)} < \infty \right\}, \quad X_{q,p}(T) = \left\{ f : \|f\|_{X_{q,p}(T)} < \infty \right\}$$

$$\mathcal{E}_{T;\sigma} = \left\{ f : \|f\|_\sigma < \infty \right\}$$

Throughout this paper we will assume that

$$\sigma \in (1, 2) \quad (2.12)$$

Therefore, Sobolev embeddings imply $Y_{q,p}^\sigma(T) \subset X_{q,p}(T)$. Actually such embeddings would take place assuming the weaker condition $\sigma > \frac{1}{2}$. The main reason for the choice of σ as in (2.12) is purely technical, and it is due to the fact that the Theorem proved in [7] to solve a suitable linearized problem (cf. for instance (3.5)) requires such a regularity. It is likely that using the "almost half derivatives" that we introduce now would be possible to weaken the condition on σ to $\frac{1}{2} < \sigma < 1$ both for the results of [7] and this paper (cf. Remark 6.4 in [7]).

We will solve (1.1)-(1.3) using a functional space that measures in a natural way the regularizing effects of the coagulation equation as $x \rightarrow \infty$ that have been studied in [7]. Let $\eta \in C^\infty(\mathbb{R}^+)$ a cutoff function satisfying $\eta(x) = 1$ for $x \in (\frac{1}{4}, 3)$, $\eta(x) = 0$ for $x \notin (\frac{1}{8}, 4)$. Given $f \in C(\mathbb{R}^+)$, $t_0 \in [0, T]$, $R \geq 1$ we define:

$$F_{R,t_0}(\theta, X) = \eta(RX) f\left(t_0 + \theta R^{-(\lambda-1)/2}, RX\right) \quad (2.13)$$

and:

$$[f]_p^{\sigma; \frac{1}{2}} = \sup_{R \geq 1} \sup_{0 \leq t_0 \leq T} R^p \left(\int_{t_0}^{\min(t_0+R^{-(\lambda-1)/2}, T)} \int_{\mathbb{R}} |\widehat{F}_{R,t_0}(\theta, k)|^2 Q_{R,\sigma}(k) dk d\theta \right)^{1/2} \quad (2.14)$$

where $Q_{R,\sigma}(k) = (1 + |k|^{2\sigma})(1 + \min\{|k|, R\})$.

$$\|f\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} = \|f\|_{L^2((0,T); H_x^\sigma(0,2))} + [f]_p^{\sigma; \frac{1}{2}} + \sup_{0 \leq t \leq T} \|f\|_{\frac{3}{2}, p} + \|f\|_{Y_{\frac{3}{2}, p}^\sigma(T)} \quad (2.15)$$

$$\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T) = \left\{ f : \|f\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} < \infty \right\}$$

The intuition behind these spaces is the following. As it has been seen in [7] the main terms in the coagulation equation for solutions that are close to the power law $x^{-\frac{3+\lambda}{2}}$ as $x \rightarrow \infty$ can be thought as a perturbation of the half-derivative operator. However, since the integral operator $Q[f]$ in (1.2) is an integral operator the equation (1.1), cannot be expected to have smoothing effects. Nevertheless, it has been seen in [7] that the equation (1.1) has some kind of regularizing effect due to the fact that the right hand side of (1.1) can be thought, for solutions close to $x^{-\frac{3+\lambda}{2}}$ as $x \rightarrow \infty$ as the half-derivative operator, if we restrict ourselves to incremental quotients with length x larger than one. This is the source of the regularizing effects that will be studied using the functionals (2.14), (2.15).

In order to gain some intuition about the spaces $X_{q,p}(T)$, $Y_{q,p}^\sigma(T)$, $\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)$ it is useful to think about them as functions that can be estimated like x^{-p} as $x \rightarrow \infty$ and x^{-q} as $x \rightarrow 0$ in the case of the spaces $X_{q,p}(T)$, $Y_{q,p}^\sigma(T)$ and $x^{-\frac{3}{2}}$ in the case of $\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)$. Concerning regularity, the functions in $X_{q,p}(T)$ are estimated pointwise, the functions in $Y_{q,p}^\sigma(T)$ have σ derivatives in space and the functions in $\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)$ have almost $(\sigma + \frac{1}{2})$ derivatives in the sense of the definition (2.14).

The main result of this paper is the following:

Theorem 1 *Suppose that f_0 satisfies (2.2)-(2.5), σ is as in (2.12) and K is as in (1.5). Then, there exists a classical solution $f \in \mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma;\frac{1}{2}}$ of (1.1)-(1.3) with $f_t \in L^\infty((0,T) \times \mathbb{R}^+)$. Moreover, this solution is unique in the class of functions satisfying:*

$$f(t, x) = \lambda(t) \xi(x) x^{-\frac{3+\lambda}{2}} + h(t, x)$$

with $\lambda \in C[0, T]$, $h \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$, $\lim_{\bar{t} \rightarrow 0} \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(\bar{t})} = 0$, where $\bar{p} = \frac{3+\lambda}{2} + \bar{\delta}$ with $0 < \bar{\delta} < r$, and T small enough.

Remark 2 *Assumptions (2.2)-(2.5) seem a very strong condition. However, this condition is analogous to the type of compatibility conditions that must be assumed solving boundary value problems in order to obtain smooth solutions, or also to assume that the initial data has as many derivatives appear in the equation solving a parabolic problem. It is likely that (2.2)-(2.5) could be weakened to the form $f_0(x) = D_1 x^{-\frac{3+\lambda}{2}} + O(x^{-\frac{3+\lambda}{2}-\delta})$ as $x \rightarrow \infty$ for some $\delta > 0$. However, to prove this would require to obtain some delicate regularizing effects that we have preferred to avoid in this paper that is already rather technical. The specific value of r will play a role in the proof of Proposition 18 (cf. Remark 29) as well as in the Proof of Proposition 27.*

3 GENERAL STRATEGY OF THE PROOF.

The general plan that we will use to prove Theorem 1 is the following. We look for a solution of (1.1)-(1.3) in the form:

$$f(t, x) = \lambda(t) f_0(x) + h(t, x) \tag{3.1}$$

where f_0 is the initial data (cf. (1.3)) and h will be a small perturbation for short times. The function λ is a differentiable function to be prescribed satisfying $\lambda(0) = 1$. Then h , λ solve:

$$h_t = \lambda(t) \mathcal{L}_{f_0}[h] + Q[h] + (\lambda(t))^2 Q[f_0] - \lambda_t f_0(x) \tag{3.2}$$

where the linear operator \mathcal{L}_{f_0} is as in [7]:

$$\begin{aligned} \mathcal{L}_{f_0}[h] = & \int_0^x (x-y)^{\lambda/2} f_0(x-y) y^{\lambda/2} h(y) dy - \\ & - x^{\lambda/2} f_0(x) \int_0^\infty y^{\lambda/2} h(y) dy - x^{\lambda/2} h(x) \int_0^\infty y^{\lambda/2} f_0(y) dy. \end{aligned} \tag{3.3}$$

Our strategy is to solve (3.2) by means of a fixed point argument for a suitable operator \mathcal{T} defined in $\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$ with r as in (2.1), σ as in (2.12) and T sufficiently small (cf. (2.15)). It is convenient first, in order to apply the well-posedness results in [7] to introduce a new time scale. We will assume in all the paper that $|\lambda(t) - 1| \leq \frac{1}{2}$. We can then define a new time scale τ and a new function Λ by means of:

$$d\tau = \lambda(t) dt \quad , \quad \tau = 0 \quad \text{at} \quad t = 0 \quad , \quad \Lambda(\tau) = \lambda(t) \quad (3.4)$$

Then (3.2) becomes:

$$h_\tau = \mathcal{L}_{f_0}[h] + \frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau) Q[f_0] - \Lambda_\tau f_0(x)$$

where we will write $h(t, x) = h(\tau, x)$ by convenience.

Given $h \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$ and $\Lambda \in C^1([0, T])$ we will define $\tilde{h} = \tilde{h}[\Lambda]$ as the unique solution of:

$$\tilde{h}_\tau = \mathcal{L}_{f_0}[\tilde{h}] + \frac{Q[\tilde{h}]}{\Lambda(\tau)} + \Lambda(\tau) Q[f_0] - \Lambda_\tau f_0(x) \quad (3.5)$$

in $\mathcal{E}_{T, \sigma}$. The existence of such a solution will be a consequence of the results in [7]. In order to apply such a results we will need to show that $Q[f_0], Q[h] \in Y_{\frac{3}{2}, (2+\delta)}^\sigma(T)$. In the case of $Q[f_0]$ this will be a consequence of (2.6), (2.7). In order to derive this property for $Q[h]$ we will use the decay and regularity properties of the functions $h \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$. The details will be given in Section 6.

After obtaining $\tilde{h} = \tilde{h}[\Lambda]$ we proceed to determine $\Lambda(\tau)$. To this end we will argue as follows. The asymptotic behaviour of \tilde{h} as $x \rightarrow \infty$ is given by:

$$\tilde{h}(\tau, x) \sim \left[\mathcal{G}[\tau; h, \Lambda] - \int_0^\tau a(\tau - s) \Lambda_\tau(s) ds \right] x^{-\frac{3+\lambda}{2}} \quad \text{as} \quad x \rightarrow \infty \quad , \quad 0 \leq \tau \leq T \quad (3.6)$$

where $a(\cdot)$ is a function depending on f_0 and $\mathcal{G}[\cdot; h, \Lambda]$ a functional that will be precised later (cf. Proposition 24, Proposition 27 and Lemma 32 for a precise formulation of this result).

The asymptotics (3.6) will be obtained using the properties of the fundamental solution constructed in [7]. In order to close the fixed point argument, we need to choose $\Lambda(\tau)$ in such a way that $\tilde{h}(\tau, x) = o\left(x^{-\frac{3+\lambda}{2}}\right)$ as $x \rightarrow \infty$. This can be achieved assuming that Λ solves the equation:

$$\int_0^\tau a(\tau - s) \Lambda_\tau(s) ds - \mathcal{G}[\tau; h, \Lambda] = 0 \quad , \quad 0 \leq \tau \leq T \quad (3.7)$$

A detailed analysis of the function $a(\tau)$ (see Subsection 8.2) will allow to transform (3.7) in something more like a first order Volterra integral equation:

$$a(0) \Lambda(\tau) - \int_0^\tau \frac{da}{d\tau}(\tau - s) \Lambda(s) ds - a(\tau) - \mathcal{G}[\tau; h, \Lambda] = 0 \quad , \quad 0 \leq \tau \leq T \quad (3.8)$$

with $a(0) = 1$. This equation can be solved by means of a standard fixed point argument, and this gives the desired Λ that will be denoted as $\tilde{\Lambda}$. We then define $\mathcal{T}[h] = \tilde{h}[\tilde{\Lambda}]$. Notice that $\mathcal{T}[h](\tau, x) = o\left(x^{-\frac{3+\lambda}{2}}\right)$ as $x \rightarrow \infty$. Actually, a more careful analysis of (3.5), (3.8) shows that $\mathcal{T}[h] \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$. Moreover, the operator \mathcal{T} is contractive in $\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$ if T is sufficiently small and a suitable choice of δ .

4 SUMMARY OF SOME OF THE RESULTS IN [6], [7].

We recall in this Section several results that have been obtained in [6], [7] and that will be used repeatedly in this paper.

In order to study the asymptotic behaviour of \tilde{h} defined in the previous Section, we will need some properties of the semigroup defined by the operator:

$$\begin{aligned} L(h) = & \int_0^{\frac{x}{2}} \left[(x-y)^{\lambda/2} G(x-y) - x^{\lambda/2} G(x) \right] y^{\lambda/2} h(y) dy + \\ & + \int_0^{\frac{x}{2}} \left[(x-y)^{\lambda/2} h(x-y) - x^{\lambda/2} h(x) \right] y^{-\frac{3}{2}} dy - x^{-\frac{3}{2}} \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} h(y) dy - 2\sqrt{2} x^{\frac{\lambda-1}{2}} h(x) \end{aligned} \quad (4.1)$$

where $G(x) = \frac{1}{x^{\frac{3+\lambda}{2}}}$. We have studied in [6] the solution of the following problem:

$$\partial_{\tau} g(\tau, x) = L[g](\tau, x) \quad , \quad x > 0 \quad , \quad \tau > 0 \quad , \quad g(0, x, x_0) = \delta(x - x_0) \quad (4.2)$$

In particular we have proved there the following results:

Theorem 3 (cf. Theorem 3.8 in [6]) *There exists a unique solution $g(\tau, \cdot, x_0) \in C^{\infty}(\mathbb{R}^+)$ of (4.2) that has the following properties. There exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ depending only on λ such that, for any $0 < \varepsilon < \varepsilon_1$ the following statements hold.*

The function $g(\tau, \cdot, x_0)$ has the following self-similar structure:

$$g(\tau, x, x_0) = \frac{1}{x_0} g\left(\tau x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right) \quad (4.3)$$

For all $\tau \geq 1$:

$$g(\tau, x, 1) = \tau^{\frac{2}{\lambda-1}} \varphi_1(\rho) + \varphi_2(\tau, \rho) \quad , \quad \rho = \tau^{\frac{2}{\lambda-1}} x \quad (4.4)$$

with:

$$\varphi_1(\rho) = \begin{cases} a_1 \rho^{-\frac{3}{2}} + O_{\varepsilon}\left(\rho^{-\frac{4-\lambda}{2}+\varepsilon}\right) & , \quad 0 \leq \rho \leq 1 \\ a_2 \rho^{-\frac{3+\lambda}{2}} + O_{\varepsilon}\left(\rho^{-(1+\lambda-\varepsilon)}\right) & , \quad \rho > 1 \end{cases} \quad (4.5)$$

where a_1, a_2 are two explicit constants.

$$\varphi_2(\tau, \rho) = \begin{cases} b_1(\tau) \rho^{-\frac{3}{2}} + O\left(\tau^{\frac{2}{\lambda-1}-\varepsilon_2} \rho^{-\frac{3}{2}+\varepsilon_2}\right) & , \quad 0 \leq \rho \leq 1 \\ b_2(\tau) \rho^{-\frac{3+\lambda}{2}} + O\left(\tau^{\frac{2}{\lambda-1}-\varepsilon_2} \rho^{-\frac{3+\lambda}{2}-\varepsilon_2}\right) & , \quad \rho > 1 \end{cases} \quad (4.6)$$

where $b_1, b_2 \in$ are two continuous functions such that $|b_1(\tau)| + |b_2(\tau)| \leq C\tau^{\frac{2}{\lambda-1}-\varepsilon_2}$.

For $0 < \tau \leq 1$ we have:

$$g(\tau, x, 1) = \begin{cases} \tau x^{-\frac{3}{2}} + b_3(\tau) x^{-\frac{3}{2}} + O\left(\tau x^{-\frac{3}{2}+\varepsilon_2}\right) & , \quad 0 \leq x \leq \frac{1}{2} \\ a_3 \tau x^{-\frac{3+\lambda}{2}} + b_4(\tau) x^{-\frac{3+\lambda}{2}} + O\left(\tau x^{-\frac{3+\lambda}{2}-\varepsilon_2}\right) & , \quad x \geq \frac{3}{2} \\ O_{\varepsilon}\left(\frac{t^{1-2\varepsilon}}{|x-1|^{\frac{3}{2}-\varepsilon}}\right) & \text{for } t^2 < |x-1| < \frac{1}{2} \end{cases} \quad (4.7)$$

where a_3 is an explicit numerical constant and b_3, b_4 are continuous functions such that $|b_3(\tau)| + |b_4(\tau)| \leq C\tau^{1+\varepsilon_2}$. Moreover:

$$\lim_{t \rightarrow 0} t^2 g(t, 1 + t^2 \chi, 1) = \Psi(\chi) \quad \text{uniformly on compact subsets of } \mathbb{R}$$

where the function Ψ is given by:

$$\Psi(\chi) = \frac{2}{\pi} \frac{\exp\left(-\frac{\pi}{\chi^{3/2}}\right)}{\chi^{3/2}} \quad \text{for } \chi > 0 \quad , \quad \Psi(\chi) = 0 \quad \text{for } \chi < 0 \quad (4.8)$$

Remark 4 The functions $O_\varepsilon(\cdot)$ depend on ε .

Remark 5 Notice that (4.5)-(4.7) imply the existence of a function $\Theta = \Theta(\tau)$ and $\varepsilon > 0$ such that:

$$\left| g(\tau, x, 1) - \Theta(\tau) x^{-\frac{3+\lambda}{2}} \right| \leq C\tau x^{-\frac{3+\lambda}{2}-\varepsilon}, \quad \tau \leq 1, \quad x \geq 1 \quad (4.9)$$

$$\left| g(\tau, x, 1) - \Theta(\tau) x^{-\frac{3+\lambda}{2}} \right| \leq \frac{C}{\tau^{\frac{\lambda+1}{\lambda-1} + \frac{2\varepsilon}{\lambda-1}} x^{\frac{3+\lambda}{2} + \varepsilon}}, \quad \tau \geq 1, \quad x \geq 1 \quad (4.10)$$

where:

$$\Theta(\tau) = \begin{cases} a_4\tau + b_4(\tau), & |b_4(\tau)| \leq C\tau^{1+\varepsilon}, \quad \tau \leq 1 \\ a_2\tau^{-\frac{\lambda+1}{\lambda-1}} + b_2(\tau), & |b_2(\tau)| \leq C\tau^{-\frac{\lambda+1}{\lambda-1}-\varepsilon}, \quad \tau \geq 1 \end{cases} \quad (4.11)$$

We will need improved estimates for $g(\tau, x, 1)$. More precisely we need to compute the next order in the expansion of g as $x \rightarrow \infty$. To this end we obtain the representation formulas for the function $g(\tau, x, 1)$ that we have obtained in the Proof of Lemma 7.10 of [6].

Theorem 6 (cf. Lemma 5.1 in [6]) The function $g(\tau, x, 1)$ described in Theorem 3 can be written as $g(\tau, x, 1) = G(\tau, X)$, $x = e^X$ with:

$$\begin{aligned} G(\tau, X) = & -\frac{\mathcal{V}(2i)i}{2\pi(\lambda-1)} e^{-\frac{3+\lambda}{2}X} \int_{\text{Im}(Y)=-\gamma_1} dY \frac{\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}\left(\frac{(3+\lambda)i}{2} + Y\right)} \Gamma\left(\frac{2iY}{\lambda-1}\right) + \\ & + \frac{i}{\pi(\lambda-1)} \int_{\text{Im}(\xi)=\beta} d\xi e^{i\xi X} \int_{\text{Im}(Y)=-\gamma_1} dY \frac{\mathcal{V}(\xi)\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}(\xi+Y)} \Gamma\left(\frac{2iY}{\lambda-1}\right) \end{aligned} \quad (4.12)$$

where $(\beta - \frac{3+\lambda}{2}) > 0$ and $\gamma_1 > 0$ are sufficiently small. The function $\mathcal{V}(\xi)$ is given by:

$$\begin{aligned} \mathcal{V}(\xi) = & \exp\left(-\frac{2i}{\lambda-1} \int_{\text{Im}(\xi)=\beta_1} \log(-\Phi(\eta)) \left[\frac{1}{1 - e^{\frac{4\pi(\xi-\eta)}{\lambda-1}}} - \frac{1}{1 + e^{-\frac{4\pi\eta}{\lambda-1}}} \right] d\eta\right), \quad \beta_1 \in \left(\frac{2+\lambda}{2}, \frac{3+\lambda}{2}\right) \\ \Phi(\eta) = & -\frac{2\sqrt{\pi} \Gamma(i\eta + 1 + \frac{\lambda}{2})}{\Gamma(i\eta + \frac{\lambda+1}{2})}, \quad \lim_{\text{Re}(\eta) \rightarrow \infty} \arg(-\Phi(\eta)) = \frac{\pi}{4} \end{aligned}$$

On the other hand we have proved the following results in [7]:

Theorem 7 (cf Theorem 2.1 in [7]) For any $\sigma \in (1, 2)$, $\bar{\delta} > 0$ and any f_0 satisfying (2.6), (2.7) there exists $T > 0$ such that for all $\mu \in Y_{3/2, 2+\bar{\delta}}^\sigma$ the Cauchy problem

$$h_\tau = \mathcal{L}_{f_0}(h) + \mu, \quad h(0) = 0 \quad (4.13)$$

has a unique solution h in $\mathcal{E}_{T, \sigma}$. Moreover $\|h\|_\sigma \leq C\|\mu\|_{Y_{3/2, (2+\bar{\delta})}^\sigma}$ for some positive constant C depending on $T, \sigma, \bar{\delta}$ as well as A, B and γ in (2.6), (2.7) but not on μ .

Theorem 8 (cf. Theorem 2.2 in [7]) For any $\sigma \in (1, 2)$, $\bar{\delta} > 0$ and for any f_0 satisfying (2.6), (2.7), the solution of the Cauchy problem (4.13) obtained in Theorem 7 satisfies

$$[h]_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}} \leq C\|\mu\|_{Y_{3/2, 2+\bar{\delta}}^\sigma}$$

for some positive constant C depending on $T, \sigma, \bar{\delta}$ as well as A, B and γ in (2.6), (2.7) but not on μ .

This is a regularity result proved in [7] that will be used repeatedly in the following:

Theorem 9 (cf. Theorem 3.1 in [7]) (i) Suppose that $Q \in L_t^2(0, 1; H_x^\sigma(1/2, 2))$, $P \in L_t^2(0, 1; H_x^{\sigma-1/2}(1/2, 2))$ with $\sigma \in (1/2, 2)$, $\kappa \in (0, 1]$ and $f \in L^\infty((1/4, 2) \times (0, 1)) \cap L^2(0, 1; H^{1/2}(1/4, 2)) \cap H^1(0, 1; L^2(1/4, 2))$ is such that $f = 0$ if $x < 1/8$ or $x > 4$ and satisfies

$$\frac{\partial f}{\partial t} = \kappa T_{\varepsilon, R} (M_{\lambda/2} f) + Q + P$$

for all $x \in (1/4, 2)$, $t \in (0, 1)$ and $f(x, 0) = 0$. Then:

$$\|f\|_{L_t^2(0, 1; H_x^\sigma(3/4, 5/4))} \leq C \left(\|Q\|_{L_t^2(0, 1; H_x^\sigma(1/2, 2))} + \frac{1}{\varepsilon \kappa} \|P\|_{L_t^2(0, 1; H_x^{\sigma-1/2}(1/2, 2))} + \|f\|_{L^\infty((1/4, 2) \times (0, 1))} \right)$$

for some positive constant C independent of ε and R .

(ii) Suppose that $Q \in L_t^2(0, T_{max}; H_x^\sigma(1/2, 2))$, $P \in L_t^2(0, T_{max}; H_x^{\sigma-1/2}(1/2, 2))$, $f \in L^\infty((1/4, 2) \times (0, T_{max})) \cap C_t^1(0, T_{max}; H_x^{1/2}(1/4, 2))$ for some $T_{max} > 0$ is such that $f = 0$ if $x < 1/8$ or $x > 4$ and satisfies

$$\frac{\partial f}{\partial t} = T_{\varepsilon, R} (M_{\lambda/2} f) + Q + P - a(x, t) f, \quad x \in (1/4, 2), t > 0 \quad (4.14)$$

$$f(x, 0) = 0 \quad (4.15)$$

for some function $a \in L^\infty(0, T_{max}; H^\sigma(1/2, 2))$, $a \geq A > 0$. Then, for all $T \in [0, T_{max} - 1]$:

$$\begin{aligned} \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} \|f(t)\|_{H^\sigma(3/4, 5/4)}^2 dt \right)^{1/2} &\leq \\ C \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} \|Q(t)\|_{H^\sigma(1/2, 2)}^2 dt \right)^{1/2} &+ \\ + \frac{C}{\varepsilon} \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} \|P(t)\|_{H^{\sigma-1/2}(1/2, 2)}^2 dt \right)^{1/2} &+ \\ + C \|f\|_{L^\infty((1/4, 2) \times (0, T_{max}))} & \end{aligned} \quad (4.16)$$

(iii) Suppose that for some $T_{max} > 0$, $Q \in L_t^2(0, T_{max}; H_x^\sigma(1/2, 2))$, $f \in L^\infty((1/4, 2) \times (0, T_{max})) \cap C_t^1(0, T_{max}; H_x^{1/2}(1/4, 2))$ is such that $f = 0$ if $x < 1/8$ or $x > 4$ and satisfies (4.14) (4.15) with $P = 0$ and $\varepsilon = 0$. Then

$$\begin{aligned} \left(\int_T^{\min(T+1, T_{max})} \int_{\mathbb{R}} |\widehat{F}(k, t)|^2 |k|^{2\sigma} \min\{|k|, R\} dk \right)^{1/2} &\leq \\ C \sup_{0 \leq T \leq T_{max}} \left(\int_T^{\min(T+1, T_{max})} \|Q(t)\|_{H^\sigma(1/2, 2)}^2 dt \right)^{1/2} &+ C \|f\|_{L^\infty((1/4, 2) \times (0, T_{max}))} \end{aligned} \quad (4.17)$$

where $F(x, t) = \eta(x) f(x, t)$, $\eta \in C^\infty$ is a cutoff satisfying $\eta(x) = 1$ if $x \in (\frac{3}{4}, \frac{5}{4})$ and $\eta(x) = 0$ if $x \notin (\frac{1}{8}, \frac{1}{4})$. The constant C is independent of R .

5 SOME AUXILIARY RESULTS.

In this Section we collect two estimates that will be used in the Proof of Theorem 1.

5.1 Remarks about notation.

We will use in the arguments several different symbols. Specific letters have been reserved for quantities with precise meanings. We write them shortly here as a guide for the reader.

The letter $r = \frac{\lambda-1}{2}$ will denote the first order correction to the asymptotics of f_0 as $x \rightarrow \infty$ (cf. (2.1)-(2.5)). We will use δ to denote the exponent of the second order correction of f_0 as $x \rightarrow \infty$. It will be assumed in the whole paper that $\delta < \min\{r, \frac{2-\lambda}{2}\}$.

The parameter $\bar{\delta}$ characterizes the functional space where the solution of the equation will be obtained (cf. Theorem 1). It will be always assumed that $\bar{\delta} < \min\{r, \delta\}$. We will use also the notation $\bar{p} = \frac{3+\lambda}{2} + \bar{\delta}$.

The symbols ε 's will be used for the fundamental solution associated to $g_t = L[g]$ (cf. Theorem 3).

We use σ to denote the spatial regularity of the solutions. We assume $\sigma \in (1, 2)$.

5.2 A general estimate for the functions in $\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$.

Lemma 10 *Suppose that $\phi \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$ for $\sigma \in (1, 2)$, $p > 0$. Let us define:*

$$\omega(t, x) = \int_0^t \phi(s, x) ds \quad , \quad x \in \mathbb{R}^+, \quad 0 \leq t \leq T \quad (5.1)$$

Then, there exists $C > 0$ independent of T , ϕ such that:

$$\|\omega\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \leq 4\sqrt{T} \|\phi\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \quad , \quad (5.2)$$

Proof. Due to (2.15) to estimate $\|\omega\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}$ we need to obtain bounds for $\|\omega\|_{L^2((0, T); H_x^\sigma(0, 2))}$, $[\omega]_p^{\sigma; \frac{1}{2}}$, $\sup_{0 \leq t \leq T} \|\omega\|_{\frac{3}{2}, p}$, $\|\omega\|_{Y_{\frac{3}{2}, p}^\sigma(T)}$. Using (5.1) and Cauchy-Schwartz we obtain:

$$\|\omega\|_{L^2((0, T); H_x^\sigma(0, 2))} \leq T \|\phi\|_{L^2((0, T); H_x^\sigma(0, 2))} \quad (5.3)$$

Using (2.10):

$$\sup_{0 \leq t \leq T} \|\omega\|_{\frac{3}{2}, p} \leq T \|\phi\|_{\frac{3}{2}, p} \quad (5.4)$$

To estimate $\|\omega\|_{Y_{\frac{3}{2}, p}^\sigma(T)}$ we need to control $N_{2; \sigma}(\omega; t_0, R)$, $M_{2; \sigma}(\omega; R)$ (cf. (2.8), (2.9)). Using again Cauchy-Schwartz inequality we arrive at:

$$N_{2; \sigma}(\omega; t_0, R) \leq \sqrt{T} N_{2; \sigma}(\phi; t_0, R) \quad , \quad R > 1 \quad , \quad M_{2; \sigma}(\omega; R) \leq \sqrt{T} M_{2; \sigma}(\phi; R) \quad , \quad R \leq 1 \quad (5.5)$$

Finally we can estimate $[\omega]_p^{\sigma; \frac{1}{2}}$ using also Cauchy-Schwartz for each value of R (cf. (2.14)):

$$[\omega]_p^{\sigma; \frac{1}{2}} \leq \sqrt{T} [\phi]_p^{\sigma; \frac{1}{2}} \quad (5.6)$$

where we use that $t_0 + \theta R^{-\frac{(\lambda-1)}{2}}$ (cf. (2.13)) is bounded by T . Combining (5.2)-(5.5) we obtain (5.2). ■

5.3 Improved estimates for $g(\tau, x, 1)$.

We will need to compute detailed asymptotics for the function $g(\tau, x, 1)$ in Theorem 3 as $x \rightarrow \infty$, since the main corrective terms coming from the asymptotics of $g(\tau, x, 1)$ have the same order of magnitude as the ones due to the natural sources in the problem for the approach indicated in Section 3.

Proposition 11 *Let $g(\tau, x, 1)$ be as in Theorem 3. Suppose that $\tau \geq 1$. Then:*

$$g(\tau, x, 1) = \tau^{\frac{2}{\lambda-1}} \varphi_1(\rho) + \varphi_2(\tau, \rho) \quad , \quad \rho = \tau^{\frac{2}{\lambda-1}} x$$

with:

$$\varphi_1(\rho) = a_2 \rho^{-\frac{3+\lambda}{2}} + a_5 \rho^{-(\frac{3+\lambda}{2}+r)} + O\left(\rho^{-(1+\lambda+\varepsilon_1)}\right) \quad , \quad \rho > 1 \quad (5.7)$$

for some $\varepsilon_1 > 0$. Moreover:

$$\varphi_2(\tau, \rho) = b_2(\tau) \rho^{-\frac{3+\lambda}{2}} + O_{\varepsilon_2}\left(\tau^{\frac{2}{\lambda-1}-\varepsilon_2} \rho^{-(\frac{3+\lambda}{2}+r)}\right) \quad , \quad \rho > 1 \quad (5.8)$$

where $|b_2(\tau)| \leq C_{\varepsilon_2} \tau^{\frac{2}{\lambda-1}-\varepsilon_2}$ for any $\varepsilon_2 > 0$.

Suppose now that $\tau \leq 1$. Then:

$$g(\tau, x, 1) = a_3 \tau x^{-\frac{3+\lambda}{2}} + b_4(\tau) x^{-\frac{3+\lambda}{2}} + O\left(\tau x^{-(\frac{3+\lambda}{2}+r)}\right) \quad , \quad x \geq \frac{3}{2} \quad (5.9)$$

where $|b_4(\tau)| \leq C \tau^{1+\varepsilon_3}$ for some $\varepsilon_3 > 0$ sufficiently small.

Proof. The argument is similar to the one in [6]. More precisely we deform the contour of integration in (4.12). Crossing the singularities of the integrand we obtain contributions using residues that yield the main terms in the asymptotics. The only difference with the argument in [6] is that we have to cross also the singularity at $\xi = (\frac{3+\lambda}{2} + r) i$. This yields the second term on the right-hand side of (5.7).

More precisely. Suppose first that $\tau \geq 1$. We then use the representation formula (cf. [6], Subsection 9.2):

$$G(\tau, X) = (\tau)^{\frac{2}{\lambda-1}} \Psi_1(\theta) + G_1(\tau, X) \quad , \quad \theta = X + \frac{2}{\lambda-1} \log(\tau)$$

where:

$$\begin{aligned} \Psi_1(\theta) &= \frac{1}{\pi(\lambda-1)i\mathcal{V}\left(\frac{\lambda+1}{2}i\right)} \int_{\text{Im}(\xi)=\beta_2} d\xi e^{i\xi\theta} \mathcal{V}(\xi) \Gamma\left(-\frac{2i}{\lambda-1}(\xi-i)\right) \\ G_1(\tau, X) &= \frac{i}{\pi(\lambda-1)} \int_{\text{Im}(\xi)=\beta_2} d\xi e^{i\xi\theta} \int_{\text{Im}(y)=\beta_3} dy \frac{\mathcal{V}(\xi) \tau^{-\frac{2iy}{\lambda-1}}}{\mathcal{V}(y)} \Gamma\left(-\frac{2i}{\lambda-1}(\xi-y)\right) \end{aligned}$$

with $\beta_2 \in (\beta_0, \frac{3-\lambda}{2})$, $\beta_0 \in (\frac{3}{2}, 2)$, $\beta_3 \in (\frac{3-\lambda}{2}, 1)$. The asymptotics of the function $\Psi_1(\theta)$ as $\theta \rightarrow \infty$ has been obtained in [6], Proposition 9.8 moving upwards the contour of integration $\{\text{Im}(\xi) = \beta_2\}$ in order to make it cross the first singularity found of \mathcal{V} at $\xi = \frac{3+\lambda}{2}i$. To obtain better estimates we just move the contour of integration above the line $\{\text{Im}(\xi) = \lambda + 1\}$. We then obtain the following generalization of formula (9.27) in [6]:

$$\begin{aligned} \Psi_1(\theta) &= -\frac{\Gamma\left(\frac{\lambda+1}{\lambda-1}\right) \mathcal{V}(2i) e^{-\frac{3+\lambda}{2}\theta}}{2\pi i \mathcal{V}\left(\frac{\lambda+1}{2}i\right)} - \frac{2\pi i \Gamma\left(\frac{\lambda}{\lambda-1}\right) \mathcal{V}(2i) e^{-(\frac{3+\lambda}{2}+r)\theta}}{\pi(\lambda-1)i\mathcal{V}\left(\frac{\lambda+1}{2}i\right) \Phi((\lambda+1)i) \Phi'\left(\left(\frac{3+\lambda}{2}i\right)\right)} + \\ &+ \frac{1}{\pi(\lambda-1)i\mathcal{V}\left(\frac{\lambda+1}{2}i\right)} \int_{\text{Im}(\xi)=1+\lambda+\varepsilon_1} d\xi e^{i\xi\theta} \mathcal{V}(\xi) \Gamma\left(-\frac{2i}{\lambda-1}(\xi-i)\right) \end{aligned} \quad (5.10)$$

with $\varepsilon > 0$ small. We have computed $\text{Res}(\mathcal{V}; \xi = (\lambda + 1) i)$ using Proposition 4.1 and (5.11) in [6].

The first term on the right-hand side of (5.10) is the first one on the right-hand side of (5.7). The last one can be estimated by $Ce^{-(\frac{3+\lambda}{2}+r+\varepsilon_1)\theta}$ for $\theta > 0$. This gives (5.7). We now estimate $G_1(\tau, X)$. This can be made as the estimate of G_1 in Lemma 9.9 of [6]. Deforming the contour $\{\text{Im}(\xi) = \beta_2\}$ as in the derivation of (9.36) of [6], but moving it above the line $\{\text{Im}(\xi) = \lambda + 1\}$ we obtain:

$$G_1(\tau, X) = b_2(\tau) e^{-\frac{3+\lambda}{2}\theta} + \tilde{b}_2(\tau) e^{-(\frac{3+\lambda}{2}+r)\theta} + \tilde{Q}_1(\tau, X)$$

where the function $b_2(\tau)$ is exactly as in [6], the function $\tilde{b}_2(\tau)$ has a similar formula, with slightly different terms arising from the integration by residues, and $\tilde{Q}_1(\tau, X)$ is similar to (9.37) in [6] with the only difference that $\beta_6 = (1 + \lambda) + \varepsilon_1$, with $\varepsilon_1 > 0$. Arguing exactly as in [6] we obtain:

$$\begin{aligned} |b_2(\tau)| + |\tilde{b}_2(\tau)| &\leq C(\tau)^{\frac{2}{\lambda-1}-\varepsilon_2} \quad \text{for } \tau \geq 1 \\ |\tilde{Q}_1(\tau, X)| &\leq C(\tau)^{\frac{2}{\lambda-1}-\varepsilon_2} e^{-[(1+\lambda)+\varepsilon_1]\theta} \end{aligned}$$

This gives (5.8). On the other hand, in order to derive (5.9) we argue as in [6], Proof of Lemma 9.10, (9.45). Indeed, moving the contour of integration $\{\text{Im}(\xi) = \beta_7\}$ to $\{\text{Im}(\xi) = (\lambda + 1 + \varepsilon_1)\}$, with $\varepsilon_1 > 0$ we obtain:

$$G(\tau, X) = -\frac{\mathcal{V}(2i)i}{2\pi(\lambda-1)} e^{-\frac{3+\lambda}{2}X} \int_{\text{Im}(Y)=-\gamma_1} dY \frac{\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}\left(\frac{(3+\lambda)i}{2} + Y\right)} \Gamma\left(\frac{2iY}{\lambda-1}\right) + \quad (5.11)$$

$$- \frac{2\mathcal{V}(2i)e^{-(\frac{3+\lambda}{2}+r)X}}{(\lambda-1)\Phi((\lambda+1)i)\Phi'\left(\frac{3+\lambda}{2}i\right)} \int_{\text{Im}(Y)=-\gamma_1} dY \frac{\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}((1+\lambda)i+Y)} \Gamma\left(\frac{2iY}{\lambda-1}\right) \quad (5.12)$$

$$+ \frac{i}{\pi(\lambda-1)} \int_{\text{Im}(\xi)=\lambda+1+\varepsilon_1} d\xi e^{i\xi X} \int_{\text{Im}(Y)=-\gamma_1} dY \frac{\mathcal{V}(\xi)\tau^{-\frac{2iY}{\lambda-1}}}{\mathcal{V}(\xi+Y)} \Gamma\left(\frac{2iY}{\lambda-1}\right)$$

The time dependence of the integral terms can be obtained as in [6], since this one comes from the integration in the Y variable. ■

6 ESTIMATES FOR THE QUADRATIC TERM $Q[h]$.

The following Lemma will be used to show smallness of the quadratic terms $Q[h]$.

Proposition 12 *For any $\sigma \in (1, 2)$, and any $\bar{\delta} > 0$, there exists $C = C(\sigma, \bar{\delta})$ such that for any $h \in \mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)$:*

$$\|Q[h]\|_{Y_{\frac{\sigma}{2}, (2+\bar{\delta})}(T)} \leq C \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)}^2$$

with $Q[\cdot]$ as in (1.2), (1.5).

In order to prove Proposition 12 we rewrite $Q[h]$ as:

$$Q[h](\tau, x) = I_1 + I_2 \quad (6.1)$$

where:

$$I_1 = - \int_{\frac{\pi}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} h(\tau, x) h(\tau, y) dy \quad (6.2)$$

$$I_2 = - \int_{\frac{\pi}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} h(\tau, x) h(\tau, y) dy + \int_0^{\frac{\pi}{2}} y^{\frac{\lambda}{2}} h(\tau, y) \left[(x-y)^{\frac{\lambda}{2}} h(\tau, x-y) - x^{\frac{\lambda}{2}} h(\tau, x) \right] dy \quad (6.3)$$

We begin estimating I_1 :

Lemma 13 *Let I_1 be as in (6.3) and $\bar{\delta} > 0$. Then:*

$$\|I_1\|_{Y_{\frac{\sigma}{2}, (2+\bar{\delta}_1)}(T)} \leq C \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)}^2 \quad (6.4)$$

where C is uniformly bounded for $0 \leq T \leq 1$.

Proof of Lemma 13. We just need to estimate the functionals $N_{2;\sigma}(I_1; \tau_0, R)$ for $R \geq 1$ and $M_{2;\sigma}(I_1; R)$ for $R \leq 1$ (cf. (2.8), (2.9)). Suppose first that $R > 1$. We introduce the rescaling $x = RX$, $y = RY$, $\tau = \tau_0 + R^{-\frac{\lambda-1}{2}}\theta$, $H_R(\theta, X) = h(\tau, x)$. Then:

$$I_1 = -R^{\lambda+1} X^{\frac{\lambda}{2}} H_R(\theta, X) \int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}} H_R(\theta, Y) dY - \quad (6.5)$$

$$\begin{aligned} & - R^{\lambda+1} X^{\frac{\lambda}{2}} H_R(\theta, X) \int_2^\infty Y^{\frac{\lambda}{2}} H_R(\theta, Y) dY \\ & \equiv I_{1,1} + I_{1,2} \end{aligned} \quad (6.6)$$

We begin estimating $I_{1,2}$. Notice that:

$$\left| R^{\frac{\lambda}{2}+1} \int_2^\infty Y^{\frac{\lambda}{2}} H_R(\theta, Y) dY \right| \leq \frac{C}{R^{\frac{1}{2}+\delta_1}} \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}$$

Then:

$$N_{2;\sigma}(I_{1,2}; \tau_0, R) \leq \frac{C}{R^{(2+2\delta)}} \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}^2, \quad R \geq 1, \quad \tau_0 \in [0, T] \quad (6.7)$$

On the other hand using the inequality:

$$\|fg\|_{H_x^\sigma(\frac{1}{2}, 2)} \leq C \left(\|f\|_{H_x^\sigma(\frac{1}{2}, 2)} \|g\|_{H_x^\sigma(\frac{1}{2}, 2)} + \|f\|_{H_x^\sigma(\frac{1}{2}, 2)} \|g\|_{H_x^\sigma(\frac{1}{2}, 2)} \right) \quad (6.8)$$

for $\sigma > \frac{1}{2}$ (cf. [16], Theorem 1, Section 4.6.4, p. 221) we obtain:

$$N_{2;\sigma}(I_{1,1}; \tau_0, R) \leq \frac{C}{R^{(2+2\delta)}} \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}^2, \quad R \geq 1, \quad \tau_0 \in [0, T] \quad (6.9)$$

Therefore, combining (6.7), (6.9):

$$N_{2;\sigma}(I_1; \tau_0, R) \leq \frac{C}{R^{(2+2\delta)}} \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}^2, \quad R \geq 1, \quad \tau_0 \in [0, T] \quad (6.10)$$

Suppose now that $R \leq 1$. We introduce now the rescaling $x = RX$, $y = RY$, $H_R(\tau, X) = h(\tau, x)$. Then:

$$\begin{aligned} I_1 &= -R^{\lambda+1} X^{\frac{\lambda}{2}} H_R(\tau, X) \int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}} H_R(\tau, Y) dY - \\ & - R^{\lambda+1} X^{\frac{\lambda}{2}} H_R(\tau, X) \int_2^\infty Y^{\frac{\lambda}{2}} H_R(\tau, Y) dY \\ & \equiv I_{1,1} + I_{1,2} \end{aligned}$$

Notice that:

$$\left| R^{\frac{\lambda}{2}+1} \int_2^\infty Y^{\frac{\lambda}{2}} H_R(\tau, Y) dY \right| \leq C \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \quad (6.11)$$

On the other hand using $\|D_X^\sigma H_R(\tau, \cdot)\|_{L^2(\frac{1}{2}, 2)}^2 = R^{2\sigma-1} \|D_x^\sigma h(\tau, \cdot)\|_{L^2(\frac{R}{2}, 2R)}^2$ we obtain:

$$\begin{aligned} \left(\int_0^T \|D_X^\sigma H_R(\tau, \cdot)\|_{L^2(\frac{1}{2}, 2)}^2 d\tau \right)^{\frac{1}{2}} &\leq \left(R^{2\sigma-1} \int_0^T \|D_x^\sigma h(\tau, \cdot)\|_{L^2(\frac{R}{2}, 2R)}^2 d\tau \right)^{\frac{1}{2}} \\ &\leq R^{-\frac{3}{2}} \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \end{aligned} \quad (6.12)$$

Moreover:

$$\left(\int_0^T \|H_R(\tau, \cdot)\|_{L^2(\frac{1}{2}, 2)}^2 d\tau \right)^{\frac{1}{2}} \leq CT \sup_{0 \leq \tau \leq T} \|H_R(\tau, \cdot)\|_{L^\infty(\frac{1}{2}, 2)} \leq CTR^{-\frac{3}{2}} \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \quad (6.13)$$

henceforth, using (6.11), (6.12), (6.13):

$$R^{\frac{3}{2}} M_{2; \sigma}(I_{1, 2}; R) + \frac{R^{\frac{3}{2}} M_{2; 0}(I_{1, 2}; R)}{T} \leq CR^{\frac{\lambda}{2}} \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}^2$$

On the other hand:

$$\begin{aligned} & \left(\int_0^T \left\| D_X^\sigma \left[\int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}} H_R(\tau, Y) dY \right] \right\|_{L^2(\frac{1}{2}, 2)}^2 d\tau \right)^{\frac{1}{2}} + \\ & + \sup_{0 \leq \tau \leq T} \left\| \int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}} H_R(\tau, Y) dY \right\|_{L^\infty(\frac{1}{2}, 2)} \\ & \leq CR^{-\frac{3}{2}} \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \end{aligned}$$

where we just estimate the L^2 norm of D_X^σ combining the estimates of the derivative and the function itself by interpolation. Then, using also (6.12), (6.13) we obtain $R^{\frac{3}{2}} M_{2; \sigma}(I_{1, 1}; R) + R^{\frac{3}{2}} M_{2; 0}(I_{1, 1}; R) \leq CR^{\lambda - \frac{1}{2}} \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}^2$ whence:

$$R^{\frac{3}{2}} M_{2; \sigma}(I_1; R) + R^{\frac{3}{2}} M_{2; 0}(I_1; R) \leq C \|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}^2, \quad R \leq 1 \quad (6.14)$$

Combining (6.10), (6.14) we obtain (6.4) and the Lemma follows. ■

In order to estimate I_2 we will need the following auxiliary Lemma:

Lemma 14 *Suppose that $\phi \in C_0^\infty(\mathbb{R}^+)$. There exists $C > 0$ depending only on ϕ and its derivatives such that the following inequality holds for any $R > 1$:*

$$\begin{aligned} & \int_{\mathbb{R}} \left(1 + |\xi|^{2\sigma}\right) \left| \left(\hat{\phi} * \hat{G}\right)(\xi) \right|^2 \left(\min\{\sqrt{|\xi|}, \sqrt{R}\}\right)^2 d\xi \\ & \leq C \int_{\mathbb{R}} \left(1 + |\xi|^{2\sigma}\right) \left| \hat{G}(\xi) \right|^2 \left(1 + \min\{\sqrt{|\xi|}, \sqrt{R}\}\right)^2 d\xi \end{aligned} \quad (6.15)$$

The proof of Lemma 14 will be based in the following inequality:

Lemma 15 *Let $W_R(\xi) = \min\{\sqrt{|\xi|}, \sqrt{R}\} = \sqrt{R} \min\left\{\sqrt{\frac{|\xi|}{R}}, 1\right\}$. There exists a constant $C > 0$ such that, for any $R > 0$ and any $\xi, \eta \in \mathbb{R}$:*

$$|W_R(\xi) - W_R(\eta)| \leq C \frac{|\xi - \eta|}{|\eta|} W_R(\eta) \quad (6.16)$$

Proof of Lemma 15. This Lemma can be thought as a particular case of Lemma 3.6 in [7]. However, we give here an elementary proof. Due to the scale invariance of the inequality (6.16) we can restrict ourselves to the case $R = 1$. The inequality is then elementary if $\max\{|\xi|, |\eta|\} \geq 1$. Suppose then that $\max\{|\xi|, |\eta|\} \leq 1$. Then (6.16) reduces to $|\sqrt{|\xi|} - \sqrt{|\eta|}| \leq C \frac{|\xi - \eta|}{|\eta|} \sqrt{|\eta|}$ which follows immediately multiplying both sides of the inequality by $(\sqrt{|\xi|} + \sqrt{|\eta|})$. ■

Proof of Lemma 14. Using the regularity properties of ϕ we have:

$$\left| (\hat{\phi} * \hat{G})(\xi) \right| \leq C_m \int_{\{|\eta| \leq 1\}} \frac{|\hat{G}(\eta)|}{1 + |\xi - \eta|^m} + C_m \int_{\{|\eta| > 1\}} \frac{|\hat{G}(\eta)|}{1 + |\xi - \eta|^m} \equiv J_1(\xi) + J_2(\xi)$$

where m can be assumed to be arbitrarily large. Using then $\frac{(1+|\xi|^\sigma)W_R(\xi)}{1+|\xi-\eta|^m} \leq \frac{C}{1+|\xi|}$ for $|\eta| \leq 1$ we obtain:

$$\begin{aligned} & \int_{\mathbb{R}} (1 + |\xi|^{2\sigma}) \left| (\hat{\phi} * \hat{G})(\xi) \right|^2 \left(\min \left\{ \sqrt{|\xi|}, \sqrt{R} \right\} \right)^2 d\xi \\ & \leq C \int_{\{|\eta| \leq 1\}} |\hat{G}(\eta)|^2 d\eta + \\ & + \int_{\mathbb{R}} (1 + |\xi|^{2\sigma}) \left(1 + (W_R(\xi))^2 \right) (J_2(\xi))^2 d\xi \\ & \leq C \int_{\mathbb{R}} (1 + |\xi|^{2\sigma}) \left(1 + (W_R(\xi))^2 \right) |\hat{G}(\xi)|^2 d\xi + \\ & + C \int_{\mathbb{R}} (1 + |\xi|^\sigma)^2 (W_R(\xi))^2 (J_2(\xi))^2 d\xi \\ & \equiv K_1 + K_2 \end{aligned}$$

In order to estimate K_2 we use Lemma 15 to obtain for $|\eta| \geq 1$:

$$\begin{aligned} & |(1 + |\xi|^\sigma) W_R(\xi) - (1 + |\eta|^\sigma) W_R(\eta)| \\ & \leq C \frac{|\xi - \eta|}{|\eta| + 1} W_R(\eta) + C \frac{|\xi - \eta|}{|\eta|^{1-\sigma} + 1} W_R(\eta) + C |\xi - \eta|^\sigma W_R(\xi) \\ & \leq C |\xi - \eta| W_R(\eta) + C |\xi - \eta|^\sigma W_R(\eta) + C \frac{|\xi - \eta|^{\sigma+1}}{|\eta| + 1} W_R(\eta) \\ & \leq C \left(|\xi - \eta|^\sigma + |\xi - \eta|^{\sigma+1} \right) W_R(\eta) \end{aligned}$$

Using this inequality to estimate the terms in $W_R(\xi) \cdot J_2(\xi)$ and using Young's inequality, as well as the fact that the integration $J_2(\xi)$ takes place in $|\eta| \geq 1$ we obtain:

$$K_2 \leq C \int_{\mathbb{R}} (1 + |\xi|^{2\sigma}) (W_R(\xi))^2 |\hat{G}(\xi)|^2 d\xi$$

whence Lemma 14 follows. ■

We now estimate I_2 . The bounds for this operator are the crucial step in the argument from the point of view of the regularity of the functions, because this operator can be estimated as some regularized version of the half-derivative operator. It will be essential to use the seminorm $[\cdot]_{\frac{3+\lambda}{2}}^{\sigma, \frac{1}{2}}$ (cf. (2.14)).

Lemma 16 *Suppose that I_2 is as in (6.3) and $\bar{\delta} > 0$. Then:*

$$\|I_2\|_{Y_{\frac{3}{2}, 2+\bar{\delta}}^\sigma} \leq CR^{-(2+\bar{\delta})} \|h\|_{Z_{\bar{p}}^{\sigma, \frac{1}{2}}(T)}^2$$

Proof of Lemma 16. Suppose first that $R \geq 1$. Using the rescaling $x = RX$, $y = RY$, $\tau = \tau_0 + R^{-\frac{\lambda-1}{2}}\theta$, $h(\tau, x) = R^{-(\frac{3+\lambda}{2} + \bar{\delta})} G_R(\theta, X)$

$$I_2 = R^{-(2+2\bar{\delta})} \int_0^{\frac{x}{R}} Y^{-\frac{3}{2}} G_R(\theta, Y) \left[(X - Y)^{\frac{3}{2}} G_R(\theta, X - Y) - X^{\frac{3}{2}} G_R(\theta, X) \right] dY \quad (6.17)$$

Notice that:

$$|G_R(\theta, Y)| \leq \|h\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)} \min\left(Y^{-(\frac{3+\lambda}{2}+\delta)}, R^{\frac{\lambda}{2}+\delta}Y^{-\frac{3}{2}}\right) \quad (6.18)$$

We rewrite (6.17) as:

$$\begin{aligned} I_2 &= I_{2,-} + \sum_{\{k=0,1,\dots;\frac{2^k}{R} \leq \frac{1}{4}\}} I_{2,k} + I_{2,+} \\ I_{2,-} &= R^{-(2+2\delta)} \int_0^{\frac{1}{R}} Y^{\frac{\lambda}{2}} G_R(\theta, Y) J(G_R; \theta, X, Y) dY \\ I_{2,k} &= R^{-(2+2\delta)} \int_{\frac{2^{k-1}}{R}}^{\frac{2^k}{R}} Y^{\frac{\lambda}{2}} G_R(\theta, Y) J(G_R; \theta, X, Y) dY \quad , \quad k = 0, 1, \dots \\ I_{2,+} &= R^{-(2+2\delta)} \int_{\frac{2^{k_{\max}}}{R}}^{\frac{X}{2}} Y^{\frac{\lambda}{2}} G_R(\theta, Y) J(G_R; \theta, X, Y) dY \\ J(G; \theta, X, Y) &= \left[(X-Y)^{\frac{\lambda}{2}} G(\theta, X-Y) \eta(X-Y) - X^{\frac{\lambda}{2}} G(\theta, X) \eta(X) \right] \end{aligned} \quad (6.19)$$

where $\frac{2^{k_{\max}}}{R} \leq \frac{1}{4} < \frac{2^{k_{\max}+1}}{R}$ and $\eta(X)$ is the cutoff function used in (2.13). (Notice that $\eta(X) = 1$ in all the regions of integration, since $X \in (\frac{1}{2}, 2)$). Let us write $\psi_R(\theta, X) = X^{\frac{\lambda}{2}} G_R(\theta, X) \eta(X)$. In order to estimate these terms in $H_X^\sigma(\frac{1}{2}, 2)$ we use Fourier:

$$\psi_R(\theta, X) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}_R(\theta, \xi) e^{i\xi X} d\xi$$

Since the functions $I_{2,-}(\theta, X)$, $I_{2,k}(\theta, X)$, $I_{2,+}(\theta, X)$ are defined for $X \in \mathbb{R}$ we can compute their Fourier transforms. Using the convolution property for Fourier transforms we have:

$$\hat{I}_{2,k}(\theta, \xi) = \hat{\psi}_R(\theta, \xi) \left[R^{-(2+2\delta)} \int_{\frac{2^{k-1}}{R}}^{\frac{2^k}{R}} Y^{\frac{\lambda}{2}} G_R(\theta, Y) (e^{-i\xi Y} - 1) dY \right]$$

Using (6.18):

$$\begin{aligned} &\|D_X^\sigma I_{2,k}\|_{L^2(\frac{1}{2}, 2)} \\ &\leq \|D_X^\sigma I_{2,k}\|_{L^2(\mathbb{R})} \\ &\leq \|h\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)} \frac{R^{-(2+\delta)}}{(2^{k-1})^\delta} \left(\int_{\mathbb{R}} |\xi|^{2\sigma} |\hat{\psi}_R(\theta, \xi)|^2 \left(\int_{\frac{2^{k-1}}{R}}^{\frac{2^k}{R}} \frac{|e^{-i\xi Y} - 1|}{Y^{\frac{3}{2}}} dY \right)^2 d\xi \right)^{\frac{1}{2}} \end{aligned}$$

We now use that:

$$\int_{\frac{2^{k-1}}{R}}^{\frac{2^k}{R}} \frac{|e^{-i\xi Y} - 1|}{Y^{\frac{3}{2}}} dY \leq C \min\{\sqrt{|\xi|}, \sqrt{R}\}$$

whence:

$$\|D_X^\sigma I_{2,k}\|_{L^2(\frac{1}{2}, 2)} \leq C \|h\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)} \frac{R^{-(2+\delta)}}{(2^{k-1})^\delta} \left(\int_{\mathbb{R}} |\xi|^{2\sigma} |\hat{\psi}_R(\theta, \xi)|^2 \left(\min\{\sqrt{|\xi|}, \sqrt{R}\} \right)^2 d\xi \right)^{\frac{1}{2}}$$

Using Lemma 14 it follows that:

$$\|D_X^\sigma I_{2,k}\|_{L^2(\frac{1}{2}, 2)} \leq \frac{CR^{-(2+\delta)}}{(2^{k-1})^\delta} \|h\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)}^2 \quad (6.20)$$

The term $I_{2,+}$ can be estimated similarly:

$$\|D_X^\sigma I_{2,+}\|_{L^2(\frac{1}{2},2)} \leq \frac{CR^{-(2+\bar{\delta})}}{(2^{k_{\max}-1})^{\bar{\delta}}} \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}^2 \quad (6.21)$$

We now estimate $I_{2,-}$. A similar argument yields:

$$\|D_X^\sigma I_{2,-}\|_{L^2(\frac{1}{2},2)} \leq \|D_X^\sigma I_{2,-}\|_{L^2(\mathbb{R})} \leq \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} R^{-(2+\bar{\delta})} \left(\int_{\mathbb{R}} |\xi|^{2\sigma} \left| \hat{\psi}_R(\theta, \xi) \right|^2 (\Omega_R(Y))^2 d\xi \right)^{\frac{1}{2}}$$

where:

$$\Omega_R(Y) = \int_0^{\frac{1}{R}} \frac{|e^{-i\xi Y} - 1|}{Y^{\frac{3}{2}}} (RY)^{\frac{\lambda}{2}} dY = \sqrt{R} \int_0^1 \frac{|e^{-i\frac{\xi}{R}y} - 1|}{y^{\frac{3}{2}}} (y)^{\frac{\lambda}{2}} dy \leq CW_R(\xi) \quad (6.22)$$

with $W_R(\xi)$ as in Lemma 15. The last inequality follows computing the asymptotics of the second integral in (6.22) for $\frac{\xi}{R} \rightarrow 0$ and $\frac{\xi}{R} \rightarrow \infty$.

Therefore

$$\|D_X^\sigma I_{2,-}\|_{L^2(\frac{1}{2},2)} \leq C \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} R^{-(2+\bar{\delta})} \left(\int_{\mathbb{R}} |\xi|^{2\sigma} \left| \hat{\psi}_R(\theta, \xi) \right|^2 (W_R(\xi))^2 d\xi \right)^{\frac{1}{2}}$$

whence:

$$\|D_X^\sigma I_{2,-}\|_{L^2(\frac{1}{2},2)} \leq C \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}^2 R^{-(2+\bar{\delta})} \quad (6.23)$$

To conclude the proof of Lemma 16 it only remains to estimate the contributions of the region where $R \leq 1$. The estimate of $R^{\frac{3}{2}} M_{2,\sigma}(I_2; R)$, $R^{\frac{3}{2}} M_{2,0}(I_2; R)$ can be made in exactly the same way as the estimate (6.14) for I_1 . Notice that the two terms in I_2 yield integrals that converge separately since $h(\tau, y)$ can be estimated as $\frac{1}{y^{\frac{3}{2}}}$ for $y \leq 1$ and then, the term $y^{\frac{\lambda}{2} - \frac{3}{2}}$ gives integrability.

Therefore:

$$R^{\frac{3}{2}} M_{2,\sigma}(I_2; R) + R^{\frac{3}{2}} M_{2,0}(I_2; R) \leq C \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}^2, \quad R \leq 1 \quad (6.24)$$

Combining (6.20), (6.21), (6.23), (6.24) Lemma 16 follows. ■

Proof of Proposition 12. It is just a consequence of (6.2), (6.3), Lemma 13, Lemma 16. ■

We can also prove the following Lipschitz property for the functional $Q[\cdot]$:

Proposition 17 For any $\sigma \in (1, 2)$ and any $\bar{\delta} > 0$ there exists $C = C(\sigma, \bar{\delta})$ such that for any $h_1, h_2 \in \mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)$:

$$\|Q[h_1] - Q[h_2]\|_{Y_{\frac{3}{2},(2+\bar{\delta})}^\sigma(T)} \leq C \left(\sum_{k=1}^2 \|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma;\frac{1}{2}}(T)}$$

with $Q[\cdot]$ as in (1.2), (1.5).

Proof. We have $Q[h_1](\tau, x) - Q[h_2](\tau, x) = I_1 + I_2$ with:

$$\begin{aligned} I_1 &\equiv - \left(\int_{\frac{x}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} h_1(\tau, x) h_1(\tau, y) dy - \int_{\frac{x}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} h_2(\tau, x) h_2(\tau, y) dy \right) \\ I_2 &\equiv \left(\int_0^{\frac{x}{2}} y^{\frac{\lambda}{2}} h_1(\tau, y) \left[(x-y)^{\frac{\lambda}{2}} h_1(\tau, x-y) - x^{\frac{\lambda}{2}} h_1(\tau, x) \right] dy - \right. \\ &\quad \left. - \int_0^{\frac{x}{2}} y^{\frac{\lambda}{2}} h_2(\tau, y) \left[(x-y)^{\frac{\lambda}{2}} h_2(\tau, x-y) - x^{\frac{\lambda}{2}} h_2(\tau, x) \right] dy \right) \end{aligned}$$

To estimate I_1 we need to estimate the functionals $N_{2;\sigma}(I_1; \tau_0, R)$ for $R \geq 1$ and $M_{2;\sigma}(I_1; R)$ for $R \leq 1$ (cf. (2.8), (2.9)). Suppose first that $R > 1$. We introduce the rescaling $x = RX$, $y = RY$, $\tau = \tau_0 + R^{-\frac{\lambda-1}{2}}\theta$, $H_{R,1}(\theta, X) = h_1(\tau, x)$, $H_{R,2}(\theta, X) = h_2(\tau, x)$. Then:

$$\begin{aligned} I_1 &= -R^{\lambda+1}X^{\frac{\lambda}{2}} [H_{R,1}(\theta, X) - H_{R,2}(\theta, X)] \int_{\frac{X}{2}}^{\infty} Y^{\frac{\lambda}{2}} H_{R,1}(\theta, Y) dY - \\ &\quad - R^{\lambda+1}X^{\frac{\lambda}{2}} H_{R,2}(\theta, X) \int_{\frac{X}{2}}^{\infty} Y^{\frac{\lambda}{2}} [H_{R,1}(\theta, Y) - H_{R,2}(\theta, Y)] dY \end{aligned} \quad (6.25)$$

Notice that:

$$\begin{aligned} &\sup_{X \in (\frac{1}{2}, 2)} \left| \int_X^{\infty} Y^{\frac{\lambda}{2}} H_{R,1}(\theta, Y) dY \right| + \left\| \int_{(\cdot)}^{\infty} Y^{\frac{\lambda}{2}} H_{R,1}(\theta, Y) dY \right\|_{H_x^{\sigma}(\frac{1}{2}, 2)} \\ &\leq \frac{C}{R^{\frac{3+\lambda}{2}+\delta}} \|h_1\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)} \\ &\sup_{X \in (\frac{1}{2}, 2)} \left| \int_{\frac{X}{2}}^{\infty} Y^{\frac{\lambda}{2}} [H_{R,1}(\theta, Y) - H_{R,2}(\theta, Y)] dY \right| + \\ &+ \left\| \int_{(\cdot)}^{\infty} Y^{\frac{\lambda}{2}} [H_{R,1}(\theta, Y) - H_{R,2}(\theta, Y)] dY \right\|_{H_x^{\sigma}(\frac{1}{2}, 2)} \leq \frac{C}{R^{\frac{3+\lambda}{2}+\delta}} \|h_1 - h_2\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)} \end{aligned}$$

Using (6.8) in (6.25) we obtain:

$$N_{2;\sigma}(I_1; \tau_0, R) \leq \frac{C}{R^{(2+2\delta)}} \left(\sum_{k=1}^2 \|h_k\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)} \quad (6.26)$$

for $R \geq 1$, $\tau_0 \in [0, T]$. Suppose now that $R \leq 1$. We introduce the rescaling $x = RX$, $y = RY$, $H_{R,1}(\tau, X) = h_1(\tau, x)$, $H_{R,2}(\tau, X) = h_2(\tau, x)$. Then:

$$I_1 = I_{1,1} + I_{1,2}$$

$$\begin{aligned} I_{1,1} &\equiv - \left[R^{\lambda+1}X^{\frac{\lambda}{2}} [H_{R,1}(\tau, X) - H_{R,2}(\tau, X)] \int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}} H_{R,1}(\tau, Y) dY + \right. \\ &\quad \left. + R^{\lambda+1}X^{\frac{\lambda}{2}} H_{R,2}(\tau, X) \int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}} [H_{R,1}(\tau, Y) - H_{R,2}(\tau, Y)] dY \right] \end{aligned} \quad (6.27)$$

$$\begin{aligned} I_{1,2} &\equiv - \left[R^{\lambda+1}X^{\frac{\lambda}{2}} [H_{R,1}(\tau, X) - H_{R,2}(\tau, X)] \int_2^{\infty} Y^{\frac{\lambda}{2}} H_{R,1}(\tau, Y) dY + \right. \\ &\quad \left. + R^{\lambda+1}X^{\frac{\lambda}{2}} H_{R,2}(\tau, X) \int_2^{\infty} Y^{\frac{\lambda}{2}} [H_{R,1}(\tau, Y) - H_{R,2}(\tau, Y)] dY \right] \end{aligned} \quad (6.28)$$

Notice that:

$$\left| R^{\frac{\lambda}{2}+1} \int_2^{\infty} Y^{\frac{\lambda}{2}} H_{R,1}(\tau, Y) dY \right| \leq C \|h_1\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)} \quad (6.29)$$

$$\left| R^{\frac{\lambda}{2}+1} \int_2^{\infty} Y^{\frac{\lambda}{2}} [H_{R,1}(\tau, Y) - H_{R,2}(\tau, Y)] dY \right| \leq C \|h_1 - h_2\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)} \quad (6.30)$$

On the other hand, using the definition of $H_{R,1}$, $H_{R,2}$ we arrive at:

$$\int_0^T \|D_X^{\sigma} H_{R,2}(\tau, \cdot)\|_{L^2(\frac{1}{2}, 2)}^2 d\tau \leq \|h_2\|_{\mathcal{Z}_p^{\sigma, \frac{1}{2}}(T)}^2 \quad (6.31)$$

$$\int_0^T \|D_X^\sigma [H_{R,1}(\tau, X) - H_{R,2}(\tau, X)]\|_{L^2(\frac{1}{2}, 2)}^2 d\tau \leq \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)}^2 \quad (6.32)$$

Moreover:

$$\left(\int_0^T \|H_{R,2}(\tau, \cdot)\|_{L^2(\frac{1}{2}, 2)}^2 d\tau \right)^{\frac{1}{2}} \leq CTR^{-\frac{3}{2}} \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \quad (6.33)$$

$$\left(\int_0^T \| [H_{R,1}(\tau, \cdot) - H_{R,2}(\tau, \cdot)] \|_{L^2(\frac{1}{2}, 2)}^2 d\tau \right)^{\frac{1}{2}} \leq CTR^{-\frac{3}{2}} \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \quad (6.34)$$

henceforth, using (6.8), (6.28)-(6.34):

$$R^{\frac{3}{2}} M_{2;0}(I_{1,2}; R) + R^{\frac{3}{2}} M_{2;\sigma}(I_{1,2}; R) \leq R^{\frac{\lambda}{2}} \left(\sum_{k=1}^2 \|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \quad (6.35)$$

On the other hand for $R \leq 1$ we have:

$$\begin{aligned} & \left(\int_0^T \left\| D_X^\sigma \left[\int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}} [H_{R,1}(\tau, \cdot) - H_{R,2}(\tau, \cdot)] dY \right] \right\|_{L^2(\frac{1}{2}, 2)}^2 d\tau \right)^{\frac{1}{2}} + \\ & + \sup_{0 \leq \tau \leq T} \left\| \int_{\frac{X}{2}}^2 Y^{\frac{\lambda}{2}} H_{R,1}(\tau, \cdot) - H_{R,2}(\tau, \cdot) dY \right\|_{L^\infty(\frac{1}{2}, 2)} \\ & \leq CR^{-\frac{3}{2}} \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \end{aligned}$$

where we just estimate the L^2 norm of D_X^σ by D_X and the function itself by interpolation. Then, using also (6.12), (6.13) as well as the fact that $R \leq 1$:

$$R^{\frac{3}{2}} M_{2;\sigma}(I_{1,1}; R) + R^{\frac{3}{2}} M_{2;0}(I_{1,1}; R) \leq C \left(\sum_{k=1}^2 \|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \quad (6.36)$$

Combining (6.35), (6.36):

$$\|I_1\|_{Y_{\frac{\sigma}{2}, (2+\delta)}^\sigma(T)} \leq \left(\sum_{k=1}^2 \|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \right) C \|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \quad (6.37)$$

where C is uniformly bounded for $0 \leq T \leq 1$.

We now estimate I_2 . Suppose first that $R \geq 1$. Using the rescaling $x = RX$, $y = RY$, $\tau = \tau_0 + R^{-\frac{\lambda-1}{2}}\theta$, $h_1(\tau, x) = R^{-(\frac{3+\lambda}{2}+\delta)} G_{R,1}(\theta, X)$, $h_2(\tau, x) = R^{-(\frac{3+\lambda}{2}+\delta)} G_{R,2}(\theta, X)$ and using (6.19) we obtain:

$$I_2 = I_{2,1} + I_{2,1}$$

$$\begin{aligned} I_2 \equiv & \left(\int_0^{\frac{\sigma}{2}} y^{\frac{\lambda}{2}} h_1(\tau, y) \left[(x-y)^{\frac{\lambda}{2}} h_1(\tau, x-y) - x^{\frac{\lambda}{2}} h_1(\tau, x) \right] dy - \right. \\ & \left. - \int_0^{\frac{\sigma}{2}} y^{\frac{\lambda}{2}} h_2(\tau, y) \left[(x-y)^{\frac{\lambda}{2}} h_2(\tau, x-y) - x^{\frac{\lambda}{2}} h_2(\tau, x) \right] dy \right) \end{aligned}$$

$$I_{2,1} = R^{-(2+2\delta)} \int_0^{\frac{X}{2}} Y^{\frac{\lambda}{2}} (G_{R,1}(\theta, Y) - G_{R,2}(\theta, Y)) J(G_{R,1}; \theta, X, Y) dY \quad (6.38)$$

$$I_{2,2} = R^{-(2+2\delta)} \int_0^{\frac{X}{2}} Y^{\frac{\lambda}{2}} G_{R,2}(\theta, Y) J(G_{R,1} - G_{R,2}; \theta, X, Y) dY \quad (6.39)$$

Notice that:

$$|G_{R,1}(\theta, Y) - G_{R,2}(\theta, Y)| \leq \|h_1 - h_2\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \min\left(Y^{-(\frac{3+\lambda}{2}+\delta)}, R^{\frac{\lambda}{2}+\delta} Y^{-\frac{3}{2}}\right) \quad (6.40)$$

We now argue exactly as in the proof of Lemma 16 in order to estimate $I_{2,1}$, $I_{2,2}$. Notice that estimating these terms it is crucial to use the boundedness of the seminorm $[\cdot]_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}$ in (2.14) for the sources. On the other hand, the argument in the Proof of Lemma 16 shows that the pointwise estimate (6.40) is needed. A similar argument and estimate allows to estimate the terms $I_{2,1}$, $I_{2,2}$ in (6.38), (6.39). Therefore, after some computations:

$$N_{2,0}(I_2, \tau_0, R) + N_{2,\sigma}(I_2, \tau_0, R) \leq \frac{C}{R^{2+\delta}} \left(\sum_{k=1}^2 \|h_k\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \quad (6.41)$$

It only remains to estimate the region where $R \leq 1$. The estimate of $R^{\frac{3}{2}} M_{2,\sigma}(I_2; R)$, $R^{\frac{3}{2}} M_{2,0}(I_2; R)$ can be made exactly in the same way as the estimate of similar terms for I_1 . Notice that the two terms in I_2 yield integrals that converge separately since $h(\tau, y)$ can be estimated as $\frac{1}{y^{\frac{3}{2}}}$ for $y \leq 1$ and then, the term $y^{\frac{\lambda}{2}-\frac{3}{2}}$ is integrable near the origin. Then:

$$R^{\frac{3}{2}} M_{2,0}(I_2; R) + R^{\frac{3}{2}} M_{2,\sigma}(I_2; R) \leq C \left(\sum_{k=1}^2 \|h_k\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \quad (6.42)$$

Combining (6.41), (6.42) we obtain:

$$\|I_2\|_{Y_{\frac{3}{2}, (2+\delta)}^\sigma(T)} \leq C \left(\sum_{k=1}^2 \|h_k\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \right) \|h_1 - h_2\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \quad (6.43)$$

The proof of the Lemma is then concluded using (6.37) and (6.43). ■

7 DERIVATION OF THE ASYMPTOTICS $x^{-\frac{3+\lambda}{2}}$ AS $x \rightarrow \infty$.

The main result in this Section is the following.

Proposition 18 *Suppose that $\varphi \in \mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}(T)$ solves:*

$$\varphi_\tau = \mathcal{L}_{f_0}[\varphi] + F(\tau, x), \quad x > 0, \quad 0 \leq t \leq T, \quad \varphi(0, x) = 0 \quad (7.1)$$

where $F \in Y_{\frac{3}{2}, 2+\delta}^\sigma(T)$ and $\bar{\delta} < r$. Then, the following asymptotics holds:

$$\varphi(\tau, x) = \mathcal{W}(\tau) x^{-\frac{3+\lambda}{2}} + \varphi_R(\tau, x) \quad \text{as } x \rightarrow \infty \quad (7.2)$$

where:

$$\mathcal{W}(\tau) = \int_0^\tau ds \int_0^\infty \frac{dx_0}{x_0} \Theta\left((\tau-s)x_0^{\frac{\lambda-1}{2}}\right) x_0^{\frac{3+\lambda}{2}} [F(s, x_0) + (\mathcal{L}_{f_0} - L)[\varphi](s, x_0)] \quad (7.3)$$

with $\Theta(\cdot)$ as in (4.11).and:

$$\varphi_R \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T) \quad (7.4)$$

The proof of Proposition 18 is based in dealing with the operator \mathcal{L}_{f_0} as a perturbation of the operator L . To this end, we rewrite (7.1) as:

$$\varphi_\tau = L[\varphi] + (\mathcal{L}_{f_0} - L)[\tilde{h}] + F(\tau, x)$$

Using variations of constants and Theorem 3 we have the representation formula:

$$\varphi(\tau, x) = \int_0^\tau ds \int_0^\infty g\left((\tau-s)x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right) [(\mathcal{L}_{f_0} - L)[\varphi](s, x) + F(s, x)] \frac{dx_0}{x_0} \quad (7.5)$$

In order to prove Proposition 18 we will derive some auxiliary Lemmas. We begin estimating the term $(\mathcal{L}_{f_0} - L)[\varphi]$ (cf. (7.5)). Most of the estimates in the next Lemma have been already obtained in [7], but we recall them here for convenience.

Lemma 19 *Suppose that f_0 satisfies (2.2)-(2.5) and $\varphi \in \mathcal{E}_{T,\sigma}$. Then:*

$$N_\infty((\mathcal{L}_{f_0} - L)[\varphi]; \tau_0, R) \leq \frac{C \|\varphi\|_\sigma}{R^{2+r}}, \quad \tau_0 \in (0, T), \quad R \geq 1 \quad (7.6)$$

$$M_\infty((\mathcal{L}_{f_0} - L)[\varphi]; R) \leq \frac{C \|\varphi\|_\sigma}{R^{\frac{3}{2}}}, \quad R \leq 1 \quad (7.7)$$

$$N_{2,\sigma}((\mathcal{L}_{f_0} - L)[\varphi], \tau_0, R) \leq \frac{C}{R^{2+r}} \|\varphi\|_{\mathcal{Z}^{\sigma, \frac{1}{2}}_{(\frac{3+\lambda}{2}+r)}}(T), \quad \tau_0 \in (0, T), \quad R \geq 1 \quad (7.8)$$

$$M_{2,\sigma}((\mathcal{L}_{f_0} - L)[\varphi], R) \leq \frac{C}{R^{\frac{3}{2}}} \|\varphi\|_{\mathcal{Z}^{\sigma, \frac{1}{2}}_{(\frac{3+\lambda}{2}+r)}}(T), \quad R \leq 1 \quad (7.9)$$

Proof of Lemma 19. We write

$$(\mathcal{L}_{f_0} - L)[\varphi](s, x_0) = A_1 + A_2$$

where:

$$A_1 = \int_0^{\frac{x}{2}} (H(x-y) - H(x)) y^{\frac{\lambda}{2}} \varphi(\tau, y) dy - H(x) \int_{\frac{x}{2}}^\infty y^{\frac{\lambda}{2}} \varphi(\tau, y) dy - x^{\frac{\lambda}{2}} \varphi(\tau, x) \int_{\frac{x}{2}}^\infty H(y) dy$$

$$A_2 = \int_0^{\frac{x}{2}} \left((x-y)^{\frac{\lambda}{2}} \varphi(\tau, x-y) - x^{\frac{\lambda}{2}} \varphi(\tau, x) \right) H(y) dy$$

and

$$H(y) = y^{\frac{\lambda}{2}} f_0(y) - y^{-\frac{3}{2}}$$

It has been proved in [7] (cf. Section 5, Lemmas 5.1, 5.2, 5.4):

$$|A_1| \leq \frac{C \|\varphi\|_{\frac{3}{2}, \frac{3+\lambda}{2}}}{x^{\frac{3}{2}}}, \quad x \leq 1, \quad |A_1| \leq \frac{C \|\varphi\|_{\frac{3}{2}, \frac{3+\lambda}{2}}}{x^{2+r}}, \quad x \geq 1 \quad (7.10)$$

$$N_\infty(A_2; \tau_0, R) \leq \frac{C \|\varphi\|_\sigma}{R^{2+r}}, \quad R \geq 1; \quad M_\infty(A_2; R) \leq \frac{C \|\varphi\|_\sigma}{R^{\frac{3}{2}}}, \quad R \leq 1, \quad \tau_0 \in (0, T) \quad (7.11)$$

$$N_{2,\sigma}(A_1, \tau_0, R) \leq \frac{C}{R^{2+r}} \|\varphi\|_{Y^{\sigma, \frac{3+\lambda}{2}}_{\frac{3}{2}}}, \quad \tau_0 \in (0, T), \quad R \geq 1 \quad (7.12)$$

$$M_{2,\sigma}(A_1, R) \leq \frac{C}{R^{\frac{3}{2}}} \|\varphi\|_{Y^{\sigma, \frac{3+\lambda}{2}}_{\frac{3}{2}}}, \quad R \leq 1 \quad (7.13)$$

where r might be chosen as in (2.1) and the norms $\|\cdot\|_{\frac{3}{2}, \frac{3+\lambda}{2}}$, $\|\cdot\|_{\sigma}$ are as in (2.10), (2.11).

On the other hand, arguing as in the Proof of Lemma 16 it is possible to prove the following estimates:

$$N_{2,\sigma}(A_2, \tau_0, R) \leq \frac{C}{R^{2+r}} \|\varphi\|_{Z_{(\frac{3+\lambda}{2}+r)}^{\sigma; \frac{1}{2}}(T)} \quad , \quad \tau_0 \in (0, T) \quad , \quad R \geq 1 \quad (7.14)$$

$$M_{2,\sigma}(A_2, R) \leq \frac{C}{R^{\frac{3}{2}}} \|\varphi\|_{Z_{(\frac{3+\lambda}{2}+r)}^{\sigma; \frac{1}{2}}(T)} \quad , \quad R \leq 1 \quad (7.15)$$

The only difference in the argument is that instead of (6.18) the estimate that must be used is:

$$|H(y)| \leq C \min\left(y^{-\frac{3}{2}}, y^{-(\frac{3}{2}+r)}\right)$$

that implies that the function $H_R(Y) = R^{(\frac{3+\lambda}{2}+r)}H(RY)$ satisfies:

$$|H_R(Y)| \leq C \min\left(Y^{-(\frac{3+\lambda}{2}+r)}, R^{\frac{\lambda}{2}+r}Y^{-\frac{3}{2}}\right)$$

Notice that for $R < 1$ we obtain estimates with the dependence $\frac{1}{R^{\frac{3}{2}}}$ on the right hand side (cf. (7.11), (7.13), (7.15)) due to the fact that the term $H(x) \int_{\frac{x}{2}}^{\infty} y^{\frac{\lambda}{2}} \varphi(\tau, y) dy$ in the definition of A_1 yields such a power law dependence for small x . Combining (7.12)-(7.15) the Lemma follows. ■

Remark 20 *The estimate for the term A_2 cannot be improved to the decay in the norm H_x^{σ} except if we obtain instead the decay R^{-2} . Such a decay has been obtained in [7]. The main novelty in the estimate for A_2 obtained in Lemma 19 is the decay like $R^{-(2+r)}$ in (7.14) for large R , that can be obtained using the estimate for the seminorm $[\varphi]_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}$ contained in the spaces $Z_{(\frac{3+\lambda}{2}+\bar{\delta})}^{\sigma; \frac{1}{2}}$.*

In the proof of the following results the notation will become simpler using the following definitions:

$$\int_{\tau_0-\rho}^{\tau_0} f(s) ds = \frac{1}{\rho} \int_{(\tau_0-\rho)_+}^{\tau_0} f(s) ds$$

The next Lemma shows how to compute the asymptotics as $x \rightarrow \infty$ of the solutions of:

$$J_{\tau} = L[J] + F(\tau, x) \quad , \quad J(0, x) = 0$$

To this end we will use the following auxiliary functional:

$$\sup_{R \geq 1, \tau_0 \in (0, T)} \left[N_{\infty}(F; \tau_0, R) R^{2+\bar{\delta}} \right] + \sup_{R < 1} \left[M_{\infty}(F; R) R^{\frac{3}{2}} \right] \equiv \mathcal{H}(F) \quad (7.16)$$

Lemma 21 *Let $0 < \bar{\delta} < \min\{\varepsilon, r\}$, with ε as in (4.10) and r as in (2.1). Suppose that $F \in X_{\frac{3}{2}, 2+\bar{\delta}}(T)$. Let $J = J(\tau, x)$ be:*

$$J(\tau, x) = \int_0^{\tau} ds \int_0^{\infty} g\left(\tau - s, x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right) F(s, x_0) \frac{dx_0}{x_0} \quad (7.17)$$

Then:

$$J(\tau, x) - I(\tau; F) x^{-\frac{3+\lambda}{2}} \xi(x) = J_R(\tau, x) \quad (7.18)$$

where $\xi(\cdot)$ is as in (2.4) and:

$$I(\tau; F) = \int_0^{\infty} \frac{dx_0}{x_0} \int_0^{\tau} ds F(s, x_0) \Theta\left(\tau - s, x_0^{\frac{\lambda-1}{2}}\right) x_0^{\frac{3+\lambda}{2}} \quad (7.19)$$

with $\Theta(\cdot)$ as in (4.11) and

$$\|J_R\|_{\frac{3}{2}, \frac{3+\lambda}{2}+\delta} \leq C \|F\|_{X_{\frac{3}{2}, 2+\delta}(T)} \quad (7.20)$$

Moreover:

$$|I(\tau; F)| \leq C \tau^{\frac{2\delta}{\lambda-1}} \|F\|_{X_{\frac{3}{2}, 2+\delta}(T)} \quad (7.21)$$

Proof. We split the integral in (7.17) as:

$$J = J_1 + J_2 + J_3 + J_4$$

$$J_1 \equiv \int_{\frac{2x}{3}}^{\infty} dx_0 \int_{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+}^{\tau} ds [\dots] \quad , \quad J_2 \equiv \int_0^{\frac{2x}{3}} dx_0 \int_{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+}^{\tau} ds [\dots] \quad (7.22)$$

$$J_3 \equiv \int_{\frac{2x}{3}}^{\infty} dx_0 \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+} ds [\dots] \quad , \quad J_4 \equiv \int_0^{\frac{2x}{3}} dx_0 \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+} ds [\dots] \quad (7.23)$$

In the term J_1 we use the fact that (4.7) implies:

$$\begin{aligned} |g(\tau, x, 1)| &\leq C \tau x^{-\frac{3}{2}} \quad , \quad 0 < x \leq \frac{1}{2} \\ |g(\tau, x, 1)| &\leq C \tau^{-2} \varphi\left(\frac{x-1}{\tau^2}\right) \quad , \quad |x-1| \leq \frac{1}{2} \quad \text{with} \quad \varphi(\xi) = \frac{1}{1 + \xi^{\frac{3}{2}-\varepsilon_1}} \end{aligned}$$

for some $\varepsilon_1 > 0$. Then:

$$|J_1| \leq J_{1,1} + J_{1,2}$$

where:

$$J_{1,1} \equiv C \int_{\frac{x}{2}}^{\frac{3x}{2}} dx_0 \int_{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+}^{\tau} (\tau-s)^{-2} ds x_0^{-\lambda} \varphi\left(\frac{x-x_0}{(\tau-s)^2 x_0^\lambda}\right) |F(s, x_0)| \quad (7.24)$$

$$J_{1,2} \equiv \frac{C}{x^{3/2}} \int_{2x}^{\infty} dx_0 \int_{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+}^{\tau} (\tau-s) ds x_0^{\frac{\lambda}{2}} |F(s, x_0)| \quad (7.25)$$

In order to estimate J_3 we use that (4.4)-(4.6) implies:

$$|g(\tau, x, 1)| \leq C \tau^{\frac{2}{\lambda-1}} \min \left\{ \left(\tau^{\frac{2}{\lambda-1}} x \right)^{-\frac{3}{2}}, \left(\tau^{\frac{2}{\lambda-1}} x \right)^{-\frac{3+\lambda}{2}} \right\} \quad , \quad \tau \geq 1$$

Then:

$$|J_3| \leq C \int_{\frac{2x}{3}}^{\infty} dx_0 \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+} ds |F(s, x_0)| \min \left\{ (\tau-s)^{-\frac{1}{\lambda-1}} x^{-\frac{3}{2}}, (\tau-s)^{-\frac{1+\lambda}{\lambda-1}} x^{-\frac{3+\lambda}{2}} \right\} \quad (7.26)$$

To obtain the leading asymptotics of J_2 as $x \rightarrow \infty$ we will use the detailed asymptotics of $g(\tau, x, 1)$. Using (4.9), (4.11) we have:

$$\left| J_2 - x^{-\frac{3+\lambda}{2}} \int_0^{\frac{2x}{3}} dx_0 \int_{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+}^{\tau} ds F(s, x_0) \Theta\left((\tau-s) x_0^{\frac{\lambda-1}{2}}\right) x_0^{\frac{1+\lambda}{2}} \right| = J_{2,R} \quad (7.27)$$

where, due to (5.9) in Proposition 11:

$$|J_{2,R}| \leq C x^{-\frac{3+\lambda}{2}-r} \int_0^{\frac{2x}{3}} dx_0 \int_{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+}^{\tau} (\tau-s) ds |F(s, x_0)| x_0^{\lambda+r} \quad (7.28)$$

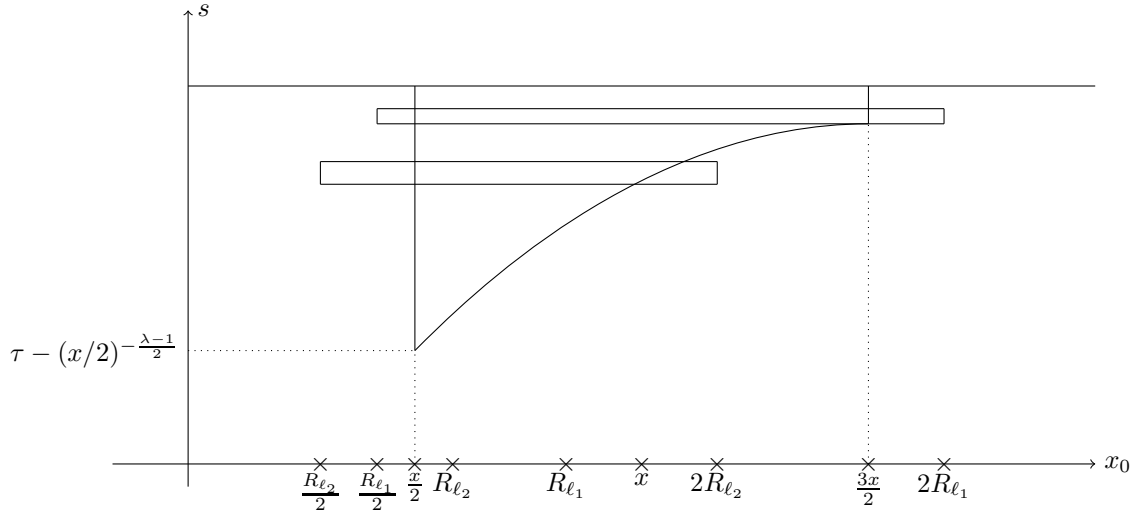


Figure 2: Two cubes of the covering $\mathcal{C}_{x,\tau}^{(1)}$

To estimate J_4 we use (4.4)-(4.6):

$$|J_4| \leq \frac{C}{x^{\frac{3+\lambda}{2}}} \int_0^{\frac{2x}{3}} dx_0 \int_0^{\left(\tau - x_0^{-\frac{\lambda-1}{2}}\right)_+} \frac{ds}{(\tau-s)^{\frac{\lambda+1}{\lambda-1}}} |F(s, x_0)| \quad (7.29)$$

where we use that, since $x_0 \leq x$, we have $(\tau-s)x^{\frac{\lambda-1}{2}} \geq \frac{x^{\frac{\lambda-1}{2}}}{x_0^{\frac{\lambda-1}{2}}} \geq 1$. We now proceed to estimate all the terms J_k , $k = 1, 2, 3, 4$ in terms of $\mathcal{H}(F)$ in (7.16). We are interested in the behaviour of all these quantities for large values of x . We begin with the term $J_{1,1}$ (cf. (7.24)). Notice that we can cover the domain $\left\{ (x_0, s) : x_0 \in \left[\frac{x}{2}, \frac{3x}{2}\right], s \in \left[\left(\tau - x_0^{-\frac{\lambda-1}{2}}\right)_+, \tau\right] \right\}$ by a set of rectangles with the form $\left[\frac{R_\ell}{2}, 2R_\ell\right] \times \left[\tau_0, \tau_0 + (R_\ell)^{-\frac{\lambda-1}{2}}\right]$, with $R_\ell \in \left[\frac{x}{2}, \frac{3x}{2}\right]$, $\tau_0 \in \left[\tau - 2^{\frac{\lambda-1}{2}} x^{-\frac{\lambda-1}{2}}, \tau\right]$ with $\ell = 1, \dots, N_{x,\tau}$ and $N_{x,\tau} \leq n_0 < \infty$ with n_0 independent on x, τ . The fact that the number of rectangles can be estimated uniformly on x follows from the self-similarity of the problem. Let us denote such finite covering as $\mathcal{C}_{x,\tau}^{(1)}$.

The specific rescaling chosen for the rectangles is due to the fact that they are the ones appearing in the definition of the functionals $N_\infty(F; \tau_0, R)$. We then have:

$$\begin{aligned} J_{1,1} &\leq C \sum_{\mathcal{C}_{x,\tau}^{(1)}} \int_{\left(\tau - CR_x^{-\frac{\lambda-1}{2}}\right)_+}^{\tau} (\tau-s)^{-2} ds \int_{\frac{x}{2}}^{\frac{3x}{2}} dx_0 x_0^{-\lambda} \varphi\left(\frac{x-x_0}{(\tau-s)^2 x_0^\lambda}\right) \|F(s, \cdot)\|_{L^\infty\left(\frac{R_x}{2}, 2R_x\right)} \\ &\leq CR_x^{-\frac{\lambda-1}{2}} \sum_{\mathcal{C}_{x,\tau}^{(1)}} \int_{\left(\tau - CR_x^{-\frac{\lambda-1}{2}}\right)_+}^{\tau} \|F(s, \cdot)\|_{L^\infty\left(\frac{R_x}{2}, 2R_x\right)} ds \\ &\leq CR_x^{-\frac{\lambda-1}{2}} \sup_{0 \leq \tau \leq T} N_\infty(F; \tau, R) \leq Cx^{-(\frac{3+\lambda}{2} + \delta)} \mathcal{H}(F) \end{aligned} \quad (7.30)$$

We now estimate the term $J_{1,2}$ in (7.25). Let us denote as $\mathcal{C}_{x,\tau}^{(2)}$ a covering of the set $\left\{ (x_0, s) : x_0 \geq 2x, \left(\tau - x_0^{-\frac{\lambda-1}{2}}\right)_+ \leq s \leq \tau \right\}$ by means of boxes with the form $\left[\frac{R}{2}, 2R\right] \times \left[\tau_0, \tau_0 + (R)^{-\frac{\lambda-1}{2}}\right]$, with $R \geq x$, $\tau_0 \in [0, \tau]$ in which each point is covered at most by a finite number of boxes (independent on x, τ), and where the sequence of sizes R increases exponentially.

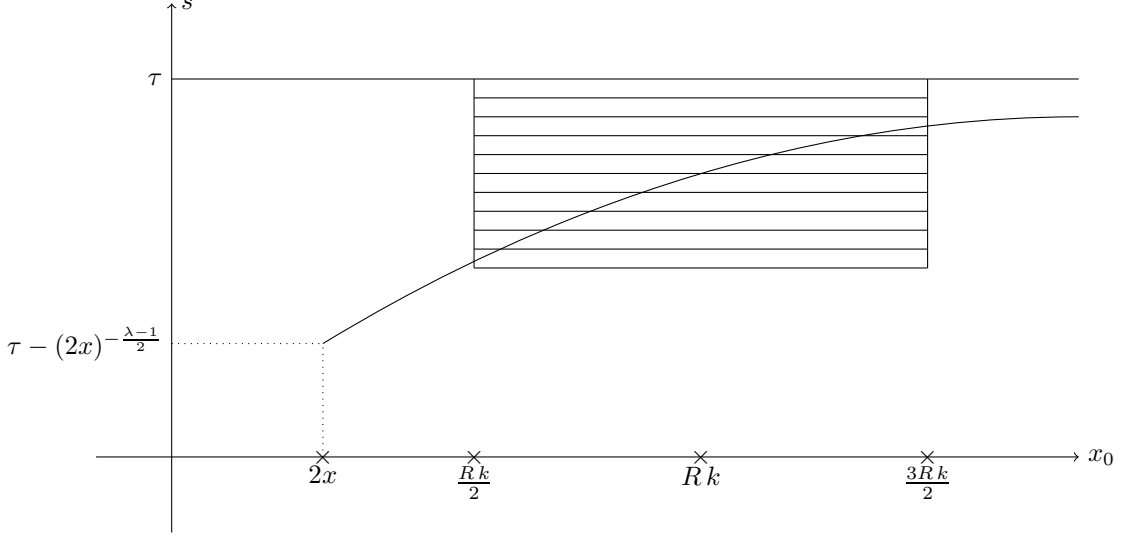


Figure 3: The sub-family \mathcal{R}_k of cubes of the covering $\mathcal{C}_{x,\tau}^{(2)}$

Then:

$$\begin{aligned}
& \frac{1}{x^{3/2}} \int_{2x}^{\infty} dx_0 \int_{\tau - x_0^{-\frac{\lambda-1}{2}}}^{\tau} ds (\tau - s) x_0^{\frac{\lambda}{2}} |F(s, x_0)| \\
& \leq \frac{C}{x^{3/2}} \sum_{\mathcal{C}_{x,\tau}^{(2)}} R^{\frac{\lambda}{2}+1} \int_{\tau_0 - R^{-\frac{\lambda-1}{2}}}^{\tau_0} ds (\tau - s) \|F(s, \cdot)\|_{L^\infty(\frac{R}{2}, 2R)} \\
& \leq \frac{C}{x^{3/2}} \sum_{\mathcal{C}_{x,\tau}^{(2)}} R^{2-\frac{\lambda}{2}} \left(\int_{\tau_0 - R^{-\frac{\lambda-1}{2}}}^{\tau_0} ds \|F(s, \cdot)\|_{L^\infty(\frac{R}{2}, 2R)}^2 \right)^{\frac{1}{2}} \\
& \leq \frac{C}{x^{3/2}} \sum_{\mathcal{C}_{x,\tau}^{(2)}} R^{2-\frac{\lambda}{2}} N_\infty(F; \tau, R) \leq \frac{C\mathcal{H}(F)}{x^{3/2}} \sum_{\mathcal{C}_{x,\tau}^{(2)}} R^{2-\frac{\lambda}{2}} R^{-(2+\delta)} \leq \frac{C\mathcal{H}(F)}{x^{\frac{3+\lambda}{2}+\delta}}
\end{aligned}$$

where we use the fact that the series $\sum_{\mathcal{C}_{x,\tau}^{(2)}} [\cdot \cdot \cdot]$ is a geometric series, due to our choice of the sizes of the boxes. Therefore:

$$|J_1| \leq Cx^{-(\frac{3+\lambda}{2}+\delta)} \mathcal{H}(F) \quad (7.31)$$

We now estimate J_3 (cf. (7.26)). To this end we introduce a new covering $\mathcal{C}_{x,\tau}^{(3)}$ of the set $\mathcal{D}_{x,\tau} = \left\{ (x_0, s) : x_0 \geq 2x, 0 \leq s \leq \left(\tau - x_0^{-\frac{\lambda-1}{2}} \right)_+ \right\}$ by means of boxes of the form $[\frac{R}{2}, 2R] \times [\tau_0, \tau_0 + (R)^{-\frac{\lambda-1}{2}}]$ with $R \geq x$, $\tau_0 \in [0, \tau]$.

We will assume that the rectangles in the set $\mathcal{C}_{x,\tau}^{(3)}$ have the following properties. (i) Each point in the set $\mathcal{D}_{x,\tau}$ is contained in at most a finite number of boxes independent on x , τ for $\tau \in [0, T]$. (ii) There exist a sequence of sizes $\{R_k\}_{k \in \mathbb{N}}$ that increases exponentially on k , such that the number of cubes with R comparable to a given R_k (i. e. $R \in (\frac{R_k}{2}, 2R_k)$) is bounded by $CR_k^{\frac{\lambda-1}{2}}$, with C independent on x , τ for $\tau \in [0, T]$. Notice that the construction of the covering implies that, for $(x_0, s) \in [\frac{R}{2}, 2R] \times [\tau_0, \tau_0 + (R)^{-\frac{\lambda-1}{2}}]$ we have $\frac{1}{2}(\tau - \tau_0) \leq (\tau - s) \leq 2(\tau - \tau_0)$.

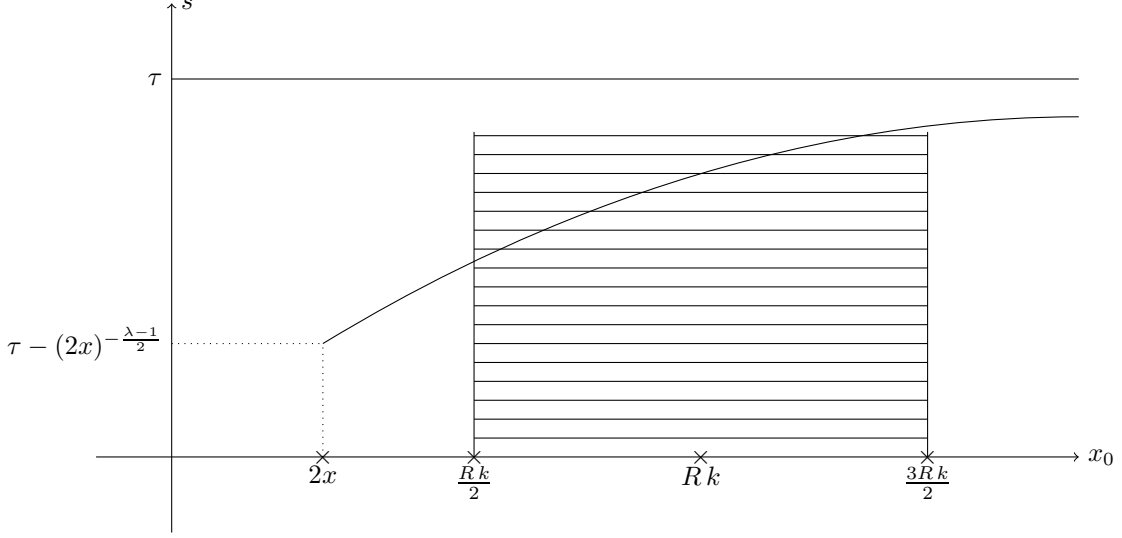


Figure 4: The sub-family \mathcal{R}_k of cubes of the covering $\mathcal{C}_{x,\tau}^{(3)}$

Then, using $\min \left\{ 1, \frac{1}{(\tau-s)^{\frac{\lambda-1}{2}} x} \right\} \leq 1$:

$$\begin{aligned}
|J_3| &\leq \frac{C}{x^{\frac{3}{2}}} \sum_{\mathcal{C}_{x,\tau}^{(3)}} \int_{\tau_0}^{\tau_0+R^{-\frac{\lambda-1}{2}}} ds \int_{\frac{R}{2}}^{2R} dx_0 \frac{\|F(s, \cdot)\|_{L^\infty(\frac{R}{2}, 2R)}}{(\tau - \tau_0)^{\frac{1}{\lambda-1}}} \\
&\leq \frac{C}{x^{\frac{3}{2}}} \sum_{\mathcal{C}_{x,\tau}^{(3)}} \left[\frac{R}{(\tau - \tau_0)^{\frac{1}{\lambda-1}}} R^{-\frac{\lambda-1}{2}} \right] \int_{\tau_0}^{\tau_0+R^{-\frac{\lambda-1}{2}}} \|F(s, \cdot)\|_{L^\infty(\frac{R}{2}, 2R)} ds \\
&\leq \frac{C}{x^{\frac{3}{2}}} \sum_{\mathcal{C}_{x,\tau}^{(3)}} \left[R \int_{\tau_0}^{\tau_0+R^{-\frac{\lambda-1}{2}}} \frac{ds}{(\tau - s)^{\frac{1}{\lambda-1}}} \right] N_\infty(F; \tau, R)
\end{aligned}$$

where we have used the fact that the length of the time integration in each box is of order $R^{-\frac{\lambda-1}{2}}$ as well as Cauchy-Schwartz inequality in the last step. Therefore, using the definition of $\mathcal{H}(F)$ as well as the properties of the covering $\mathcal{C}_{x,\tau}^{(3)}$ and in particular the properties of the sequence $\{R_k\}$:

$$\begin{aligned}
|J_3| &\leq \frac{C\mathcal{H}(F)}{x^{\frac{3}{2}}} \sum_{\mathcal{C}_{x,\tau}^{(3)}} \left[R^{-(1+\delta)} \int_{\tau_0}^{\tau_0+R^{-\frac{\lambda-1}{2}}} \frac{ds}{(\tau - s)^{\frac{1}{\lambda-1}}} \right] \\
&\leq \frac{C\mathcal{H}(F)}{x^{\frac{3}{2}}} \sum_{\{R_k\}} \left[R_k^{-(1+\delta)} \int_0^{\tau - CR_k^{-\frac{\lambda-1}{2}}} \frac{ds}{(\tau - s)^{\frac{1}{\lambda-1}}} \right] \leq \frac{C\mathcal{H}(F)}{x^{\frac{3}{2}}} \sum_{\{R_k\}} R_k^{-\frac{3}{2}-\delta} \leq \frac{C\mathcal{H}(F)}{x^{\frac{3+\lambda}{2}+\delta}}
\end{aligned}$$

... Add Figure?. ...

Then:

$$|J_3| \leq \frac{C\mathcal{H}(F)}{x^{\frac{3+\lambda}{2}+\delta}} \quad (7.32)$$

We now estimate the term J_2 . Using (4.11), (7.27) and (7.28)) we obtain $J_2 \leq K_2$ where:

$$K_2 \leq \int_0^\infty dx_0 \int_{\left(\tau - x_0^{-\frac{\lambda-1}{2}}\right)_+}^\tau ds |F(s, x_0)| x_0^{\frac{\lambda+1}{2}}$$

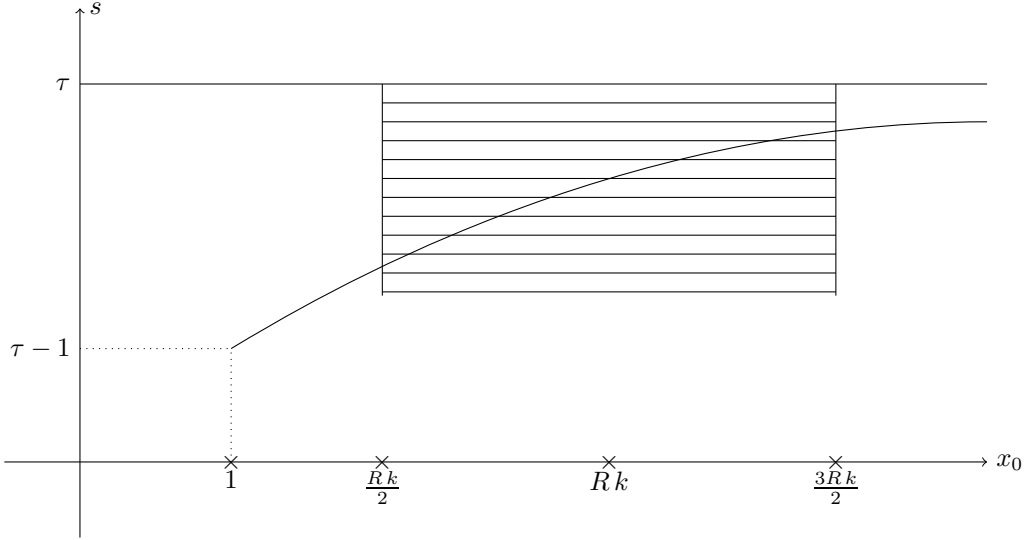


Figure 5: The sub-family \mathcal{R}_k of cubes of the covering $\mathcal{C}_{x,\tau}^{(4)}$

We introduce a covering $\mathcal{C}_{x,\tau}^{(4)}$ such that each point of the set $\left\{ (x_0, s) : x_0 \geq 1, \left(\tau - x_0^{-\frac{\lambda-1}{2}} \right)_+ \leq s \leq \tau \right\}$ is contained in a bounded number of boxes having the form $\left[\frac{R}{2}, 2R \right] \times \left[\tau_0, \tau_0 + (R)^{-\frac{\lambda-1}{2}} \right]$, and where the values of the radii R increase exponentially.

Then:

$$K_2 \leq \int_0^\tau ds \int_0^1 dx_0 |F(s, x_0)| x_0^{\frac{\lambda+1}{2}} + \sum_{\mathcal{C}_{x,\tau}^{(4)}} \int_{\frac{R}{2}}^{2R} dx_0 \int_{\left(\tau - x_0^{-\frac{\lambda-1}{2}} \right)_+}^\tau ds |F(s, x_0)| x_0^{\frac{\lambda+1}{2}} = K_{2,1} + K_{2,2}$$

Using Cauchy-Schwartz we then estimate $K_{2,2}$ as:

$$K_{2,2} \leq C \sum_{\mathcal{C}_{x,\tau}^{(4)}} R^2 N_\infty(F; \tau, R) \leq C\mathcal{H}(F) \sum_{\mathcal{C}_{x,\tau}^{(4)}} R^{-\delta} \leq C\mathcal{H}(F)$$

On the other hand we estimate $K_{2,1}$ decomposing the interval $[0, 1]$ as:

$$[0, 1] = \bigcup_{n=0}^{\infty} [R_{n+1}, R_n] \quad , \quad R_n = 2^{-n} \quad , \quad n = 0, 1, 2, \dots \quad (7.33)$$

Then, using again Cauchy-Schwartz:

$$\begin{aligned} K_{2,1} &\leq C \sum_{n=0}^{\infty} \int_0^\tau ds \int_{R_{n+1}}^{R_n} dx_0 \|F(s, \cdot)\|_{L^\infty\left(\frac{R_n}{2}, R_n\right)} R_n^{\frac{\lambda+1}{2}} \\ &\leq C\mathcal{H}(F) \sum_{n=0}^{\infty} R_n^{\frac{\lambda+3}{2}+1} R_n^{-\frac{3}{2}} \leq C\mathcal{H}(F) \end{aligned}$$

Therefore:

$$J_2 \leq K_2 \leq C\mathcal{H}(F)$$

Actually we can derive a more precise approximation for J_2 rewriting it as $J_2 = I_2(\tau) x^{-\frac{3+\lambda}{2}} + J_{2;R,1} + J_{2;R}$ where:

$$I_2(\tau) = \int_0^\infty \frac{dx_0}{x_0} \int_{\left(\tau - x_0^{-\frac{\lambda-1}{2}} \right)_+}^\tau ds F(s, x_0) \Theta \left((\tau - s) x_0^{\frac{\lambda-1}{2}} \right) x_0^{\frac{3+\lambda}{2}}$$

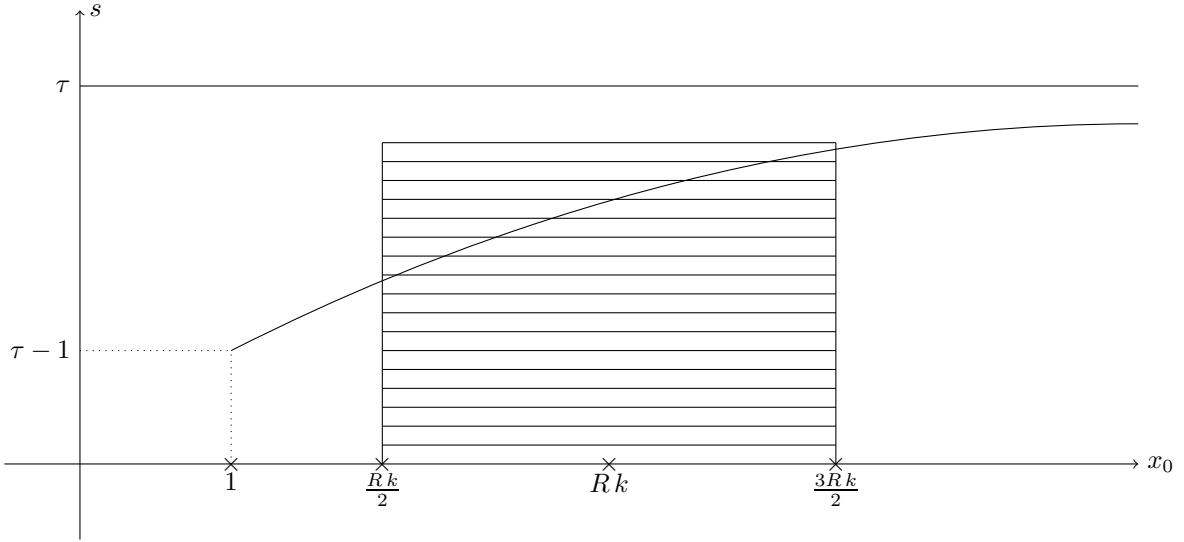


Figure 6: The sub-family \mathcal{R}_k of cubes of the covering $\mathcal{C}_{x,\tau}^{(5)}$

$$J_{2;R,1} = -x^{-\frac{3+\lambda}{2}} \int_{\frac{2x}{3}}^{\infty} \frac{dx_0}{x_0} \int_{\left(\tau-x_0-\frac{\lambda-1}{2}\right)_+}^{\tau} ds F(s, x_0) \Theta\left((\tau-s)x_0^{\frac{\lambda-1}{2}}\right) x_0^{\frac{3+\lambda}{2}}$$

$$J_{2,R} = \int_0^{\tau} ds \int_0^{\infty} \left[g\left((\tau-s)x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right) - \Theta\left((\tau-s)x_0^{\frac{\lambda-1}{2}}\right) \left(\frac{x_0}{x}\right)^{\frac{3+\lambda}{2}} \right] F(s, x_0) \frac{dx_0}{x_0}$$

The terms $J_{2,R}$ can be estimated using Proposition 11 (cf. (5.7), (5.8)) arguing as in the estimate of K_2 exactly as the previous estimate for J_2 since $r > \bar{\delta}$. Therefore, we obtain the estimate:

$$|J_{2,R}| \leq C x^{-\frac{3+\lambda}{2}-r} \int_0^{\frac{2x}{3}} \frac{dx_0}{(x_0)^{1+\delta-r}} \leq C x^{-\frac{3+\lambda}{2}-\bar{\delta}}$$

where the constants C could be very large for small $\bar{\delta}$. on the other hand:

$$|J_{2;R,1}| \leq C x^{-\frac{3+\lambda}{2}} \int_{\frac{2x}{3}}^{\infty} \frac{dx_0}{x_0} \int_{\left(\tau-x_0-\frac{\lambda-1}{2}\right)_+}^{\tau} ds (\tau-s) x_0^{\frac{\lambda-1}{2}} x_0^{\frac{3+\lambda}{2}} (x_0)^{-2-\bar{\delta}} \leq C x^{-\frac{3+\lambda}{2}-\bar{\delta}}$$

whence:

$$\left| J_2 - I_2(\tau) x^{-\frac{3+\lambda}{2}} \right| \leq C x^{-\frac{3+\lambda}{2}-\bar{\delta}} \quad (7.34)$$

We now estimate the integrand in J_4 in (7.29). In order to apply Lebesgue's Theorem we need to prove that:

$$K_4 = \int_0^{\infty} dx_0 \int_0^{\left(\tau-x_0-\frac{\lambda-1}{2}\right)_+} \frac{ds}{(\tau-s)^{\frac{\lambda+1}{\lambda-1}}} |F(s, x_0)| < \infty$$

We define a new covering $\mathcal{C}_{x,\tau}^{(5)}$ of the set $\left\{ (x_0, s) : x_0 \geq 1, 0 \leq s \leq \left(\tau-x_0-\frac{\lambda-1}{2}\right)_+ \right\}$ having the same properties as the covering $\mathcal{C}_{x,\tau}^{(3)}$.

Then, using that the integrand is empty for $x_0 \leq 1$ due to the fact that $\tau \leq T \leq 1$, as well as Cauchy-Schwartz and the properties of the covering $\mathcal{C}_{x,\tau}^{(5)}$ we obtain:

$$\begin{aligned}
K_4 &\leq C \sum_{\mathcal{C}_{x,\tau}^{(5)}} \frac{R^{1-\frac{\lambda-1}{2}}}{(\tau-\tau_0)^{\frac{\lambda+1}{\lambda-1}}} \int_{\tau_0}^{\left(\tau-CR^{-\frac{\lambda-1}{2}}\right)^+} \|F(s,\cdot)\|_{L^\infty\left(\frac{R}{2},2R\right)} ds \\
&\leq C\mathcal{H}(F) \sum_{\mathcal{C}_{x,\tau}^{(5)}} \frac{R^{-1-\frac{\lambda-1}{2}-\bar{\delta}}}{(\tau-\tau_0)^{\frac{\lambda+1}{\lambda-1}}} \\
&\leq C\mathcal{H}(F) \sum_{\mathcal{C}_{x,\tau}^{(5)}} R_k^{-(1+\bar{\delta})} \int_{\tau_0}^{\left(\tau-CR_k^{-\frac{\lambda-1}{2}}\right)^+} (\tau-s)^{-\frac{\lambda+1}{\lambda-1}} ds \\
&\leq C\mathcal{H}(F) \sum_{\{R_k\}} R_k^{-\bar{\delta}} \leq C\mathcal{H}(F)
\end{aligned}$$

whence:

$$K_4 \leq C\mathcal{H}(F) \quad (7.35)$$

In order to obtain (7.18) we need to subtract from J_4 the leading contribution and to estimate the remainder. Using (4.10) we obtain:

$$\begin{aligned}
J_4 &= \int_0^{\frac{2x}{3}} \frac{dx_0}{x_0} \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)^+} ds g\left((\tau-s)x_0^{\frac{\lambda-1}{2}}, \frac{x}{x_0}, 1\right) F(s, x_0) \\
&= \int_0^{\frac{2x}{3}} \frac{dx_0}{x_0} \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)^+} ds \Theta\left((\tau-s)x_0^{\frac{\lambda-1}{2}}\right) (x)^{-\frac{3+\lambda}{2}} F(s, x_0) + J_{4,R}
\end{aligned}$$

where:

$$|J_{4,R}| \leq C \int_0^{\frac{2x}{3}} \frac{dx_0}{x_0} \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)^+} ds \left((\tau-s)x_0^{\frac{\lambda-1}{2}}\right)^{-\frac{2\varepsilon}{\lambda-1}-\frac{\lambda+1}{\lambda-1}} \left(\frac{x}{x_0}\right)^{-\frac{3+\lambda}{2}-\varepsilon} |F(s, x_0)|$$

The argument yielding (7.35) implies the existence of the integral:

$$I_4(\tau) = \int_0^\infty \frac{dx_0}{x_0} \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)^+} ds \Theta\left((\tau-s)x_0^{\frac{\lambda-1}{2}}\right) (\tau-s)^{-\frac{3+\lambda}{\lambda-1}} F(s, x_0)$$

Therefore:

$$J_4 = I_4(\tau) x^{-\frac{3+\lambda}{2}} + \int_{\frac{2x}{3}}^\infty \frac{dx_0}{x_0} \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)^+} ds \Theta\left((\tau-s)x_0^{\frac{\lambda-1}{2}}\right) (\tau-s)^{-\frac{3+\lambda}{\lambda-1}} F(s, x_0) + J_{4,R} \quad (7.36)$$

We then estimate the remainders in (7.36). Notice that the bounds for Θ and F yield:

$$\begin{aligned}
&\left| \int_{\frac{2x}{3}}^\infty \frac{dx_0}{x_0} \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)^+} ds \Theta\left((\tau-s)x_0^{\frac{\lambda-1}{2}}\right) (\tau-s)^{-\frac{3+\lambda}{\lambda-1}} F(s, x_0) \right| \\
&\leq C \left| \int_{\frac{2x}{3}}^\infty \frac{dx_0}{x_0} \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)^+} ds (\tau-s)^{\frac{2}{\lambda-1}} x_0 (\tau-s)^{-\frac{3+\lambda}{\lambda-1}} (x_0)^{-(2+\bar{\delta})} \right| \\
&\leq Cx^{-\bar{\delta}}
\end{aligned}$$

We estimate the term $J_{4,R}$ in (7.36) as:

$$|J_{4,R}| \leq Cx^{-\frac{3+\lambda}{2}-\varepsilon} \int_0^{\frac{2x}{3}} \frac{dx_0}{x_0} \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+} ds \left((\tau-s)x_0^{\frac{\lambda-1}{2}} \right)^{-\frac{2\varepsilon+\lambda+1}{\lambda-1}} (x_0)^{\frac{\lambda-1}{2}+\varepsilon-\bar{\delta}} \leq Cx^{-\frac{3+\lambda}{2}-\bar{\delta}}$$

using $\varepsilon > \bar{\delta}$ whence:

$$\left| J_4 - I_4(\tau) x^{-\frac{3+\lambda}{2}} \right| \leq Cx^{-\frac{3+\lambda}{2}-\bar{\delta}} \quad (7.37)$$

Combining (7.31), (7.32), (7.34), (7.37) we obtain that the function $J_R(\tau, x)$ defined in (7.18) satisfies

$$|J_R(\tau, x)| \leq \frac{C \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)}}{x^{\frac{3+\lambda}{2}+\bar{\delta}}} \quad \text{for } x \geq 1 \quad (7.38)$$

On the other hand, in order to estimate $J_R = J$ for $x \leq 1$ we argue as follows. We recall that $|J_1| \leq J_{1,1} + J_{1,2}$ with $J_{1,1}$, $J_{1,2}$ as in (7.24), (7.25). We have for $R \leq 1$ and $x \in (\frac{R}{2}, 2R)$:

$$J_{1,1}(\tau, x) \leq C \int_0^\tau ds \|F(s, \cdot)\|_{L^\infty(\frac{R}{4}, 4R)} \leq M_\infty(F; R) \leq C \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)}$$

We estimate $J_{1,2}$ covering the set $(s, x_0) \in (0, T) \times (0, 1)$ by means of union of the rectangles $(0, T) \times [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, $n = 0, 1, 2, \dots$. Let us denote this covering as $\mathcal{C}^{(6)}$. Then:

$$\begin{aligned} J_{1,2}(\tau, x) &\leq \frac{C}{x^{3/2}} \int_{2x}^1 dx_0 \int_0^\tau (\tau-s) ds x_0^{\frac{\lambda}{2}} |F(s, x_0)| + \\ &\quad + \frac{C}{x^{3/2}} \int_1^\infty dx_0 \int_{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+}^\tau (\tau-s) ds x_0^{\frac{\lambda}{2}} |F(s, x_0)| \\ &= J_{1,2,1}(\tau, x) + J_{1,2,2}(\tau, x) \end{aligned}$$

$$\begin{aligned} J_{1,2,1}(\tau, x) &\leq \frac{C}{x^{3/2}} \sum_{\mathcal{C}^{(6)}} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} dx_0 \int_0^T (\tau-s) ds x_0^{\frac{\lambda}{2}} \|F(s, \cdot)\|_{L^\infty(\frac{1}{2^{n+1}}, \frac{1}{2^n})} \\ &\leq \frac{C}{x^{3/2}} \sum_{\mathcal{C}^{(6)}} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} dx_0 \left(\frac{1}{2^n}\right)^{\frac{\lambda}{2}} M_\infty\left(F; \frac{1}{2^n}\right) \\ &\leq \frac{C \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)}}{x^{3/2}} \sum_{\mathcal{C}^{(6)}} \frac{1}{2^{n(\frac{\lambda-1}{2})}} \leq \frac{C \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)}}{x^{3/2}} \end{aligned}$$

The integral term in $J_{1,2,2}(\tau, x)$ can be estimated using the covering $\mathcal{C}_{x,\tau}^{(2)}$ exactly in the same way as in the estimate of $J_{1,2}$ for $x \geq 1$. It then follows that $J_{1,2,2}(\tau, x) \leq C \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)} x^{-3/2}$ whence:

$$J_{1,2}(\tau, x) \leq \frac{C \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)}}{x^{3/2}} \quad (7.39)$$

In order to estimate J_3 for $x \leq 1$ we use (7.26). Notice that for $s < \tau \leq T < 1$ and $x \leq 1$ we have $(\tau - s)^{-\frac{1}{\lambda-1}} x^{-\frac{3}{2}} \leq (\tau - s)^{-\frac{1+\lambda}{\lambda-1}} x^{-\frac{3+\lambda}{2}}$ whence (7.26) yields:

$$\begin{aligned} |J_3| &\leq \frac{C}{x^{\frac{3}{2}}} \int_{\frac{2x}{3}}^{\infty} dx_0 \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+} ds \frac{|F(s, x_0)|}{(\tau-s)^{\frac{1}{\lambda-1}}} \\ &\leq \frac{C}{x^{\frac{3}{2}}} \int_{\frac{2x}{3}}^1 dx_0 \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+} ds \frac{|F(s, x_0)|}{(\tau-s)^{\frac{1}{\lambda-1}}} + \\ &\quad + \frac{C}{x^{\frac{3}{2}}} \int_1^{\infty} dx_0 \int_0^{\left(\tau-x_0^{-\frac{\lambda-1}{2}}\right)_+} ds \frac{|F(s, x_0)|}{(\tau-s)^{\frac{1}{\lambda-1}}} \\ &\equiv J_{3,1} + J_{3,2} \end{aligned}$$

The term $J_{3,2}$ can be estimated in the same manner as J_3 for $x \geq 1$. We just use the covering $\mathcal{C}_{x,\tau}^{(3)}$ to obtain $J_{3,2} \leq C \|F\|_{X_{\frac{3}{2}, 2+\delta}(T)} x^{-3/2}$. Since $J_{3,1} = 0$ for $0 \leq \tau \leq T < 1$ we then obtain:

$$J_3 \leq C \|F\|_{X_{\frac{3}{2}, 2+\delta}(T)} x^{-3/2} \quad (7.40)$$

In order to estimate J_2 we use (7.27), (7.28) as well as (4.11)

$$\begin{aligned} J_2 &\leq C x^{-\frac{3+\lambda}{2}} \int_0^{\frac{2x}{3}} dx_0 \int_0^{\tau} ds F(s, x_0) (\tau-s) x_0^{\lambda} \\ &\leq C x^{-\frac{3}{2}} \int_0^1 dx_0 \int_0^{\tau} ds F(s, x_0) (\tau-s) (x_0)^{\frac{\lambda}{2}} \end{aligned}$$

and this integral can be estimated exactly as $J_{1,2,1}(\tau, x)$, using the covering $\mathcal{C}^{(6)}$:

$$J_2 \leq \frac{C}{x^{3/2}} \sum_{\mathcal{C}^{(6)}} \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} dx_0 \int_0^T (\tau-s) ds x_0^{\frac{\lambda}{2}} \|F(s, \cdot)\|_{L^\infty\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)}$$

whence:

$$J_2 \leq C \|F\|_{X_{\frac{3}{2}, 2+\delta}(T)} x^{-3/2} \quad (7.41)$$

Finally we notice that (7.29) implies that $J_4 = 0$ for $x \leq 1$, $0 \leq \tau \leq T < 1$. Combining (7.39), (7.40), (7.41) we obtain:

$$|J(\tau, x)| \leq C \|F\|_{X_{\frac{3}{2}, 2+\delta}(T)} x^{-3/2}$$

for $0 < x \leq 1$. Using then also (7.38) the result follows.

To conclude the proof of Lemma 21 it only remains to show (7.21). To this end we use similar covering arguments. First we decompose the expression of $I(\tau; F)$ as:

$$I(\tau; F) = I_1 + I_2 + I_3 + I_4$$

$$\begin{aligned} I_1 &= \int_0^1 \frac{dx_0}{x_0} \int_0^{\tau} ds [\dots] \quad , \quad I_2 = \int_1^{\tau^{-\frac{2}{\lambda-1}}} \frac{dx_0}{x_0} \int_0^{\tau} ds [\dots] \\ I_3 &= \int_{\tau^{-\frac{2}{\lambda-1}}}^{\infty} \frac{dx_0}{x_0} \int_{\tau-x_0^{-\frac{\lambda-1}{2}}}^{\tau} ds [\dots] \quad , \quad I_4 = \int_{\tau^{-\frac{2}{\lambda-1}}}^{\infty} \frac{dx_0}{x_0} \int_0^{\tau-x_0^{-\frac{\lambda-1}{2}}} ds [\dots] \end{aligned}$$

Using (4.11) as well as the definition of $\|\cdot\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)}$ we obtain, using Cauchy-Schwartz:

$$|I_1| \leq C\tau^{\frac{3}{2}} \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)} \sum_{n=0}^{\infty} \left(\frac{1}{2^n}\right)^{\lambda-\frac{1}{2}} \leq C\tau^{\frac{3}{2}} \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)} \quad (7.42)$$

Using the splitting of the domains of integrations in rectangles as above as well as (4.11) we also obtain the following estimates:

$$\begin{aligned} |I_2| &\leq C \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)} \int_1^{\tau^{-\frac{2}{\lambda-1}}} dx_0 (x_0)^{\lambda-(2+\bar{\delta})} \int_0^{\tau} (\tau-s) ds \\ |I_3| &\leq C \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)} \int_{\tau^{-\frac{2}{\lambda-1}}}^{\infty} dx_0 (x_0)^{\lambda-(2+\bar{\delta})} \int_{\tau-x_0}^{\tau} (\tau-s) ds \\ |I_4| &\leq C \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)} \int_{\tau^{-\frac{2}{\lambda-1}}}^{\infty} dx_0 (x_0)^{-(2+\bar{\delta})} \int_0^{\tau-x_0} (\tau-s)^{-\frac{\lambda+1}{\lambda-1}} ds \end{aligned}$$

Therefore, since the three integrals on the right-hand side are bounded by $\leq C\tau^{\frac{2\bar{\delta}}{\lambda-1}} \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)}$:

$$|I_2| + |I_3| + |I_4| \leq C\tau^{\frac{2\bar{\delta}}{\lambda-1}} \|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)}$$

Combining this with (7.42) we obtain (7.21). ■

We now derive the asymptotics (7.3) for the solutions of (7.1) in a pointwise sense. More precise estimates for the regularity of the error terms will be derived later.

Lemma 22 *Suppose that $\varphi \in \mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}(T)$ solves (7.1) with $F \in Y_{\frac{3}{2}, 2+\bar{\delta}}^{\sigma}(T)$ and $\bar{\delta} < r$. Then (7.2) holds with $\mathcal{W}(\tau)$ as in (7.3) and:*

$$\|\varphi_R\|_{\frac{3}{2}, \frac{3+\lambda}{2}+\bar{\delta}} \leq C \left(\|F\|_{X_{\frac{3}{2}, 2+\bar{\delta}}^{\sigma}(T)} + \|\varphi\|_{\mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}(T)} \right) \quad (7.43)$$

Proof of Lemma 22. Lemma 19 as well as the fact that $r > \bar{\delta}$ implies that $\|(\mathcal{L}_{f_0} - L)[\varphi]\|_{X_{\frac{3}{2}, 2+\bar{\delta}}(T)} < \infty$. Lemma 22 then follows applying 21 with F replaced by

$$\bar{F} = F + (\mathcal{L}_{f_0} - L)[\varphi] . \quad (7.44)$$

■

In order to prove suitable regularity properties of the remainder φ_R in (7.2) we will need to obtain some regularity properties for the function $I(\cdot; \bar{F})$, where the functional I is as in (7.19) and \bar{F} as in (7.44). The rationale behind the argument is that an equation for φ_R contains terms that can be estimated only if some regularity for $I(\tau; \bar{F})$ is available.

We have:

Lemma 23 *Suppose that $\sup_{\tau \in [0, T]} \|F(\tau, \cdot)\|_{\frac{3}{2}, 2+\bar{\delta}} < \infty$. Let $\Delta = \tau_1 - \tau_0$, with $0 \leq \tau_0 \leq \tau_1 \leq T$. Then:*

$$|I(\tau_1; F) - I(\tau_0; F)| \leq C(\Delta)^{\frac{2\bar{\delta}}{\lambda-1}}$$

with $C > 0$ depending on $\sup_{\tau \in [0, T]} \|F(\tau, \cdot)\|_{\frac{3}{2}, 2+\bar{\delta}}$, λ .

Proof. We write, using (7.19):

$$\begin{aligned}
& I(\tau_1; F) - I(\tau_0; F) \\
&= \int_0^\infty \frac{dx_0}{x_0} \int_{\tau_0}^{\tau_1} ds \Theta \left((\tau_1 - s) x_0^{\frac{\lambda-1}{2}} \right) x_0^{\frac{3+\lambda}{2}} F(s, x_0) + \\
&+ \int_0^\infty \frac{dx_0}{x_0} \int_0^{\tau_0} ds \left[\Theta \left((\tau_1 - s) x_0^{\frac{\lambda-1}{2}} \right) - \Theta \left((\tau_0 - s) x_0^{\frac{\lambda-1}{2}} \right) \right] x_0^{\frac{3+\lambda}{2}} F(s, x_0) \\
&= L_1 + L_2
\end{aligned}$$

We have the global estimate:

$$|F(s, x_0)| \leq C x_0^{-(2+\delta)}$$

Then:

$$\begin{aligned}
|L_1| &\leq C \int_0^\infty \frac{dx_0}{x_0} \int_{\tau_0}^{\tau_1} ds \left| \Theta \left((\tau_1 - s) x_0^{\frac{\lambda-1}{2}} \right) \right| x_0^{\frac{3+\lambda}{2}} x_0^{-(2+\delta)} \\
&= C \int_0^\infty \frac{dx_0}{(x_0)^{1+\delta}} \int_0^{\Delta x_0^{\frac{\lambda-1}{2}}} |\Theta(s)| ds = C(\Delta)^{\frac{2\delta}{\lambda-1}} \int_0^\infty \frac{dy}{(y)^{1+\delta}} \int_0^{y^{\frac{\lambda-1}{2}}} |\Theta(s)| ds
\end{aligned}$$

Using the fact that $\Psi_1(y) = \int_0^{y^{\frac{\lambda-1}{2}}} |\Theta(s)| ds$ satisfies $|\Psi_1(y)| \leq C \min\{y^{\lambda-1}, 1\}$ (cf. (4.11)) we obtain:

$$|L_1| \leq C(\Delta)^{\frac{2\delta}{\lambda-1}} \quad (7.45)$$

We now estimate $|L_2|$. We have:

$$\begin{aligned}
|L_2| &\leq \int_0^\infty \frac{dx_0}{x_0} \int_0^{\tau_0} ds \left| \Theta \left((\tau_1 - s) x_0^{\frac{\lambda-1}{2}} \right) - \Theta \left((\tau_0 - s) x_0^{\frac{\lambda-1}{2}} \right) \right| x_0^{\frac{3+\lambda}{2}} x_0^{-(2+\delta)} \\
&= \int_0^\infty \frac{dx_0}{(x_0)^{1+\delta}} \int_0^{\frac{\tau_0}{\Delta}} d\rho \left| \Theta \left((1+\rho) \Delta x_0^{\frac{\lambda-1}{2}} \right) - \Theta \left(\rho \Delta x_0^{\frac{\lambda-1}{2}} \right) \right| x_0^{\frac{\lambda-1}{2}} \Delta \\
&\leq (\Delta)^{\frac{2\delta}{\lambda-1}} \int_0^\infty \frac{\Psi_2(y) dy}{(y)^{1+\delta}} \int_0^\infty dZ \left| \Theta \left(y^{\frac{\lambda-1}{2}} + Z \right) - \Theta(Z) \right|
\end{aligned}$$

where $\Psi_2(y) = \int_0^\infty dZ \left| \Theta \left(y^{\frac{\lambda-1}{2}} + Z \right) - \Theta(Z) \right|$. We have used the change of variables $y = (\Delta)^{\frac{2}{\lambda-1}} x_0$, and $Z = y^{\frac{\lambda-1}{2}} \rho$. Using (4.11) we obtain $\Psi_2(y) \leq C \min\{y^{\frac{\lambda-1}{2}}, 1\}$. Then:

$$|L_2| \leq C(\Delta)^{\frac{2\delta}{\lambda-1}} \quad (7.46)$$

Lemma 23 follows combining (7.45), (7.46). ■

We can now prove Proposition 18 deriving suitable regularity estimates for the function φ_R .

Proof of Proposition 18. Given the function φ_R defined in (7.2) our goal is to prove (7.4). To this end, given $I(\tau; F)$ in (7.3) defined in $0 \leq \tau \leq T$ we extend it to \mathbb{R} defining $I(\tau; \bar{F}) = I(0; \bar{F})$ for $\tau \leq 0$ and $I(\tau; \bar{F}) = I(T; \bar{F})$ for $\tau \geq T$. We define a function $\tilde{I}_R(\tau)$ for $0 \leq \tau \leq T$ by means of:

$$\tilde{I}_R(\tau; \bar{F}) = (I(\cdot; \bar{F}) * \chi_R)(\tau) \quad (7.47)$$

with $\chi_R(\tau) = R^{\frac{\lambda-1}{2}} \chi_1 \left(R^{\frac{\lambda-1}{2}} \tau \right)$ where the nonnegative function $\chi_1 \in C^\infty(\mathbb{R})$ is compactly supported in $[-1, 1]$ and it satisfies $\int_{\mathbb{R}} \chi_1(\tau) d\tau = 1$. We extend $I(\tau; \bar{F})$ in (7.47) as it can be easily seen, using Lemma 23 with $\delta < r$ with r as in (2.1) that:

$$\left| \tilde{I}_R(\tau; \bar{F}) - I(\tau; \bar{F}) \right| \leq CR^{-\delta}, \quad \tau \in [0, T], \quad \left| \frac{d\tilde{I}_R(\tau; \bar{F})}{d\tau} \right| \leq R^{\frac{\lambda-1}{2}-\delta}, \quad \tau \in [0, T] \quad (7.48)$$

We define $\tilde{\varphi}_R(\tau, x) = \varphi(\tau, x) - \tilde{I}_R(\tau; \bar{F}) x^{-\frac{3+\lambda}{2}} \xi(x)$ where ξ is as in (2.4). Then, using (7.2) we obtain $|\tilde{\varphi}_R(\tau, x)| \leq |\varphi_R(\tau, x)| + \left| \tilde{I}_R(\tau; \bar{F}) - I(\tau; \bar{F}) \right| \xi(x) x^{-\frac{3+\lambda}{2}}$ whence (7.43), (7.48) yields:

$$|\tilde{\varphi}_R(\tau, x)| \leq CR^{-(\frac{3+\lambda}{2}+\delta)} \quad , \quad x \in \left[\frac{R}{2}, R \right] \quad , \quad \tau \in [0, T]$$

where $\tilde{\varphi}_R$ satisfies:

$$\begin{aligned} (\tilde{\varphi}_R)_\tau &= \mathcal{L}_{f_0} [\tilde{\varphi}_R] + \tilde{F}(\tau, x) \\ \tilde{F}(\tau, x) &= \bar{F}(\tau, x) - \frac{d\tilde{I}_R(\tau; \bar{F})}{d\tau} x^{-\frac{3+\lambda}{2}} + \tilde{I}_R(\tau; \bar{F}) \mathcal{L}_{f_0} \left[x^{-\frac{3+\lambda}{2}} \xi(x) \right] \end{aligned} \quad (7.49)$$

where \bar{F} is as in (7.44). Using (7.48) as well as the fact that $\bar{F} \in Y_{\frac{3}{2}, 2+\delta}^\sigma$ and the fact that $\mathcal{L}_{f_0} \left[x^{-\frac{3+\lambda}{2}} \xi(x) \right]$ decreases like $x^{-(2+\delta)}$ as $x \rightarrow \infty$, including derivatives, it follows that $\tilde{F} \in Y_{\frac{3}{2}, 2+\delta}^\sigma$. We now use the rescaling $\tilde{\varphi}_R(\tau, x) = R^{-(\frac{3+\lambda}{2}+\delta)} \Phi_R(\rho, X)$, $x = RX$, $\tau = \frac{\rho}{R^{\frac{\lambda-1}{2}}}$. The function $\Phi_R(\rho, X)$ then satisfies an equation of the form $\frac{\partial \Phi_R}{\partial \rho} = \mathcal{L}_{f_0}^R [\Phi_R] + \tilde{F}_R$ where $\mathcal{L}_{f_0}^R$ is the operator obtained rescaling \mathcal{L}_{f_0} as well as \tilde{F}_R that is obtained rescaling \tilde{F} (cf. (5.82) in the proof of Lemma 5.8 of [7]).

... See if it is worth to copy the formulas in [7]. ...

We can now argue as in the proof of Lemma 5.8 in [7], applying Theorem 9 in Section 4 to prove:

$$\|\tilde{\varphi}_R\|_{\mathcal{Z}_\rho^{\sigma; \frac{1}{2}}(T)} \leq C \|F\|_{Y_{\frac{3}{2}, 2+\delta}^\sigma}$$

This concludes the Proof of Proposition 18. ■

8 FIXED POINT ARGUMENT.

In this Section we prove Theorem 1 by means of a fixed point argument. Our strategy is to define a function \tilde{h} by means of (3.5) with initial data $\tilde{h}(0, x) = 0$. To this end, we define two auxiliary functions \tilde{h}_1 , \tilde{h}_2 that will be due respectively to the source terms $\left(\frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau) Q[f_0] \right)$ and $(-\Lambda_\tau f_0(x))$. The goal of this splitting is to treat in a separate way the term containing Λ_τ since this term will require some careful analysis of the time regularity of the solutions. The function h will be then defined as $\tilde{h}(t, x) = \tilde{h}_1(t, x) + \tilde{h}_2(t, x)$.

8.1 Construction of the function \tilde{h}_1 .

We define \tilde{h}_1 as the solution of the problem:

$$\tilde{h}_{1,\tau} = \mathcal{L}_{f_0} [\tilde{h}_1] + \frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau) Q[f_0] \quad (8.1)$$

$$\tilde{h}_1(0, x) = 0 \quad , \quad x > 0 \quad (8.2)$$

The existence, uniqueness and main regularity properties of the function \tilde{h}_1 are given in the following result:

Proposition 24 *Given $h \in \mathcal{Z}_\rho^{\sigma; \frac{1}{2}}(T)$ with $\bar{\delta} < r$, f_0 as in (2.2)-(2.5), $0 < T \leq 1$ and $\Lambda \in C[0, T]$ satisfying $|\Lambda(\tau) - 1| \leq \frac{1}{4}$ there exist a unique function $\tilde{h}_1 \in \mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}(T)$ solution of (8.1), (8.2) if T is sufficiently small. Moreover, there exist a function $\mathcal{G}[\cdot; h, \Lambda] \in C[0, T]$ such that the function r_1 defined by means of:*

$$\tilde{h}_1(\tau, x) - \mathcal{G}[\tau; h, \Lambda] \xi(x) x^{-\frac{3+\lambda}{2}} = r_1(\tau, x; h, \Lambda) \quad (8.3)$$

satisfies $r_1(\cdot, \cdot; h, \Lambda) \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$. The function $\xi(\cdot)$ is as in (2.4). The mappings

$$\begin{aligned} \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T) \times C[0, T] &\rightarrow C[0, T] \\ (h, \Lambda) &\mapsto \mathcal{G}[\cdot; h, \Lambda] \end{aligned} \quad (8.4)$$

$$\begin{aligned} \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T) \times C[0, T] &\rightarrow \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T) \\ (h, \Lambda) &\mapsto r_1(\cdot, \cdot; h, \Lambda) \end{aligned} \quad (8.5)$$

are Lipschitz continuous assuming that $C[0, T]$ is endowed with the uniform topology and T is sufficiently small. Moreover there exists ρ_0, T_0 such that the Lipschitz constant for the maps can be made arbitrarily small if $T < T_0$ and $\|h\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \leq \rho_0$.

We will prove Proposition 24 with the help of some auxiliary Lemmas. Estimates for $Q[f_0](x_0)$ are obtained in the next result.

Lemma 25 *Let f_0 be as in (2.2)-(2.5). We assume that this function is defined in $[0, T] \times \mathbb{R}^+$ as a function independent on τ . We then have:*

$$\|Q[f_0]\|_{Y_{\frac{3}{2}, (2+r)}^\sigma(T)} \leq C \quad (8.6)$$

$$\|Q[f_0]\|_{Y_{\frac{3}{2}, (2+\delta_1)}^\sigma(T)} \leq C \max \left\{ \sqrt{T}, T^{\frac{2(r-\delta)}{\lambda-1}} \right\}, \quad 0 < \delta_1 \leq r \quad (8.7)$$

$$|Q[f_0](x_0)| \leq \frac{C}{1+x_0^{2+r}}, \quad x_0 > 0 \quad (8.8)$$

Proof of Lemma 25. We use the decomposition (2.6)

$$f_0 = \bar{f}_0(x) + h_0(x), \quad \bar{f}_0(x) = \frac{\xi(x)}{x^{\frac{3+\lambda}{2}}}$$

with $\xi(\cdot)$ as in (2.4). Then:

$$|h_0(x)| + x|h'_0(x)| \leq \frac{CB}{x^{\frac{3+\lambda}{2}+r}}, \quad x \geq 1 \quad (8.9)$$

Then:

$$Q[f_0](x_0) = Q[\bar{f}_0](x_0) + \mathcal{L}_{\bar{f}_0}[h_0](x_0) + Q[h_0](x_0) \quad (8.10)$$

$$Q[\bar{f}_0](x) = - \int_{\frac{x}{2}}^{\infty} (xy)^{\frac{\lambda}{2}} \bar{f}_0(x) \bar{f}_0(y) dy + \int_0^{\frac{x}{2}} y^{\frac{\lambda}{2}} \bar{f}_0(y) \left[(x-y)^{\frac{\lambda}{2}} \bar{f}_0(x-y) - x^{\frac{\lambda}{2}} \bar{f}_0(x) \right] dy$$

Using the fact that $\bar{f}_0(y) = y^{-\frac{3+\lambda}{2}}$ for large y we obtain:

$$\begin{aligned} Q[\bar{f}_0](x) &= - \int_{\frac{x}{2}}^{\infty} \frac{dy}{x^{\frac{3}{2}} y^{\frac{3}{2}}} + \int_0^{\frac{x}{2}} \left[\frac{1}{(x-y)^{\frac{3}{2}}} - \frac{1}{x^{\frac{3}{2}}} \right] \frac{dy}{y^{\frac{3}{2}}} + \\ &\quad + \int_0^2 [\xi(y) - 1] \left[\frac{1}{(x-y)^{\frac{3}{2}}} - \frac{1}{x^{\frac{3}{2}}} \right] \frac{dy}{y^{\frac{3}{2}}} \end{aligned}$$

The first two terms on the right-hand side cancel out. The third one can be estimated, using Taylor's expansion, as:

$$\int_0^2 \frac{1}{y^{\frac{3}{2}}} \left| \frac{1}{(x-y)^{\frac{3}{2}}} - \frac{1}{x^{\frac{3}{2}}} \right| dy \leq \frac{C}{x^{\frac{5}{2}}}, \quad x > 1$$

whence:

$$|Q[\bar{f}_0](x)| \leq \frac{C}{x^{\frac{5}{2}}}, \quad x > 1 \quad (8.11)$$

We can estimate $Q[h_0](x_0)$ as:

$$|Q[h_0](x)| \leq \frac{CB^2}{x^{2+2r}} + \left| \int_0^{\frac{x}{2}} \frac{dq}{dy}(y) S(x, y; h_0) dy \right|$$

where:

$$q(y; h_0) = \int_y^\infty \xi^{\frac{\lambda}{2}} h_0(\xi) d\xi, \quad S(x, y; h_0) = \left[(x-y)^{\frac{\lambda}{2}} h_0(x-y) - x^{\frac{\lambda}{2}} h_0(x) \right]$$

Integrating by parts we obtain:

$$\left| \int_0^{\frac{x}{2}} \frac{dq}{dy}(y; h_0) S(x, y; h_0) dy \right| \leq \frac{CB^2}{x^{2+2r}} + CB^2 \left| \int_0^{\frac{x}{2}} \frac{1}{y^{\frac{1}{2}+r}} \frac{1}{(x-y)^{\frac{5}{2}+r}} dy \right| \leq \frac{CB^2}{x^{2+2r}}$$

Then:

$$|Q[h_0](x)| \leq \frac{CB^2}{x^{2+2r}}, \quad x > 1 \quad (8.12)$$

Finally we write $\mathcal{L}_{f_0}[h_0]$ as:

$$\begin{aligned} \mathcal{L}_{f_0}[h_0] = & - \int_0^{\frac{x}{2}} S(x, y; f_0) q'(y; h_0) dy - x^{\lambda/2} f_0(x) q\left(\frac{x}{2}; h_0\right) + \\ & - \int_0^{\frac{x}{2}} S(x, y; h_0) q'(y; f_0) dy - x^{\lambda/2} h_0(x) q\left(\frac{x}{2}; f_0\right) \end{aligned}$$

An immediate computation shows that the second and fourth terms on the right can be estimated as $CBx^{-(2+r)}$. The other two terms can be estimated integrating by parts. Then:

$$|\mathcal{L}_{\bar{f}_0}[h_0](x)| \leq \frac{CB}{x^{2+r}}, \quad x > 1 \quad (8.13)$$

In the region $x \leq 1$ we have trivially boundedness of $Q[f_0](x_0)$. Combining this with (8.10), (8.11), (8.12), (8.13) we obtain (8.8). Estimate (8.6) follows similarly using the differentiability properties assumed for f_0 .

It only remains to prove (8.7). To this end, notice that due to the definition of $\|\cdot\|_{Y_{\frac{3}{2}, (2+\delta)}^\sigma(T)}$ we need to estimate L^2 norms in time for $R \leq 1$. Since we have estimates in L^∞ for $Q[f_0]$ we then obtain a dependence on T like \sqrt{T} . Similar estimates can be obtained, using also the definition of the norms $\|\cdot\|_{Y_{\frac{3}{2}, (2+r)}^\sigma(T)}$ for all the values of $R \leq T^{-\frac{2}{\lambda-1}}$, since for these values there is not splitting of the domain of integration in the t variable. In the region where $R > T^{-\frac{2}{\lambda-1}}$ we use the fact that $Q[f_0]$ is pointwise estimated by $R^{-(2+r)}$. Therefore:

$$R^{(2+\delta)} (N_\infty(Q[f_0]; t_0, R) + N_{2;\sigma}(Q[f_0]; t_0, R)) \leq CR^{\delta-r} \leq CT^{\frac{2(r-\delta)}{\lambda-1}}$$

for $R > T^{-\frac{2}{\lambda-1}}$. Therefore (8.7) follows. ■

As a next step we estimate the quadratic terms in (8.1).

Lemma 26 *Given $h \in \mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)$ and Λ as in Proposition 24. Then:*

$$\left\| \frac{Q[h]}{\Lambda(\cdot)} \right\|_{Y_{\frac{3+\lambda}{2}, (2+\delta)}^\sigma} \leq C \|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)}^2 \quad (8.14)$$

Moreover:

$$\begin{aligned} & \left\| \frac{Q[h_1]}{\Lambda_1(\cdot)} - \frac{Q[h_2]}{\Lambda_2(\cdot)} \right\|_{Y_{\frac{3+\lambda}{2}, (2+\delta)}^\sigma} \\ & \leq C \left(\sum_{k=1}^2 \|h_k\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \right) \left(\|h_1 - h_2\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} + \|\Lambda_1 - \Lambda_2\|_{C[0, T]} \right) \end{aligned} \quad (8.15)$$

Proof of Lemma 26. Estimates (8.14), (8.15) are just a consequence of Propositions 12, 17 as well as the fact that $\frac{1}{2} \leq \Lambda(\cdot) \leq \frac{3}{2}$. ■

Proof of Proposition 24. Existence and uniqueness of the function \tilde{h}_1 follow from the results in [7] (cf. 7) combined with Lemmas 25, 26.

On the other hand, the decay and regularity properties of the the function r_1 defined in (8.3) are a consequence of Proposition 18. In order to apply this Proposition some regularity and decay for the source terms $\frac{Q[h]}{\Lambda(\tau)}$ and $\Lambda(\tau) Q[f_0]$ are needed. In the case of $\frac{Q[h]}{\Lambda(\tau)}$ such properties are a consequence of Proposition 12. The corresponding properties for $\Lambda(\tau) Q[f_0]$ follow from Lemma 25 and the fact that $r > \bar{\delta}$. The function $\mathcal{G}[\tau; h, \Lambda]$ is given by the function $\mathcal{W}(\tau)$ in (7.3) with source F given by $\frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau) Q[f_0]$. Notice that the linearity of the equation satisfied by \tilde{h}_1 as well as the Lipschitz property for $Q[h]$ in Proposition 17 implies that the map $h \rightarrow \tilde{h}_1$ is Lipschitz in h in the space $\mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}(T)$. Moreover, due to Lemma 19, the map $h \rightarrow (\mathcal{L}_{f_0} - L)(\tilde{h}_1)$ from $\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)$ to $Y_{\frac{3}{2}, \frac{3+\lambda}{2} + \bar{\delta}}^\sigma$ is Lipschitz.. Therefore the map in (8.4) has the Lipschitz dependence stated in Proposition 24. The Lipschitz property for the map in (8.5) is again a consequence of Proposition 17, the linearity of the problem under consideration and Proposition 18.

It only remains to check that the Lipschitz constant of the maps (8.4), (8.5) can be made small if $T \leq T_0$ and T_0 is small enough. Indeed, given two couples $(h^{(1)}, \Lambda^{(1)})$, $(h^{(2)}, \Lambda^{(2)})$ satisfying the hypothesis of the Proposition, let us denote as $F^{(1)}$, $F^{(2)}$, $\tilde{h}_1^{(1)}$, $\tilde{h}_1^{(2)}$ and $\mathcal{W}^{(1)}$, $\mathcal{W}^{(2)}$ the corresponding functions F , \tilde{h}_1 , \mathcal{W} respectively. The stated Lipschitz properties yield:

$$\begin{aligned} & \left\| F^{(1)} - F^{(2)} \right\|_{Y_{\frac{3}{2}, (2+\delta)}^\sigma} + \left\| (\mathcal{L}_{f_0} - L)(\tilde{h}_1^{(1)} - \tilde{h}_1^{(2)}) \right\|_{Y_{\frac{3}{2}, (2+\delta)}^\sigma} \\ & \leq C \left(\left\| \Lambda^{(1)} - \Lambda^{(2)} \right\|_{C[0, T]} + \left\| h^{(1)} - h^{(2)} \right\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \right) \end{aligned} \quad (8.16)$$

Using the inequality (8.16) combined with (7.3) and (7.21) we then obtain that the difference $\mathcal{W}^{(1)} - \mathcal{W}^{(2)}$ can be estimated as:

$$\left| \mathcal{W}^{(1)}(\tau) - \mathcal{W}^{(2)}(\tau) \right| \leq CT^{\frac{2\bar{\delta}}{\lambda-1}} \left(\left\| \Lambda^{(1)} - \Lambda^{(2)} \right\|_{C[0, T]} + \left\| h^{(1)} - h^{(2)} \right\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \right)$$

for $0 \leq \tau \leq T$.

The fact that the Lipschitz constant for the map (8.5) can be made small for small T is just a consequence of (8.7) in Lemma 25 and (8.15) in Lemma 26 if ρ_0 in the statement of Proposition 24 is sufficiently small. ■

8.2 Construction of the function \tilde{h}_2 .

It would be natural to construct \tilde{h}_2 as the solution of the problem:

$$\tilde{h}_{2, \tau} = \mathcal{L}_{f_0}[\tilde{h}_2] - \Lambda_\tau f_0(x) \quad , \quad \tilde{h}_2(0, x) = 0 \quad , \quad x > 0 \quad (8.17)$$

However, since it is more convenient from the technical point of view to avoid using the derivative Λ_τ we will use an alternative procedure that we describe shortly here. More precisely, we will obtain a solution of the initial value problem:

$$\psi_\tau = \mathcal{L}_{f_0} [\psi] \quad , \quad \psi(0, x) = f_0(x) \quad , \quad x > 0 \quad (8.18)$$

In order to solve this problem we define the change of variables:

$$\psi(\tau, x) = f_0(x) + \zeta(\tau, x) \quad (8.19)$$

The function ζ then solves:

$$\zeta_\tau = \mathcal{L}_{f_0} [\zeta] + \mathcal{L}_{f_0} [f_0] \quad , \quad \zeta(0, x) = 0 \quad , \quad x > 0 \quad (8.20)$$

This equation can be solved, assuming (2.2)-(2.5) using Theorem 7.

Variation of constants formula then suggests that \tilde{h}_2 , solution of (8.17) is given by:

$$\tilde{h}_2(\tau, x) = - \int_0^\tau \psi(\tau - s, x) \Lambda_\tau(s) ds \quad (8.21)$$

and assuming that ψ is differentiable in time we would obtain:

$$\tilde{h}_2(\tau, x) = -f_0(x) \Lambda(\tau) + \psi(\tau, x) - \int_0^\tau \frac{\partial \psi}{\partial \tau}(\tau - s, x) \Lambda(s) ds \quad (8.22)$$

This representation formula avoids using Λ_τ . However, it requires to prove that $\frac{\partial \psi}{\partial \tau}$ is well defined. We now prove the properties of ψ required to give a precise meaning to (8.22).

Proposition 27 *Suppose that f_0 satisfies (2.2)-(2.5). There exist a function $\psi \in \mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}(T)$ defined by means of (8.18). We have:*

$$\psi(\tau, x) = a(\tau) \xi(x) x^{-\frac{3+\lambda}{2}} + r_2(\tau, x) \quad (8.23)$$

where $\xi(\cdot)$ is the cutoff in (2.4) and where:

$$a(0) = 1 \quad , \quad \|r_2\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \leq C \quad (8.24)$$

Moreover:

$$\left| \frac{da}{d\tau} \right| \leq C \quad , \quad \left\| \frac{\partial r_2}{\partial \tau} \right\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \leq C \quad (8.25)$$

The proof of this result is based on the following Lemma:

Lemma 28 *Suppose that f_0 satisfies (2.2)-(2.5). There exists a constant C such that, for any $0 < T \leq 1$:*

$$\|\mathcal{L}_{f_0} [f_0]\|_{Y_{\frac{3}{2}, 2+\delta}^\sigma(T)} + \|\mathcal{L}_{f_0} [\mathcal{L}_{f_0} [f_0]]\|_{Y_{\frac{3}{2}, 2+\delta}^\sigma(T)} \leq C \quad (8.26)$$

where $\delta > 0$ is as in (2.2)-(2.5).

Proof of Lemma 28. Using (2.2)-(2.5) we obtain the asymptotics:

$$\mathcal{L}_{f_0} [f_0] = K x^{-\frac{3+\lambda}{2}} \xi(x) + w_{0,R}(x) \quad (8.27)$$

with a remainder $w_{0,R}(x)$ that can be estimated, together with its derivatives as $x^{-(\frac{3+\lambda}{2}+\delta)}$ as $x \rightarrow \infty$. The main idea to keep in mind is that the operator \mathcal{L}_{f_0} acting on power laws x^{-p} amounts to multiply then by $C_p x^r$. The constant C_p vanishes if $p = \frac{3+\lambda}{2}$. The estimate (8.26) will be then

proved multiplying by the cutoff $\eta(\tau)$ and taking the operator \mathcal{L}_{f_0} . Since the leading power law in (8.27) is $x^{-\frac{3+\lambda}{2}}$ the action of the operator \mathcal{L}_{f_0} will cancel the first order and only a remainder behaving like $x^{-(2+\delta)}$ will be left, with δ as in (2.2)-(2.5).

We now describe the details. The operator \mathcal{L}_{f_0} is defined in (3.3). We then have, rearranging the integral terms:

$$\frac{1}{2}\mathcal{L}_{f_0}[f_0](x) = Q[f_0](x) \quad (8.28)$$

Using (2.2), (2.3):

$$\mathcal{L}_{f_0}[f_0](x) = \mathcal{L}_{f_{1;2}}[f_{1;2}](x) + 2\mathcal{L}_{f_{1;2}}[f_3] + \mathcal{L}_{f_3}[f_3](x) \quad (8.29)$$

We can estimate $\mathcal{L}_{f_{1;2}}[f_3]$, as well as its derivative using (2.2), (2.3), (2.5) as well as the fact that $1 < \lambda < 2$ and $0 < \delta < r$:

$$|\mathcal{L}_{f_{1;2}}[f_3]| + (1+x) \left| \frac{d}{dx}(\mathcal{L}_{f_{1;2}}[f_3]) \right| + (1+x)^2 \left| \frac{d^2}{dx^2}(\mathcal{L}_{f_{1;2}}[f_3]) \right| \leq \frac{C}{(1+x)^{2+r+\delta}}, \quad x > 0 \quad (8.30)$$

The term $\mathcal{L}_{f_3}[f_3](x)$ can be estimated similarly:

$$\left| \frac{1}{2}\mathcal{L}_{f_3}[f_3] \right| + (1+x) \left| \frac{1}{2} \frac{d}{dx}(\mathcal{L}_{f_3}[f_3]) \right| + (1+x)^2 \left| \frac{d^2}{dx^2}(\mathcal{L}_{f_3}[f_3]) \right| \leq \frac{C}{(1+x)^{\frac{3+\lambda}{2}+r}} \leq \frac{C}{(1+x)^{2+r+\delta}} \quad (8.31)$$

for $x > 0$. Therefore, it only remains to estimate $\mathcal{L}_{f_{1;2}}[f_{1;2}]$ in (8.29).

We then only need to approximate the term $\mathcal{L}_{f_{1;2}}[f_{1;2}]$ that might be rewritten as:

$$\mathcal{L}_{f_{1;2}}[f_{1;2}](x) = \mathcal{L}_{f_1}[f_1](x) + 2\mathcal{L}_{f_1}[f_2](x) + \mathcal{L}_{f_2}[f_2](x) \quad (8.32)$$

We then have, using $r > \delta$:

$$|\mathcal{L}_{f_2}[f_2]| + (1+x) \left| \frac{d}{dx}(\mathcal{L}_{f_2}[f_2]) \right| + (1+x)^2 \left| \frac{d^2}{dx^2}(\mathcal{L}_{f_2}[f_2]) \right| \leq \frac{C}{(1+x)^{2+r+\delta}}, \quad x > 0 \quad (8.33)$$

We need to obtain precise asymptotics of the terms $\mathcal{L}_{f_1}[f_1](x)$, $2\mathcal{L}_{f_1}[f_2](x)$ in order to obtain the leading order term in (8.27). Let us write $\bar{f}_1(x) = x^{-\frac{3+\lambda}{2}}$. Notice that:

$$\mathcal{L}_{\bar{f}_1}[\bar{f}_1] = 0 \quad (8.34)$$

Moreover, we have the following identity:

$$x^{\lambda/2} f_1(x) \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} f_1(y) dy = x^{\lambda/2} \bar{f}_1(x) \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} \bar{f}_1(y) dy, \quad x \geq 1$$

Using (8.34) we then obtain for $x > 2$:

$$\mathcal{L}_{f_1}[f_1] = \int_0^2 y^{\lambda/2} \bar{f}_1(y) [\xi(y) - 1] \left[(x-y)^{\lambda/2} \bar{f}_1(x-y) - x^{\lambda/2} \bar{f}_1(x) \right] dy$$

Taylor's expansion, as well as the fact that $\delta < \frac{\lambda-2}{2}$ implies:

$$|\mathcal{L}_{f_1}[f_1]| + (1+x) \left| \frac{d}{dx}(\mathcal{L}_{f_1}[f_1]) \right| + (1+x)^2 \left| \frac{d^2}{dx^2}(\mathcal{L}_{f_1}[f_1]) \right| \leq \frac{C}{(1+x)^{\frac{3+\lambda}{2}+\delta}}, \quad x > 0 \quad (8.35)$$

where we use the fact that $\mathcal{L}_{f_1}[f_1]$ and its derivatives are trivially bounded for x bounded as it might be seen using directly using the definition of $\mathcal{L}_{f_1}[f_1]$.

It remains to estimate the term $2\mathcal{L}_{f_1}[f_2](x)$ in (8.32). Using the definition of $\mathcal{L}_{f_1}[f_2](x)$ we obtain

$$2\mathcal{L}_{f_1}[f_2](x) = \mathcal{H}_1(f_1, f_2)(x) + \mathcal{H}_2(f_1, f_2)(x) \quad (8.36)$$

where:

$$\begin{aligned} \mathcal{H}_1(f_1, f_2)(x) &= \int_0^{\frac{x}{2}} y^{\lambda/2} f_1(y) \left[(x-y)^{\lambda/2} f_2(x-y) - x^{\lambda/2} f_2(x) \right] dy + \\ &\quad + \int_0^{\frac{x}{2}} y^{\lambda/2} f_2(y) \left[(x-y)^{\lambda/2} f_1(x-y) - x^{\lambda/2} f_1(x) \right] dy \\ \mathcal{H}_2(f_1, f_2)(x) &= x^{\lambda/2} f_1(x) \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} f_2(y) dy + x^{\lambda/2} f_2(x) \int_{\frac{x}{2}}^{\infty} y^{\lambda/2} f_1(y) dy \end{aligned}$$

The term $\mathcal{H}_2(f_1, f_2)(x)$ can be explicitly computed for large values of x :

$$\mathcal{H}_2(f_1, f_2)(x) = K_1 x^{-\frac{3+\lambda}{2}}, \quad K_1 \in \mathbb{R} \text{ and } x > 1 \quad (8.37)$$

In order to approximate $\mathcal{H}_1(f_1, f_2)(x)$ we define a function $\bar{f}_2(x) = \frac{a_1}{x^{\frac{3+\lambda}{2}+r}}$. We then have:

$$\mathcal{H}_1(\bar{f}_1, \bar{f}_2)(x) = K_2 x^{-\frac{3+\lambda}{2}}, \quad K_2 \in \mathbb{R} \text{ and } x > 0 \quad (8.38)$$

On the other hand:

$$\begin{aligned} &\mathcal{H}_1(f_1, f_2)(x) - \mathcal{H}_1(\bar{f}_1, \bar{f}_2)(x) \\ &= \int_0^2 y^{\lambda/2} \bar{f}_1(y) [\xi(y) - 1] \left[(x-y)^{\lambda/2} \bar{f}_2(x-y) - x^{\lambda/2} \bar{f}_2(x) \right] dy + \\ &\quad + \int_0^2 y^{\lambda/2} \bar{f}_2(y) [\xi(y) - 1] \left[(x-y)^{\lambda/2} \bar{f}_1(x-y) - x^{\lambda/2} \bar{f}_1(x) \right] dy \end{aligned}$$

Taylor's expansion, as well as the fact that $\delta < \frac{2-\lambda}{2}$, yields:

$$\begin{aligned} &|\mathcal{H}_1(f_1, f_2)(x) - \mathcal{H}_1(\bar{f}_1, \bar{f}_2)(x)| + (1+x) \left| \frac{d}{dx} (\mathcal{H}_1(f_1, f_2)(x) - \mathcal{H}_1(\bar{f}_1, \bar{f}_2)(x)) \right| \\ &\leq \frac{C}{x^{\frac{3}{2}}} \leq \frac{C}{x^{\frac{3+\lambda}{2}+\delta}}, \quad x \geq 1 \end{aligned} \quad (8.39)$$

The boundedness of $\mathcal{L}_{f_1}[f_2](x)$ and its derivatives combined with (8.36)-(8.39) yields:

$$\left| 2\mathcal{L}_{f_1}[f_2](x) - Kx^{-\frac{3+\lambda}{2}} \right| \leq \frac{C}{1+x^{\frac{3+\lambda}{2}+\delta}}, \quad x > 0 \quad (8.40)$$

Combining (8.29)-(8.35), (8.40) we obtain (8.27) where:

$$\sum_{k=0}^3 (1+x)^k \left| \frac{d^k w_{0,R}}{dx^k} \right| \leq \frac{C}{1+x^{\frac{3+\lambda}{2}+\delta}}, \quad x > 0 \quad (8.41)$$

Applying $\mathcal{L}_{f_0}[\cdot]$ on both sides of (8.27), using $\mathcal{L}_{f_0}\left[x^{-\frac{3+\lambda}{2}}\right] = 0$, and arguing as in the proof of (8.35) we obtain $\sum_{k=0}^2 (1+x)^k \left| \frac{d^k}{dx^k} \left(\mathcal{L}_{f_0} \left[Kx^{-\frac{3+\lambda}{2}} \xi(x) \right] \right) \right| \leq \frac{C}{1+x^{2+\delta}}$. On the other hand, the action of the operator \mathcal{L}_{f_0} over functions satisfying (8.41) amounts to multiplying by x^r for large values of x . Therefore $\sum_{k=0}^2 (1+x)^k \left| \frac{d^k}{dx^k} (\mathcal{L}_{f_0}[w_{0,R}]) \right| \leq \frac{C}{1+x^{2+\delta}}$ and the result follows. ■

Proof of Proposition 27. Due to (8.26) in Lemma 28 $\mathcal{L}_{f_0}[f_0]$ is bounded in the space $Y_{3/2, 2+\delta}^\sigma(T)$. Therefore (8.20) can be solved using the results in [7]. We obtain in this way a solution $\zeta \in$

$\mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}(T)$. Then, ψ can be obtained by means of (8.19). Therefore expansion (8.23), (8.24) are just a consequence of Proposition 18.

It only remains to obtain estimates for the derivatives on time of the functions a , r_2 . Formal differentiation of (8.20) suggests that $w = \frac{\partial \mathcal{L}}{\partial \tau}$ satisfies the following initial value problem:

$$(w)_\tau = \mathcal{L}_{f_0}[w] \quad , \quad w(0, x) = \mathcal{L}_{f_0}[f_0] \quad (8.42)$$

Actually we can use the results in [7] to construct a solution of (8.42) as follows. We define a function $W(\tau, x)$ by means of:

$$w(\tau, x) = \mathcal{L}_{f_0}[f_0] + W(\tau, x) \quad (8.43)$$

Then, w solves (8.42) iff W solves:

$$W_\tau = \mathcal{L}_{f_0}[W] + \mathcal{L}_{f_0}[\mathcal{L}_{f_0}[f_0]] \quad , \quad W(0, x) = 0 \quad (8.44)$$

In order to be able to solve the problem (8.42) we use the hypothesis (2.2)-(2.5). Due to Lemma 28 we have that $\|\mathcal{L}_{f_0}[\mathcal{L}_{f_0}[f_0]]\|_{Y_{\frac{\sigma}{2}, 2+\delta}^\sigma(T)}$ bounded. Therefore, we can apply the results in [7] (cf. Theorem 7) to obtain a unique solution W of (8.44) satisfying $\|W\|_{\mathcal{Z}_{\frac{3+\lambda}{2}}^{\sigma; \frac{1}{2}}(T)} \leq C$. The function ψ , solution of (8.18) can be obtained, using also (8.19) as:

$$\psi(\tau, x) = f_0(x) + \mathcal{L}_{f_0}[f_0]\tau + \int_0^\tau W(s, x) ds \quad (8.45)$$

Using (7.2), (7.4) in Proposition 18 we obtain:

$$W(\tau, x) = \mathcal{W}(\tau) x^{-\frac{3+\lambda}{2}} \xi(x) + W_R(\tau, x) \quad (8.46)$$

with

$$|\mathcal{W}(\tau)| \leq C \quad , \quad 0 \leq \tau \leq T \quad , \quad \|W_R(\tau, x)\|_{\mathcal{Z}_{\frac{\sigma}{p}, \frac{1}{2}}^\sigma(T)} \leq C \quad (8.47)$$

Then $\psi(\tau, x) = a(\tau) x^{-\frac{3+\lambda}{2}} \xi(x) + r_2(\tau, x)$ where:

$$a(\tau) = 1 + K\tau + \int_0^\tau \mathcal{W}(s) ds \quad , \quad r_2(\tau, x) = \left[f_0(x) - x^{-\frac{3+\lambda}{2}} \xi(x) \right] + w_{0,R}(x)\tau + \int_0^\tau W_R(s, x) ds$$

Using (8.47) we obtain (8.25) and the Proposition follows. ■

Remark 29 As indicated in Section 2 the assumption (2.2)-(2.5) is very strong. However, the argument proving Proposition 27 shows that the main reason for assuming (2.2)-(2.5) is to show that $\left| \frac{da}{d\tau} \right|$, $\frac{\partial r_2}{\partial \tau}$ are bounded in a suitable sense. It would be possible to weaken (2.2), (2.3) to some assumption with the form $f_0(x) = D_1 x^{-\frac{3+\lambda}{2}} + O\left(x^{-\frac{3+\lambda}{2}-\delta}\right)$ as $x \rightarrow \infty$ for some $\delta > 0$. Making such an assumption the only difference in the argument proving Proposition 27 would be that the term $\mathcal{L}_{f_0}[f_0]$ in (8.42) would behave like $O\left(x^{-(2+\delta)}\right)$ instead of $O\left(x^{-\frac{3+\lambda}{2}}\right)$ as $x \rightarrow \infty$. Unfortunately the well-posedness theory developed in [7] cannot cover such weakest rate of decay at infinity. The expected asymptotics for $w(\tau, x)$ as $x \rightarrow \infty$ for small τ would have the form $w(\tau, x) \sim \frac{C}{\tau^{1-\frac{2\delta}{\lambda-1}}} x^{-\frac{3+\lambda}{2}}$ as $x \rightarrow \infty$, $\tau \rightarrow 0$. This type of asymptotics has been obtained in [4], [5] for an different equation, namely the Uehling-Uhlenbeck equation. Unfortunately since the well-posedness theory of classical solutions for the coagulation equation is more difficult, we have preferred not to consider such a case, at the price of assuming stronger regularity assumptions near the singular point. Nevertheless it would be an interesting question to prove analogous regularizing results in time.

With the previous construction we can define the function \tilde{h}_2 as follows.

Definition 30 For any f_0, Λ satisfying the assumptions in Proposition 24 we define \tilde{h}_2 by means of

$$\tilde{h}_2(\tau, x) = -f_0(x) \Lambda(\tau) + \psi(\tau, x) - \int_0^\tau w(\tau - s, x) \Lambda(s) ds \quad (8.48)$$

where ψ is as in (8.45) and w is as in (8.43).

Remark 31 The rationale behind Definition 30 is the following. Assuming smoothness we obtain, differentiating (8.48):

$$\left(\tilde{h}_2\right)_\tau = -f_0(x) \Lambda_\tau(\tau) + \frac{\partial \psi}{\partial \tau}(\tau, x) - \Lambda(\tau) \mathcal{L}_{f_0}[f_0] - \int_0^\tau \mathcal{L}_{f_0}[w](\tau - s, x) \Lambda(s) ds$$

where we have used (8.42). Exchanging the order of the integral in time and \mathcal{L}_{f_0} and using again (8.48) we obtain, after some cancellations $\tilde{h}_{2,\tau} = -f_0 \Lambda_\tau(\tau) + \frac{\partial \psi}{\partial \tau} + \mathcal{L}_{f_0}(\tilde{h}_2) - \mathcal{L}_{f_0}[\psi]$. Using then (8.18) we obtain that \tilde{h}_2 would solve (8.17).

The asymptotics of the function \tilde{h}_2 as $x \rightarrow \infty$ can be derived using the corresponding results for the functions ψ, w in Proposition 27.

Lemma 32 For any $\Lambda \in C[0, T]$ satisfying the assumptions in Proposition 24 we have:

$$\tilde{h}_2(\tau, x) = \mathcal{K}[\Lambda](\tau) \xi(x) x^{-\frac{3+\lambda}{2}} + \tilde{h}_{2,R}(\tau, x; \Lambda)$$

where

$$\mathcal{K}[\Lambda](\tau) = -\Lambda(\tau) + a(\tau) - \int_0^\tau \mathcal{W}(\tau - s) \Lambda(s) ds$$

with a as in Proposition 27, $\frac{da}{d\tau} = \mathcal{W}$ and $\tilde{h}_{2,R} \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$. Moreover, the map:

$$C[0, T] \rightarrow \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T) \quad : \quad \Lambda \rightarrow \tilde{h}_2 \quad (8.49)$$

is Lipschitz if $T \leq T_0$, with T_0 sufficiently small.

Proof. It is just a consequence of the definition of \tilde{h}_2 in (8.48) and Proposition 27. ■

8.3 Setting of the fixed point argument. Solution of an integral equation.

Given $h \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$ and $\Lambda \in C[0, T]$ as in Proposition 24 we can define a map $(h, \Lambda) \rightarrow \tilde{h}$ with $\tilde{h}(t, x) = \tilde{h}_1(t, x) + \tilde{h}_2(t, x)$ where \tilde{h}_1 is as in Proposition 24 and \tilde{h}_2 as in (8.48).

We now select, for any given h , the function Λ in order to have:

$$\lim_{x \rightarrow \infty} \left(x^{\frac{3+\lambda}{2}} \tilde{h}(t, x) \right) = 0 \quad (8.50)$$

Due to (8.3) and since $r_1(\cdot, \cdot; h) \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$, as well as Lemma 32 it follows that (8.50) holds if Λ solves:

$$\Lambda(\tau) = \mathcal{G}[\tau; h, \Lambda] + a(\tau) - \int_0^\tau \mathcal{W}(\tau - s, x) \Lambda(s) ds \quad (8.51)$$

We first show that for $h \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$ we can find $\Lambda = \Lambda(\cdot; h)$ such that (8.51) is satisfied.

Lemma 33 *There exist $T_0 > 0$ sufficiently small, such that, for any $h \in \mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)$ satisfying $\|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \leq \rho_0$ equation (8.51) has a unique solution for $0 \leq \tau \leq T$, assuming that $T \leq T_0$. Moreover, this solution defines a mapping:*

$$\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T) \rightarrow C[0, T] : h \longmapsto \Lambda(\cdot; h) \quad (8.52)$$

that is contractive.

Proof. The function \mathcal{W} in the integral term (8.51) is uniformly bounded due to Proposition 27. On the other hand, the function $\mathcal{G}[\tau; h, \Lambda]$ is Lipschitz contractive in Λ if T_0 is sufficiently small and $\|h\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \leq \rho_0$ due to Proposition 24. It then follows from (8.51) that the mapping (8.52) is contractive. ■

Lemma 34 *Let us denote as B_{ρ_0} the ball of radius ρ_0 in $\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)$, with ρ_0 as in Proposition 24 and let us consider the mapping from B_{ρ_0} to B_{ρ_0} given by $h \rightarrow \mathcal{T}[h]$ where $\mathcal{T}[h] = \tilde{h}_1 + \tilde{h}_2$, with \tilde{h}_1 as in Proposition 24 and \tilde{h}_2 as in (8.48) and with $\Lambda = \Lambda(\cdot; h)$ in (8.1) where $\Lambda(\cdot; h)$ is chosen as in Lemma 33. Then, there exists T_0 such that the mapping \mathcal{T} is contractive in B_{ρ_0} if $T \leq T_0$. In such a case there exists a unique fixed point of \mathcal{T} in B_{ρ_0} .*

Proof. The definition of \tilde{h} , \tilde{h}_1 , \tilde{h}_2 combined with (8.51) imply:

$$\mathcal{T}[h](\tau, x) = \tilde{h}(\tau, x) = r_1(\tau, x; h, \Lambda(\cdot; h)) + \tilde{h}_{2,R}(\tau, x; \Lambda(\cdot; h)) \quad (8.53)$$

Notice that \mathcal{T} transforms B_{ρ_0} into B_{ρ_0} for $T \leq T_0$ small. Indeed, r_1 consists of two pieces that are due to the contributions of the source terms $\frac{Q[h]}{\Lambda(\tau)}$ and $\Lambda(\tau)Q[f_0]$ in (8.1) respectively. The norm $\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)$ of the solution due to the source term $\frac{Q[h]}{\Lambda(\tau)}$ can be bounded as $C\rho_0^2$ due to Proposition 12. On the other hand in order to estimate the contribution due to the term $\Lambda(\tau)Q[f_0]$ we use (8.7) in Lemma 25. Using then Proposition 18 it follows that the contribution due to the source $\Lambda(\tau)Q[f_0]$ is smaller than $\frac{\rho_0}{8}$ if T_0 is small enough.

On the other hand, in order to see that the contribution of the term $\tilde{h}_{2,R}$ is small for small times, we use the formulas for ψ , \tilde{h}_2 . Using (8.45) and (8.48) we obtain:

$$\tilde{h}_2(\tau, x) = f_0(x)[1 - \Lambda(\tau)] + \mathcal{L}_{f_0}[f_0]\tau + \int_0^\tau W(s, x) ds - \int_0^\tau w(\tau - s, x)\Lambda(s) ds$$

We subtracts the terms behaving like $\xi(x)x^{-\frac{3+\lambda}{2}}$ in all the pieces. We then obtain:

$$\begin{aligned} \tilde{h}_{2,R}(\tau, x) &= \left[f_0(x) - \xi(x)x^{-\frac{3+\lambda}{2}} \right] [1 - \Lambda(\tau)] + \\ &+ w_{0,R}(x)\tau + \int_0^\tau W_R(s, x) ds - \int_0^\tau [w_{0,R}(x) + W_R(\tau - s, x)]\Lambda(s) ds \\ &\equiv \tilde{h}_{2,R,1}(\tau, x) + \tilde{h}_{2,R,2}(\tau, x) + \tilde{h}_{2,R,3}(\tau, x) + \tilde{h}_{2,R,4}(\tau, x) \end{aligned}$$

(cf. (8.27), (8.43), (8.46)). Using (8.27) we obtain $\left\| \tilde{h}_{2,R,2} \right\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \leq CT$. We can estimate $\tilde{h}_{2,R,3}$ and $\tilde{h}_{2,R,4}$ in the space $\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)$ using the fact that these functions are integrals on time of functions bounded in $\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)$. Using Lemma 10 we obtain $\left\| \tilde{h}_{2,R,3} \right\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} + \left\| \tilde{h}_{2,R,4} \right\|_{\mathcal{Z}_{\bar{p}}^{\sigma; \frac{1}{2}}(T)} \leq C\sqrt{T}$. It only remains to control the term $\tilde{h}_{2,R,1}$. To this end, we use here the integral equation (8.51) that yields:

$$[1 - \Lambda(\tau)] = -\mathcal{G}[\tau; h, \Lambda] + (1 - a(\tau)) + \int_0^\tau \mathcal{W}(\tau - s, x)\Lambda(s) ds \quad (8.54)$$

Due to Proposition 24 and Lemmas 25, 26 we can estimate the contributions to $\mathcal{G}[\tau; h, \Lambda]$ that are due to $\Lambda(\tau)$, $Q[f_0]$ and $\frac{Q[h]}{\Lambda(\tau)}$ respectively as $C \max\left\{\sqrt{T}, T^{\frac{2(r-\delta)}{\lambda-1}}\right\}$ and $C\rho_0^2$. Therefore this contribution can be estimated by $\frac{\rho_0}{8}$. The second term on the right-hand side of (8.54) can be estimated using the differentiability of a (cf. (8.25)). Therefore this term can be estimated as CT . On the other hand, the boundedness of \mathcal{W} (cf. (8.47)) provides a similar estimate for the last term in (8.54). Therefore, using the regularity of f_0 we obtain $\left\|\tilde{h}_{2,R,1}\right\|_{\mathcal{Z}_p^{\sigma;\frac{1}{2}}(T)} \leq \frac{\rho_0}{4}$. It then follows that \mathcal{T} transforms B_{ρ_0} into B_{ρ_0} if T_0 is sufficiently small. Combining the contractivity of the map (8.52) with the Lipschitz properties of the maps (8.5), (8.49) we obtain the contractivity of \mathcal{T} if $T \leq T_0$ sufficiently small. ■

Proof of Theorem 1. We define \tilde{f} by means of:

$$\tilde{f}(\tau, x) = \Lambda(\tau) f_0(x) + h(\tau, x) \quad (8.55)$$

where $h(\tau, x)$ is the fixed point associated to the operator \mathcal{T} obtained in Lemma 34. Notice that:

$$h(\tau, x) = \mathcal{T}(h)(\tau, x) = r_1(\tau, x; h, \Lambda) + \tilde{h}_{2,R}(\tau, x) = \tilde{h}_1(\tau, x) + \tilde{h}_2(\tau, x)$$

where \tilde{h}_1, \tilde{h}_2 are as in Proposition 24 and (8.48) respectively. Using (8.55) and (8.48) we obtain:

$$\tilde{f}(\tau, x) = \tilde{h}_1(\tau, x) + \psi(\tau, x) - \int_0^\tau w(\tau - s, x) \Lambda(s) ds \quad (8.56)$$

where ψ, w are as in (8.45), (8.43). The function \tilde{f} is differentiable with respect to τ due to Proposition 24, Proposition 27 and the continuity of Λ , and differentiability of w (cf. Lemma 33 and (8.43), (8.44) respectively). Therefore \tilde{f} solves:

$$\tilde{f}_\tau = \frac{Q[\tilde{f}]}{\Lambda(\tau)} \quad (8.57)$$

as it can be checked as follows. Differentiating (8.56), using the fact that w solves (8.42) and exchanging the integration in time with the operator $\mathcal{L}_{f_0}[\cdot]$ we obtain:

$$\tilde{f}_\tau = \left(\tilde{h}_1\right)_\tau + \psi_\tau - \Lambda(\tau) \mathcal{L}_{f_0}[f_0] - \mathcal{L}_{f_0} \left[\int_0^\tau w(\tau - s, x) \Lambda(s) ds \right]$$

Eliminating the integral in the last term by means of (8.56):

$$\tilde{f}_\tau = \left[\left(\tilde{h}_1\right)_\tau - \mathcal{L}_{f_0}[\tilde{h}_1] \right] + [\psi_\tau - \mathcal{L}_{f_0}[\psi]] - \Lambda(\tau) \mathcal{L}_{f_0}[f_0] + \mathcal{L}_{f_0}[\tilde{f}]$$

Proposition 27 yields $\psi_\tau - \mathcal{L}_{f_0}[\psi] = 0$. Using also (8.1) we obtain:

$$\tilde{f}_\tau = \frac{Q[h]}{\Lambda(\tau)} + \Lambda(\tau) Q[f_0] - \Lambda(\tau) \mathcal{L}_{f_0}[f_0] + \mathcal{L}_{f_0}[\tilde{f}]$$

where due to (8.55) $h = \tilde{f}(\tau, x) - \Lambda(\tau) f_0(x)$. Then:

$$\tilde{f}_\tau = \frac{Q[\tilde{f}]}{\Lambda(\tau)} - \mathcal{L}_{f_0}[\tilde{f}] + \Lambda(\tau) Q[f_0] + \Lambda(\tau) Q[f_0] - \Lambda(\tau) \mathcal{L}_{f_0}[f_0] + \mathcal{L}_{f_0}[\tilde{f}]$$

Using that $\mathcal{L}_{f_0}[f_0] = 2Q[f_0]$ (cf. (8.28)) we obtain that \tilde{f} solves (8.57). Using the time scale t given by means of (3.4) we deduce that $f(t, x) = \tilde{f}(\tau, x)$ solves (1.1), (1.2). Using (8.2), (8.18) and (8.56) we have that f satisfies (1.3). This concludes the Proof of the existence of the sought-for solution.

We prove uniqueness in the class of solutions stated in Theorem 1 as follows. Suppose that we have two solutions f^α, f^β of (1.1)-(1.3) such that $f^\alpha = \lambda^\alpha(t) f_0(x) + h^\alpha$, $f^\beta = \lambda^\beta(t) f_0(x) + h^\beta$ with $\lambda^\alpha, \lambda^\beta \in C[0, T]$, $h^\alpha, h^\beta \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$.

Using the change of variables (3.4), for both solutions and denoting as τ the new time scale in both case we obtain functions $\tilde{f}^\alpha, \tilde{f}^\beta$ satisfying (8.57) with $\Lambda = \Lambda^\alpha = \lambda^\alpha$ and $\Lambda = \Lambda^\beta = \lambda^\beta$ respectively. We will write, with a bit abuse of notation $\tilde{f}^\alpha = \Lambda^\alpha(\tau) f_0(x) + h^\alpha$, $\tilde{f}^\beta = \Lambda^\beta(\tau) f_0(x) + h^\beta$. We define functions $\tilde{h}_2^\alpha, \tilde{h}_2^\beta \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$ by means of (8.48) with the corresponding functions $\Lambda^\alpha, \Lambda^\beta$. We define also the functions $\tilde{h}_1^\alpha, \tilde{h}_1^\beta \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$ by means of $\tilde{h}_1^\alpha = h^\alpha - \tilde{h}_2^\alpha$, $\tilde{h}_1^\beta = h^\beta - \tilde{h}_2^\beta$.

Using arguments analogous to the ones used in the derivation of (8.57) we obtain:

$$\left(\tilde{h}_1^k\right)_\tau - \mathcal{L}_{f_0} \left[\tilde{h}_1^k\right] = \frac{Q[h^k]}{\Lambda^k(\tau)} + \Lambda^k(\tau) Q[f_0], \quad \tilde{h}_1^k(0) = 0, \quad k = \alpha, \beta$$

Using Proposition 24 we obtain that $\tilde{h}_1^\alpha, \tilde{h}_1^\beta$ have the asymptotics (8.3). Moreover, for $T \leq T_0$ small enough we have that the operator \mathcal{G} is contractive in h and Λ . Therefore:

$$\|\mathcal{G}[\cdot; h^\alpha, \Lambda^\alpha] - \mathcal{G}[\cdot; h^\beta, \Lambda^\beta]\|_{C[0, T]} \leq \theta \|\Lambda^\alpha - \Lambda^\beta\|_{C[0, T]} + \theta \|h^\alpha - h^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}$$

where $0 < \theta < 1$ can be made arbitrarily small for T_0 sufficiently small. Moreover, since $h^\alpha, h^\beta \in \mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)$, the functions $\Lambda^\alpha, \Lambda^\beta$ solve the integral equation (8.51). The function Λ depends in a Lipschitz manner on the function \mathcal{G} with a constant smaller than two if $T \leq T_0$. Therefore $\|\Lambda^\alpha - \Lambda^\beta\|_{C[0, T]} \leq 2\theta \|\Lambda^\alpha - \Lambda^\beta\|_{C[0, T]} + 2\theta \|h^\alpha - h^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}$ whence $\|\Lambda^\alpha - \Lambda^\beta\|_{C[0, T]} \leq 4\theta \|h^\alpha - h^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}$.

Using then the contractivity of the mapping $(h, \Lambda) \rightarrow \tilde{h}_1$ (cf. Proposition 24) we then obtain $\|\tilde{h}_1^\alpha - \tilde{h}_1^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \leq \frac{1}{4} \|h^\alpha - h^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}$. On the other hand (8.48) yields $\|\tilde{h}_2^\alpha - \tilde{h}_2^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \leq C\theta \|h^\alpha - h^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}$. Therefore, choosing θ small enough:

$$\|h^\alpha - h^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \leq \|\tilde{h}_1^\alpha - \tilde{h}_1^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} + \|\tilde{h}_2^\alpha - \tilde{h}_2^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)} \leq \frac{1}{2} \|h^\alpha - h^\beta\|_{\mathcal{Z}_p^{\sigma; \frac{1}{2}}(T)}$$

whence $h^\alpha = h^\beta$. Then $\Lambda^\alpha = \Lambda^\beta$ and the uniqueness follows. ■

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