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Letter

Information Theory

Gaussian inputs: Performance limits over non-coherent SISO and MIMO channels

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SUMMARY

Performance limits of information transfer over a discrete time memoryless Rayleigh fading channel with neither the receiver nor the transmitter knowing the fading coefficients except its statistics is an important problem in information theory. We derive closed form expressions for the mutual information of single input single output (SISO) and multiple input multiple output (MIMO) Rayleigh fading channels for any antenna number at any signal to noise ratio (SNR). Using these expressions, we show that the maximum mutual information of non-coherent Rayleigh fading MIMO channels is achieved with a single transmitter and multiple receivers when the input distribution is Gaussian. We show that the addition of transmit antennas for a fixed number of receivers result in a reduction of mutual information. Furthermore, we argue that the mutual information is bounded by the SNR in both SISO and MIMO systems showing the sub-optimality of Gaussian signalling in non-coherent Rayleigh fading channels. Copyright © 2006 AEIT

1. INTRODUCTION

The independent and identically distributed (i.i.d.) Gaussian distribution is the optimal input in the additive white Gaussian noise (AWGN) channel [1], as well as in the SISO Rayleigh fading channel when the receiver has the perfect channel state information (CSI) [2, 3]. The Gaussian distribution is the capacity achieving input distribution when the transmitter has perfect CSI in addition to the receiver with optimal power distribution over the channel, usually called *water filling* at the transmitter [2, 4]. However, when neither the receiver nor the transmitter has CSI (non-coherent), the optimal input distribution which achieves channel capacity is not Gaussian [5]. In this paper, we evaluate how well the Gaussian distributed input perform at any SNR over non-coherent Rayleigh fading SISO and MIMO channels.

A plethora of literature is available on the capacity and achievable rates over fading channels for example References [6, 7], and a significant amount of effort has been expended to study fading channel models where side information about the fading is available at either the receiver or the transmitter or both [2, 6, 8, 9]. Fading channel capacity of SISO systems with perfect receiver CSI was originally shown by Ericson [10] and later by Lee [3] who showed the degradation of the channel capacity in a Rayleigh fading environment compared to the AWGN channel. They also showed how to use diversity schemes to improve the performance. In particular, Reference [7] describes the statistical models of fading channels and focuses on the information

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theory of fading channels, by emphasising capacity as the most important measure. Even when the non-coherent capacity or its supremum is known, it is difficult to identify the input distribution which provides the capacity based on the input constraints [11]. Even though, the Gaussian input offers the channel capacity in coherent Rayleigh fading channels, the capacity achieving input distribution in non-coherent Rayleigh fading channels is discrete with a finite number of mass points [5, 11]. This was originally conjectured by Richters [12] who considered the problem of communicating over an average power limited discrete memoryless SISO Rayleigh fading channels without any CSI. Abou-Faycal, Trott and Shamai [5] given a rigorous proof for Richters' original thoughts. This result shows that in non-coherent Rayleigh fading SISO channels, the optimal input has a significant different character than in the unfaded Gaussian or the coherent Rayleigh fading channels. Effects on Gaussian signalling in non-coherent Rayleigh fading SISO channels is shown in Reference [13]. It is proven that the mutual information is asymptotically bounded by SNR. The results are useful in purging Gaussian distributed inputs at high SNR. Asymptotic lower and upper bounds are derived in Reference [14] using a Gauss-Markov fading model in terms of the first-order Markov process. It is shown that the mutual information by the Gaussian input is bounded above by a constant at high SNR, generalising the result of [13].

Much of the interest in MIMO systems was motivated by the work done in References [15, 16] which shows the potential increase in the channel capacity when the CSI is perfectly known at the receiver using multiple antennas for both transmission and receiving. The main outcome is linear growth in capacity with the minimum of the number of transmit and receive antennas. In MIMO communication systems, the capacity achieving input distribution in Rayleigh fading with CSI perfectly known at the receiver is circularly symmetric complex Gaussian [15] having equal power at each transmitter. Similar to SISO Rayleigh fading channel [3, 10], the Gaussian input achieves the channel capacity in MIMO systems when the channel matrix is fixed or random [15, 17] with perfect CSI at the receiver. The additional capacity gain when the transmitter has the knowledge of the underlying channel can be achieved as in SISO channels by *water filling* the transmit antennas where the optimal input distribution is Gaussian [18].

Hochwald and Marzetta [19] analysed MIMO channels operating in a Rayleigh flat fading environment assuming the fading coefficients remain constant for a coherence interval of T symbol periods. Under this assumption they concluded that further increasing the number of transmit antennas beyond T cannot increase the capacity for non-coherent channels. Also, they characterised the certain structure of the optimal input distribution being mutually orthogonal with respect to time among the transmitter antennas. This work is extended by Zheng and Tse [20]. It provides the asymptotic capacity at high SNR in terms of T and antenna numbers interpreting the problem as sphere packing in the Grassmann manifold. Also, they showed that having more transmit antennas than receive antennas provide no capacity gain at high SNR, while having more receive antennas does yield a capacity gain. At low SNR, it is proven that the non-coherent and coherent capacities are asymptotically equal, and there is no capacity penalty for not knowing the channel at the receiver.

Even the optimal input for the non-coherent SISO Rayleigh fading channel is known to be discrete, the optimal input in non-coherent Rayleigh fading with multiple antennas is difficult to obtain. Preliminary results demonstrating capacity bounds [21] and input distributions [22] exist, however, these are for high SNR or asymptotic in antenna numbers. As the Gaussian distributed input is optimal in coherent SISO [2, 3] and MIMO [15] Rayleigh fading channels, it is of interest to find the mutual information in the absence of CSI. In particular, the interest in Gaussian input is studied in Reference [23], and shows the numerical analysis of the mutual information of non-coherent SISO Rayleigh fading channel with an analytical lower bound.

In this paper, we extend the work carried out in SISO systems [23] and provide a *closed form*† expression for the mutual information using Gaussian quadrature formulas and demonstrate the performance of Gaussian signalling at any SNR. This work supersedes the result given in References [13, 14] for the asymptotic value since the closed form expression can be used for any SNR. In addition, we express the mutual information of the non-coherent, uncorrelated Rayleigh fading MIMO channel and derive a closed form expression when the input distribution is complex Gaussian for a finite number of antennas and SNR. Evaluating the results in this paper, with both [19, 20], we confirm the increase in mutual information with receiver diversity. Furthermore, we show the reduction in mutual information with additional transmitters disputing the undisturbed capacity under multiple transmitters claimed in Reference [19].

† In this paper, we use the term *closed form* to a computationally efficient and numerically tractable expression where the outcome is strictly defined.

2. SYSTEM MODEL

2.1. SISO channel notation

Consider the discrete time, frequency flat, SISO Rayleigh fading channel,

$$y = ax + n \tag{1}$$

where x, y are the complex channel input and output respectively, a is the fading coefficient, and n represent background noise samples associated with the channel. The time index in Equation (1) is omitted for simplicity. Both a and n are assumed to be independent zero mean circular complex Gaussian random variables with equal unit variance in each dimension. Furthermore, they are assumed to be independent of each other and of the channel input. We also assume that the input x is average power limited, that is, $\int x^2 p_X(x)\mathrm{d}x = \Omega_x \le P$, where $p_X(x)$ is the probability density function (pdf) of the channel input.

2.2. MIMO channel notation

The input output relationship of a MIMO channel with n_t transmitters and n_r receivers is given by

$$\boldsymbol{Y} = \mathbf{H}\boldsymbol{X} + \boldsymbol{N} \tag{2}$$

where $\boldsymbol{Y}$ is the vector of received signals of $n_\mathrm{r} \times 1$, $\mathbf{H}$ is the channel gain matrix of $n_\mathrm{r} \times n_\mathrm{t}$ where each element h_{ij}, $i = 1, \ldots, n_\mathrm{r}$, $j = 1, \ldots, n_\mathrm{t}$ is assumed to be zero mean circular complex Gaussian random variables with a unit variance in each dimension, $\boldsymbol{X}$ is the vector of transmitted signals of $n_\mathrm{t} \times 1$, and $\boldsymbol{N}$ is the zero mean circular complex Gaussian unit variance noise vector of $n_\mathrm{r} \times 1$ elements.

Let $X = |\boldsymbol{X}|$ and $Y = |\boldsymbol{Y}|$ where $|\cdot|$ is the Euclidean norm, and also let x and y represent each realisation of X and Y (i.e. $x \epsilon X$ and $y \epsilon Y$). The input is power limited with an average power constraints $\int x^2 p_X(x)\mathrm{d}x \le P$, where $p_X(x)$ is the pdf of the channel input X. We use $\Gamma(\cdot)$ and $\Psi(\cdot)$ to indicate Gamma and Psi functions respectively, $h(\boldsymbol{X})$ denotes the differential entropy of $\boldsymbol{X}$, and $I(\boldsymbol{X}; \boldsymbol{Y})$ designates the mutual information between $\boldsymbol{X}$ and $\boldsymbol{Y}$. All the differential entropies and the mutual information are defined to the base 'e', hence the results are expressed in 'nats'. Also let $\gamma = -\int_0^\infty \mathrm{e}^{-y} \log y \mathrm{d}y \approx 0.5772\ldots$, to denote the Euler's constant.

2.3. Channel model

The Rayleigh channel model is particularly appropriate when there is no direct line of sight (LOS) component except the faded components arising from multipath propagation. In both the channel models noted by Equations (1) and (2), fading is assumed to be flat with a multiplicative effect on the channel input. This assumption is valid if the channel coherence bandwidth $B_\mathrm{coh} = 1/T_\mathrm{m}$ is much greater than the signal bandwidth $B_\mathrm{s} = 1/T_\mathrm{s}$ applied on the input, where T_m and T_s are the delay spread of the channel impulse response and the symbol duration respectively. In particular, each fading coefficient assumed to be independent realisations at every symbol period, and the reliable estimation of fading coefficients could be quite difficult due to the short duration between independent fades. Therefore, we consider the non-coherent scenario where neither the receiver nor the transmitter knows the fading coefficients other than the statistics.

3. SISO MUTUAL INFORMATION

Mutual information between the input and output of a Rayleigh fading channel can be written as [11]

$$I(X;Y) = \int_0^\infty \int_0^\infty p_{Y|X}(y|x) p_X(x) \times \log\left[\frac{p_{Y|X}(y|x)}{\int_0^\infty p_{Y|V}(y|v) p_V(v) dv}\right] \mathrm{d}x\mathrm{d}y \tag{3}$$

using the probability distribution of the magnitudes of the random variables X and Y. The output conditional pdf, $p_{Y|X}(y|x)$ [11, 24] for the channel model (1) with complex Gaussian random variables a and n having a unit variance in each dimension is given by Reference [23]

$$p_{Y|X}(y|x) = \frac{y}{1+x^2} \exp\left[\frac{-y^2}{2(1+x^2)}\right] \tag{4}$$

Without loss of generality, the magnitude sign is omitted in Equation (4) for simplicity and likewise in the rest of this paper.

Mutual information $I(X;Y) = h(Y) - h(Y|X)$, in Equation (3) is found using

$$h(Y) = -\int_0^\infty p_Y(y) \log p_Y(y) \mathrm{d}y \tag{5}$$

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and

$$h(Y|X) = \frac{1}{2}\mathrm{E}_x\{\log(1+x^2)\} - \frac{1}{2}\log 2 + \left(1+\frac{\gamma}{2}\right) \quad (6)$$

Equations (5) and (6) were first used by Taricco [11] to apply Lagrange optimisation to obtain the channel capacity.

3.1. Output conditional entropy

When the input distribution is complex Gaussian, the pdf of $|x|$ is Rayleigh and given by

$$p_X(x) = \frac{2x}{\Omega_x}\exp\left(\frac{-x^2}{\Omega_x}\right), x \geq 0 \quad (7)$$

where Ω_x is the average input power constraint. Then the output conditional entropy [23]

$$h(Y|X) = \frac{1}{2}C_{\mathrm{rcsi}} - \frac{1}{2}\log 2 + \left(1+\frac{\gamma}{2}\right) \quad (8)$$

where C_{rcsi} is the channel capacity when the CSI is perfectly known at the receiver [3, 7, 10].

3.2. Output entropy

The output probability density function

$$p_Y(y) = \int_0^\infty p_X(x)p_{Y|X}(y|x)\mathrm{d}x \quad (9)$$

for the Gaussian distributed input is given by

$$p_Y(y) = \int_0^\infty \frac{2x}{\Omega_x}\exp\left(\frac{-x^2}{\Omega_x}\right)\frac{y}{1+x^2}\exp\left[\frac{-y^2}{2(1+x^2)}\right]\mathrm{d}x \quad (10)$$

We substitute Equation (10) in (5) to get

$$h(Y) = -\int_0^\infty \int_0^\infty \frac{2xy}{\Omega_x(1+x^2)}\exp\left[\frac{-x^2}{\Omega_x} - \frac{y^2}{2(1+x^2)}\right]\mathrm{d}x \times \log\left\{\int_0^\infty \frac{2xy}{\Omega_x(1+x^2)}\exp\left[\frac{-x^2}{\Omega_x} - \frac{y^2}{2(1+x^2)}\right]\mathrm{d}x\right\}\mathrm{d}y \quad (11)$$

Since this integral cannot be evaluated analytically, we show how the Gauss–Hermite quadrature is used to arrive at a closed form expression.

3.2.1. Gaussian quadrature and Hermite polynomials

Gaussian quadrature formulas are useful in numerical integrations and provides fast and accurate results [25]. The common investigated method for approximating a definite integral is $\int_a^b \omega(x)f(x)\mathrm{d}x \simeq \sum_{i=1}^q A_i f(x_i)$, assuming the moments of the function $\omega(x)$ are defined and finite or bounded. The Gaussian quadrature formula has a degree of precision or exactness m if the solution is exact whenever $f(x)$ is a polynomial of degree $\leq m$ or equivalently, whenever $f(x) = \{1, x, \ldots, x^m\}$ and it is not exact for $f(x) = x^{m+1}$. The x_i are called the nodes of the formula and A_i are called coefficients (or weights).

If $\omega(x)$ is non-negative in $[a, b]$, then n points and coefficients can be found to make the solution exact for all polynomials of degree $\leq 2q-1$ and it is the highest degree of precision which can be obtained using q points [25].

If the function $\omega(x)$ is in the form of e^{-x^2}, the solution to the integral can be found using the roots (nodes) of the Hermite polynomial $H_q(x)$ [26] and the weights are given by

$$\omega_i = \frac{2^{q-1}q!\sqrt{\pi}}{q^2\left[H_{q-1}(x_i)\right]^2}$$

The roots and the weights are excessively given in Reference [26] for $a = 0$ and $b = \infty$ up to $q = 15$.

3.2.2. Closed form expression

Let $t^2 = x^2/\Omega_x$, where $\mathrm{d}x = \sqrt{\Omega_x}\mathrm{d}t$, and substitute in Equation (10) to get

$$p_Y(y) = \int_{t=0}^{t=\infty} \exp(-t^2)\frac{2ty}{(1+\Omega_x t^2)}\exp\left[\frac{-y^2}{2(1+\Omega_x t^2)}\right]\mathrm{d}t. \quad (12)$$

This integral is in the form of $\int_a^b \phi(v)\omega(v)\mathrm{d}v$ where $\omega(v) \equiv \exp(-t^2)$. Therefore it can be evaluated using Hermite polynomials in the form of $p_Y(y) = \sum_{j=1}^q \omega_j f(v_j)$. The quantities v_j and ω_j are the roots and the weights of the Hermite polynomials respectively. Applying these weights and roots in Equation (12) we obtain

$$p_Y(y) = \sum_{j=1}^q \omega_j \frac{2v_j y}{(1+\Omega_x v_j^2)}\exp\left[\frac{-y^2}{2(1+\Omega_x v_j^2)}\right] \quad (13)$$

Using this result, we get the output entropy

$$h(Y) = -\int_0^\infty \left\{\sum_{\ell=1}^q \omega_\ell \frac{2v_\ell y}{(1+\Omega_x v_\ell^2)}\exp\left[\frac{-y^2}{2(1+\Omega_x v_\ell^2)}\right]\right\}$$

$$\times \log\left\{\sum_{j=1}^{q}\omega_j \frac{2v_j y}{(1+\Omega_x v_j^2)}\exp\left[\frac{-y^2}{2(1+\Omega_x v_j^2)}\right]\right\}\mathrm{d}y \tag{14}$$

Taking the integration inside the summation and substituting $t^2 = y^2/(2(1+\Omega_x v_j^2))$ in Equation (14), the ℓth term where $\ell = 1,\ldots,q$ can be written as

$$h(Y)_\ell = -\int_0^\infty \exp(-t^2)(4\omega_\ell v_\ell t)\log\left\{\sum_{j=1}^{q}\frac{2\sqrt{2}\omega_j v_j t}{(1+\Omega_x v_j^2)}\right.$$
$$\left.\times\sqrt{(1+\Omega_x v_\ell^2)}\exp\left[-t^2\frac{(1+\Omega_x v_\ell^2)}{(1+\Omega_x v_j^2)}\right]\right\}\mathrm{d}t \tag{15}$$

The integral in the ℓth term has the form of $\int_a^b \phi(v)\omega(v)\mathrm{d}v$ where $\omega(v) \equiv \exp(-t^2)$. We can now simplify Equation (15) using Hermite polynomials as

$$h(Y)_\ell = \sum_{i=1}^{r}\omega_i(4\omega_\ell v_\ell v_i)\log\left\{\sum_{j=1}^{q}\frac{2\sqrt{2}\omega_j v_j v_i}{(1+\Omega_x v_j^2)}\right.$$
$$\left.\times\sqrt{(1+\Omega_x v_\ell^2)}\exp\left[-v_i^2\frac{(1+\Omega_x v_\ell^2)}{(1+\Omega_x v_j^2)}\right]\right\} \tag{16}$$

Since the output entropy is the summation of each term, finally we get

$$h(Y) = -\sum_{\ell=1}^{q}h(Y)_\ell$$
$$= -\sum_{\ell=1}^{q}\sum_{i=1}^{r}(4\omega_i v_i\omega_\ell v_\ell)\log\left\{\sum_{j=1}^{q}\frac{2\sqrt{2}\omega_j v_j v_i}{(1+\Omega_x v_j^2)}\right.$$
$$\left.\times\sqrt{(1+\Omega_x v_\ell^2)}\exp\left[-v_i^2\frac{(1+\Omega_x v_\ell^2)}{(1+\Omega_x v_j^2)}\right]\right\} \tag{17}$$

$h(Y)$ presented in closed form in Equation (17) using Gauss–Hermite quadrature is very useful in finding the mutual information for any SNR. The computational time is much less than the numerical integrations to be carried out with high accuracy. Mutual information can be computed subtracting Equation (8) from equation (17).

4. MIMO MUTUAL INFORMATION

4.1. Output conditional entropy

The conditional pdf of channel output given input of Equation (2) is given by Reference [27]

$$p_{Y|X}(y|x) = \frac{y^{2n_r-1}\exp\left[-\frac{y^2}{2(1+x^2)}\right]}{2^{n_r-1}\Gamma(n_r)(1+x^2)^{n_r}} \tag{18}$$

With Equation (18), we get the output conditional entropy

$$h(\boldsymbol{Y}|\boldsymbol{X}) = -\mathrm{E}_x\left\{\int_0^\infty p_{Y|X}(y|x)\log p_{Y|X}(y|x)\mathrm{d}y\right\}, \tag{19a}$$
$$= \tfrac{1}{2}\mathrm{E}_x\left[\log(1+x^2)\right] + \log\left[\tfrac{\Gamma(n_r)}{\sqrt{2}}\right]$$
$$-\left(n_r - \tfrac{1}{2}\right)\Psi(n_r) + n_r \tag{19b}$$

where the expectation is taken over $x \equiv (x_1, x_2, \ldots, x_{n_t})$.

In case of a single antenna system with $n_t = n_r = 1$, Equation (19b) simplifies to entropy given in Reference [11]. We can use Equation (19b) to compute the output conditional entropy of non-correlated Rayleigh fading MIMO channels when no CSI is available for a given input distribution.

4.2. Nakagami-m distribution

For a Gaussian distributed input in a MIMO system, the joint input distribution is required in order to generalise with antenna number when the input power is kept constant. Since the output conditional pdf (18) depends only on the magnitudes of input and output, we use scalar random variables. Input $X = |\boldsymbol{X}|$ can be described as the magnitude of the $2m$ zero mean and equal variance Gaussian random variables for n_t transmitters. This is the well known Nakagami-m distribution [28], which can also be considered as the square root of the sum of squares of m independent Rayleigh or $2m$ Gaussian variate [27]. The Nakagami-m pdf for $r = \sqrt{X_1^2 + X_2^2 + \cdots + X_{2m}^2}$ is described by [28]

$$f_R(r) = \frac{2m^m r^{2m-1}}{\Gamma(m)\Omega_r^m}\exp\left(-\frac{mr^2}{\Omega_r}\right) \tag{20}$$

where m is the Nakagami-m fading parameter which range varies from $1/2$ to ∞, X_i for $i = 1,\ldots,2m$ are Gaussian distributed, and $\Omega_r = \overline{r^2}$ is the average value of the random variable r.

The composite input distribution when each input signal is Gaussian can be treated as $2n_t$ Nakagami with pdf

$$p_X(x) = \frac{2{n_t}^{n_t}x^{2n_t-1}}{\Gamma(n_t)\Omega_x^{n_t}}\exp\left(-\frac{n_t x^2}{\Omega_x}\right) \tag{21}$$

Note here the number of transmitters $n_\mathrm{t} = m$ since the total dimensions are $2n_\mathrm{t}$ where Ω_x is the average mean-squared input power irrespective of the number of transmitters n_t. Now we show the derivation of the closed form expression for the mutual information with input distribution (21) using Hermite polynomials.

4.3. Mutual information in closed form

Using $h(\boldsymbol{Y}|\boldsymbol{X})$ in Equation (19b), we obtain the mutual information

$$I(\boldsymbol{X};\boldsymbol{Y}) = h(\boldsymbol{Y}) - h(\boldsymbol{Y}|\boldsymbol{X}) \tag{22a}$$

$$= -\int_0^\infty p_Y(y)\log p_Y(y)\mathrm{d}y - \tfrac{1}{2}\mathrm{E}_x\left\{\log(1+x^2)\right\} - \log\left[\tfrac{\Gamma(n_\mathrm{r})}{\sqrt{2}}\right] + \left(n_\mathrm{r} - \tfrac{1}{2}\right)\Psi(n_\mathrm{r}) - n_\mathrm{r} \tag{22b}$$

We have the following theorem.

Theorem 1. Mutual information of a non-coherent uncorrelated Rayleigh fading MIMO channel when the input distribution is complex Gaussian is given by Equation (22a) where

$$h(\boldsymbol{Y}|\boldsymbol{X}) = \frac{1}{\Gamma(n_\mathrm{t})}\sum_{j=1}^{r}\omega_j v_j^{2n_\mathrm{t}-1}\log\left(1+\frac{\Omega_x v_j^2}{n_\mathrm{t}}\right) + \log\left[\frac{\Gamma(n_\mathrm{r})}{\sqrt{2}}\right] - \left(n_\mathrm{r}-\frac{1}{2}\right)\Psi(n_\mathrm{r}) + n_\mathrm{r} \tag{23}$$

and

$$h(\boldsymbol{Y}) = -\sum_{\ell=1}^{q}\sum_{i=1}^{r}\frac{4\omega_i\omega_\ell v_i^{2n_\mathrm{r}-1}v_\ell^{2n_\mathrm{t}-1}}{\Gamma(n_\mathrm{t})\Gamma(n_\mathrm{r})}\log\left\{\frac{2\sqrt{2}}{\Gamma(n_\mathrm{t})\Gamma(n_\mathrm{r})}\times\sum_{j=1}^{q}\frac{\omega_j v_j^{2n_\mathrm{t}-1}v_i^{2n_\mathrm{r}-1}\left(1+\frac{\Omega_x v_\ell^2}{n_\mathrm{t}}\right)^{n_\mathrm{r}-\frac{1}{2}}}{\left(1+\frac{\Omega_x v_j^2}{n_\mathrm{t}}\right)^{n_\mathrm{r}}}\times\exp\left[-\frac{v_i^2\left(1+\frac{\Omega_x v_\ell^2}{n_\mathrm{t}}\right)}{\left(1+\frac{\Omega_x v_j^2}{n_\mathrm{t}}\right)}\right]\right\} \tag{24}$$

The quantities $\{\omega\}$ and $\{v\}$ represent the weights and roots of the Hermite polynomials.

Proof. The first term of Equation (19b) for the input distribution (21) can be written as

$$h'(\boldsymbol{Y}|\boldsymbol{X}) = \frac{n_\mathrm{t}^{n_\mathrm{t}}}{\Gamma(n_\mathrm{r})\Omega_x^{n_\mathrm{t}}}\int_0^\infty x^{2n_\mathrm{t}-1}\exp\left(-\frac{n_\mathrm{t}x^2}{\Omega_x}\right)\times\log(1+x^2)\mathrm{d}x \tag{25}$$

Using Gauss–Hermite quadrature and substituting $t^2 = n_\mathrm{t}x^2/\Omega_x$ in Equation (25), we get

$$h'(\boldsymbol{Y}|\boldsymbol{X}) = \int_0^\infty \frac{\Omega_x^{n_\mathrm{t}}}{n_\mathrm{t}^{n_\mathrm{t}}}\mathrm{e}^{-t^2}t^{2n_\mathrm{r}-1}\log\left(1+\frac{\Omega_x t^2}{n_\mathrm{t}}\right)\mathrm{d}t = \frac{1}{\Gamma(n_\mathrm{t})}\sum_{j=1}^{r}\omega_j v_j^{2n_\mathrm{t}-1}\log(1+\frac{\Omega_x v_j^2}{n_\mathrm{t}}) \tag{26}$$

Substituting Equation (26) in Equation (19b) we obtain (23).

The output pdf $p_Y(y)$ in non-coherent MIMO Rayleigh fading channel when the input distribution is Gaussian (i.e. the magnitude distribution is $2n_\mathrm{t}$ Nakagami),

$$p_Y(y) = \int_0^\infty \frac{\xi(n_\mathrm{r},n_\mathrm{t})n_\mathrm{t}^{n_\mathrm{t}}y^{2n_\mathrm{r}-1}}{\Omega_x^{n_\mathrm{t}}}\frac{x^{2n_\mathrm{t}-1}\exp(-\frac{n_\mathrm{t}x^2}{\Omega_x})}{(1+x^2)^{n_\mathrm{r}}}\times\exp\left[-\frac{y^2}{2(1+x^2)}\right]\mathrm{d}x \tag{27}$$

can be used to evaluate the output entropy

$$h(\boldsymbol{Y}) = -\int_0^\infty p_Y(y)\log p_Y(y) \tag{28}$$

where $\xi(n_\mathrm{r},n_\mathrm{t}) = 2/\Gamma(n_\mathrm{t})\Gamma(n_\mathrm{r})2^{n_\mathrm{t}-1}$. Again, we use the Hermite polynomials to find a closed form solution from Equation (27). Substituting $t^2 = n_\mathrm{t}x^2/\Omega_x$ in Equation (27) and using Hermite polynomial weighting factors $\{\omega\}$ and roots $\{v\}$, we get

$$p_Y(y) = \xi(n_\mathrm{t},n_\mathrm{r})\left\{\sum_{j=1}^{q}\frac{\omega_j v_j^{2n_\mathrm{t}-1}y^{2n_\mathrm{r}-1}}{\left(1+\frac{\Omega_x v_j^2}{n_\mathrm{t}}\right)}\times\exp\left[-\frac{y^2}{2\left(1+\frac{\Omega_x v_j^2}{n_\mathrm{t}}\right)}\right]\right\} \tag{29}$$

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Using Equation (29), the ℓth term, $\ell = 1, \ldots, q$ of $h(\boldsymbol{Y})$ can be written as

$$h(\boldsymbol{Y})_\ell = -\xi(n_t, n_r) \int_0^\infty \left\{ \frac{\omega_\ell v_\ell^{2n_t-1} y^{2n_r-1}}{\left(1 + \frac{\Omega_x v_\ell^2}{n_t}\right)} \times \exp\left[-\frac{y^2}{2\left(1 + \frac{\Omega_x v_\ell^2}{n_t}\right)} \right] \right\} \log \{\xi(n_t, n_r) \times \sum_{j=1}^{q} \frac{\omega_j v_j^{2n_t-1} y^{2n_r-1}}{\left(1 + \frac{\Omega_x v_j^2}{n_t}\right)} \exp\left[-\frac{y^2}{2\left(1 + \frac{\Omega_x v_j^2}{n_t}\right)} \right] \right\} \mathrm{d}y \quad (30)$$

Integral in Equation (30) can be solved using Hermite polynomials for a closed form solution by substituting $t^2 = y^2/2(1 + \Omega_x v_\ell^2/n_t)$ and applying the Hermite polynomial approximation. Using the summation $h(\boldsymbol{Y}) = \sum_{\ell=1}^{q} h(\boldsymbol{Y})_\ell$, finally we get the closed form expression (24) for the output entropy. ■

Both Equations (24) and (23) are useful to evaluate the mutual information (22a) in a non-coherent Rayleigh fading MIMO channel. The computation time is negligible when compared to other numerical integration methods and the accuracy is very high. Using the closed form expression given in Equations (23) and (24), the mutual information for any transmit and receive configurations can be found at a given SNR.

5. NUMERICAL RESULTS

5.1. *Non-coherent SISO channel*

Mutual information computed using closed form expressions (8) and (17) when the input distribution is complex Gaussian is shown in Figure 1. These expressions are very useful since the roots v and the weights ω are available in tabulated form for various degrees of m of the polynomial interested in Reference [25]. The computation time is negligible when compared to the numerical integrations to be solved and the accuracy is very high and typically known as exact on appropriate selection of the number of roots and the weighting factors [26].

Capacity of the non-coherent Rayleigh fading channel [5] attained with a discrete input is plotted in Figure 1 to compare with mutual information obtained when the input is Gaussian distributed. Also, Figure 1 has the asymptotic value of mutual information, $\lim_{\Omega_x \to \infty} (C_{\mathrm{cnf}} - C_{\mathrm{rcsi}}) = \gamma$ shown in Reference [13], when the input distribution is complex Gaussian where C_{cnf} is the Shannon capacity. Difference between the channel capacity and mutual information obtained with Gaussian input indicates that at high SNR loss in capacity due to Gaussian signalling increases. For instance, 44% of capacity loss is observed when SNR = 20 dB. Therefore, mutual information obtained with the Gaussian distributed input is very low compared to channel capacity in the absence of CSI indicating the sub-optimality of Gaussian signalling in non-coherent Rayleigh fading channels.

5.2. *Non-coherent MIMO channel*

We can numerically plot the mutual information using closed form expressions (23) and (24) when the input distribution is complex Gaussian. This is straightforward since the roots v and the weights ω are available in tabulated form for various degrees of m of the polynomial interested in Reference [25, 26].

Figure 2 shows the mutual information versus input power for various number of receivers with a single transmitter. Mutual information increases logarithmically as the

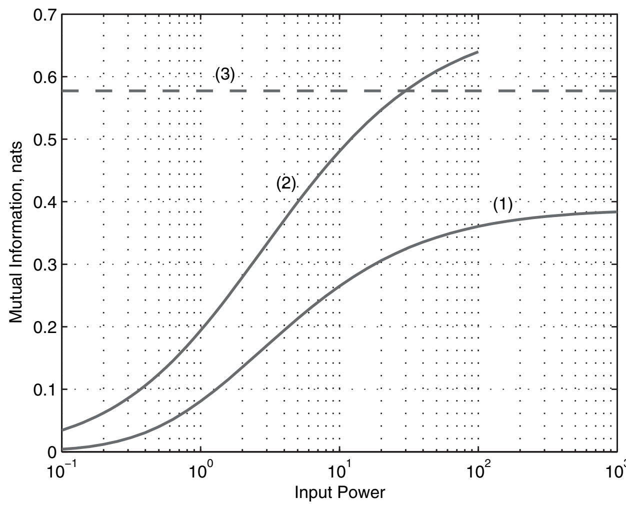

Figure 1. Mutual information, $I(X;Y)$ (1) of non-coherent SISO Rayleigh fading channel when the input is Gaussian distributed vs the channel capacity (2) simulated with two discrete inputs. Dashed line (3) shows the limit of mutual information with Gaussian input when SNR approaches infinity [12].

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number of receive antennas increases due to the enhancement of the receiver diversity (array gain). The increase in mutual information with additional receive antennas agree with the results obtained in References [18, 19] for the capacity. Also, it is trivial in coherent channels [16], where the receiver diversity improves the capacity significantly at high SNR than in non-coherent channels. However, mutual information is bounded on SNR for all combinations. It is evident from Figure 2 that mutual information approaches a limit at high SNR in all cases.

Work reported in Reference [20] indicates that having more transmit antennas than receive antennas provides no capacity gain at high SNR whilst both non-coherent and coherent capacities are asymptotically equal at high SNR. Therefore, in low-SNR regime the penalty for not knowing the channel at the receiver is negligible. As a result, at low SNR, the performance of Gaussian signalling is good since it is the capacity achieving distribution in both SISO [30] and MIMO [15] channels. This is verified from our results since the loss in mutual information compared to capacity at low SNR is minimal.

Figure 3 depicts the mutual information with SNR for various number of transmitters having a single receive antenna. Mutual information decreases with the increase of transmit antennas. The result is different to proved capacity increase with addition of transmitters in coherent scheme. Similar trends were observed with multiple receive antennas. Furthermore, Reference [19] revealed that for $T = 1$, the scenario considered in this paper, capacity is achieved by a single transmitter. Capacity remains the same for $n_t > 1$. However, the results shown in here for mutual information are contradictory. Furthermore, the mutual information is bounded by SNR for all configurations similar to multiple receivers with one transmit antenna.

In the non-coherent Rayleigh fading channels, input distribution (21) varies as transmitters increases and gets peaky [28, 29]. With a finite input power in the non-coherent Rayleigh fading MIMO channel, the allocation of equal power in all available transmitters is a waste since the channel is not perfectly known at neither the transmitter nor the receiver ensuing the optimum number of transmitters to one.

Figure 4 shows mutual information having equal number of transmitters and receivers. At low SNR, mutual information increases slightly due to the effective gain (array gain) with additional receivers over that of transmitters since the addition of transmitters for a fixed numbers of receivers reduces the mutual information of a non-coherent channel. At high SNR, mutual information decreases with the increase of equal number of transmitters and receivers since mutual information of the channel we consider is bounded at high SNR in the presence of Gaussian signalling at the input. Numerical result obtained with $n_t = 1$ and $n_r = 1$ using Equations (23) and (24) is similar to results shown in Reference [23] for SISO systems. Mutual information we derived for Gaussian signalling is 50% of the capacity shown in Reference [22] for large n_r and P with $n_t = 1$. Furthermore, mutual information is low compared to channel capacity reported in both [19, 20] at high SNR. This shows the sub-optimality of Gaussian signalling in non-coherent Rayleigh fading MIMO channels.

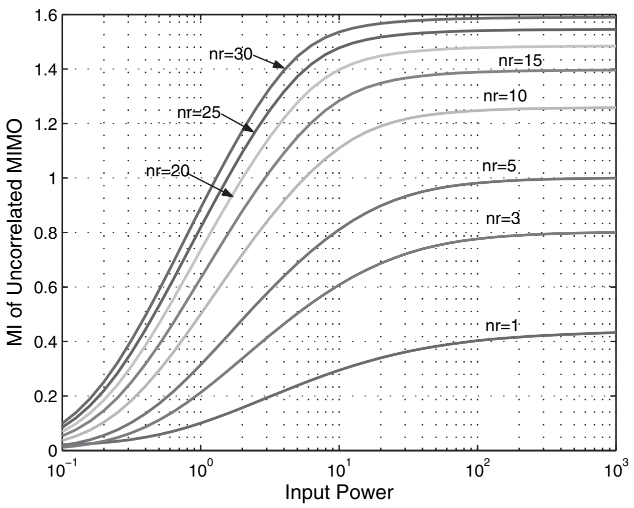

Figure 2. Mutual information of non-coherent uncorrelated Rayleigh fading MIMO channel vs input power when the input distribution is complex Gaussian for different n_r, for $n_t = 1$.

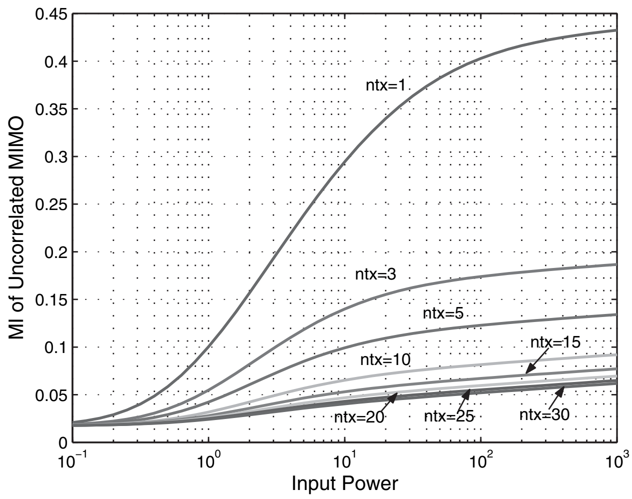

Figure 3. Mutual information of non-coherent uncorrelated Rayleigh fading MIMO channel vs input power when the input distribution is complex Gaussian for different n_t, for $n_r = 1$.

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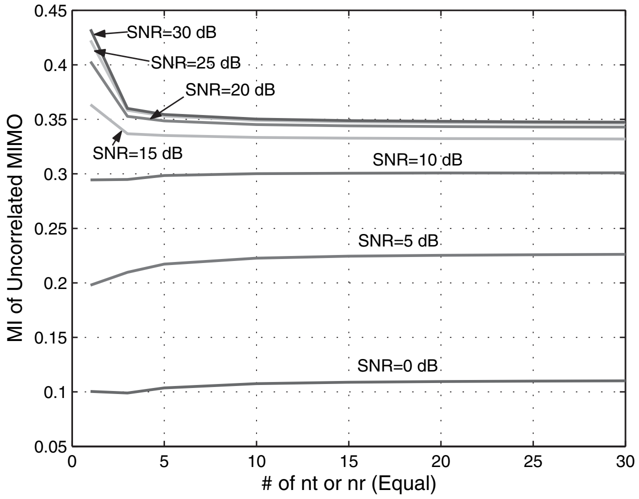

Figure 4. Mutual information of uncorrelated non-coherent Rayleigh fading MIMO channel having equal number of receivers and transmitters for different SNR.

6. CONCLUSIONS

Closed form expressions are derived to calculate mutual information of discrete time, frequency flat, Rayleigh fading SISO and MIMO channels with no CSI either at the receiver or transmitter for any SNR when the input is Gaussian distributed. It is shown that the mutual information is bounded by the SNR for both SISO and MIMO channels. In the low-SNR regime, the knowledge of CSI at the receiver is less important since the difference between channel capacity and mutual information with Gaussian input is insignificant. Since the Gaussian distribution is optimal with CSI at the receiver only or at the transmitter and receiver, Gaussian-distributed input signalling can be recommended in both coherent and non-coherent Rayleigh fading SISO and MIMO channels at low SNR. It is also concluded that additional transmitter antennas reduces mutual information with Gaussian-distributed input whilst capacity remains unchanged, a claim originated in Reference [19]. Therefore, a single transmitter with multiple receivers is desirable for Gaussian-distributed input to maximise the mutual information.

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